Kernel-based Image Reconstruction from Scattered Radon Data

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Abstract

Computerized tomography requires suitable numerical methods for the approximation of a bivariate function \( f \) from a finite set of discrete Radon data, each of whose data samples represents one line integral of \( f \). In standard reconstruction methods, specific assumptions concerning the geometry of the Radon lines are usually made. In relevant applications of image reconstruction, however, such assumptions are often too restrictive. In this case, one would rather prefer to work with reconstruction methods allowing for arbitrary distributions of scattered Radon lines.

This paper proposes a novel image reconstruction method for scattered Radon data, which combines kernel-based scattered data approximation with a well-adapted regularization of the Radon transform. This results in a very flexible numerical algorithm for image reconstruction, which works for arbitrary distributions of Radon lines. This is in contrast to the classical filtered back projection, which essentially relies on a regular distribution of the Radon lines, e.g., parallel beam geometry. The good performance of the kernel-based image reconstruction method is illustrated by numerical examples and comparisons.

1 Introduction

Computed Axial Tomography (CAT or CT) is a powerful technique to generate images from measurements of X-ray scans. One X-ray scan typically consists of several million of data samples, each of which corresponds to an X-ray beam passing through the computational domain, travelling from an emitter to a detector. The sensors of the operational CT scanner (positioned at the emitter and at the detector) then measures, for each X-ray beam, the loss of energy, resulting from the X-ray beam passing through the medium. The loss of energy reflects the ability of the medium to absorb energy, and so it depends on its specific structure and material properties. The amount of absorption can be described as a function of the computational domain \( \Omega \), termed attenuation coefficient function, \( f : \Omega \to [0, \infty) \).

Medical imaging is one relevant application for CT, where the primary goal is to reconstruct the unknown attenuation coefficient function \( f \) from given X-ray scans in order to generate clinically useful medical images. This requires robust numerical algorithms to reconstruct characteristic features of medical images at sufficiently high accuracy, on the one hand, and at sufficiently small computational costs, on the other hand. For details concerning the acquisition of X-ray scans, their underlying mathematical models, and standard computational methods for medical image reconstruction, we refer to the textbook [5] of Feeman.

To describe the mathematical problem of medical image reconstruction from X-ray scans, let us first regard the Radon transform \( \mathcal{R}f \) of \( f \), given by

\[
\mathcal{R}f(t, \theta) = \int_{\mathbb{R}} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) \, ds \tag{1}
\]

for \( t \in \mathbb{R} \) and \( \theta \in [0, \pi) \), where we assume \( f : \Omega \to \mathbb{R} \) to be a bivariate function on a compact domain \( \Omega \subset \mathbb{R}^2 \), or, in other words, we assume that \( f \) is compactly supported on \( \mathbb{R}^2 \), where we extend \( f \) to \( \mathbb{R}^2 \) by letting \( f \equiv 0 \) outside \( \Omega \). In many relevant application scenarios, we may assume that the image domain \( \Omega \) is the unit square, i.e., \( \Omega = [0, 1]^2 \), but this restriction is rather immaterial for our following discussion. We will merely assume \( f \in L^1 \), so that for any pair of \( t \in \mathbb{R} \) and \( \theta \in [0, \pi) \) the Radon integral in (1) is well-defined.

We remark that the Radon transform \( \mathcal{R}f(t, \theta) \) gives, for any fixed pair \((t, \theta)\) \(\in\mathbb{R}\times[0,\pi)\), a line integral for \( f \) over a specific straight line \( \ell \equiv \ell_{t, \theta} \). In order to see this, let \( \ell_{t, \theta} \subset \mathbb{R}^2 \) denote the unique straight line, which is perpendicular to unit vector \( n_\theta = (\cos \theta, \sin \theta) \) and which passes through point \( p = (t \cos \theta, t \sin \theta) = tn_\theta \). In this case, the line \( \ell_{t, \theta} \) can be parameterized as

\[
(x_1(s), x_2(s)) = (t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta). \tag{2}
\]

By this specific choice for a parameterization of \( \ell_{t, \theta} \) in (2), we immediately see that, for \( t \in \mathbb{R} \) and \( \theta \in [0, \pi) \),

\[
\int_{\ell_{t, \theta}} f(x) \, dx = \mathcal{R}f(t, \theta),
\]

where we let \( x = (x_1, x_2) \), and so the line integral of \( f \) over \( \ell_{t, \theta} \) coincides with the Radon transform (1) of \( f \) at \( (t, \theta) \).

On the other hand, any straight line \( \ell \) in the plane can be described by a unique pair \((t, \theta)\) of a radial parameter \( t \in \mathbb{R} \) and an angular parameter \( \theta \in [0, \pi) \) satisfying \( \ell \equiv \ell_{t, \theta} \). In this way, the Radon transform \( \mathcal{R}f \) of \( f \) can be viewed as a linear integral

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transformation, which maps any bivariate function \( f \in L^1 \) (in Cartesian coordinates) onto a bivariate function \( \mathcal{R}f \) (in polar coordinates), where the image \( \mathcal{R}f \) contains all line integrals of \( f \) over the set of straight lines in the plane.

Due to the seminal work \([12]\) of Johann Radon (in 1917), any (sufficiently regular) function \( f \) can be reconstructed from its Radon transform \( \mathcal{R}f \). The inversion of the Radon transform is given by the filtered back projection (FBP) formula (see \([5, \text{Chapter 6}]\)),

\[
f(x) = \frac{1}{2\pi} \mathcal{F}^{-1} \left[ |S| \mathcal{F}(\mathcal{R}f)(S, \theta) \right](x),
\]

where \( \mathcal{F} \) is, for any fixed angle \( \theta \), the univariate Fourier transform w.r.t. the radial variable \( t \), and so is \( \mathcal{F}^{-1} \) the univariate inverse Fourier transform w.r.t. the frequency variable \( S \). Moreover, the back projection \( B \) is, for any function \( h \equiv h(t, \theta) \) (in polar coordinates), given by the average

\[
B h(x) = \frac{1}{2\pi} \int_0^\pi h(x_1 \cos \theta + x_2 \sin \theta, \theta) d\theta
\]

of \( h(t, \theta) \) over the angular variable \( \theta \), where we let

\[
t = x_1 \cos \theta + x_2 \sin \theta = x \cdot n_\theta
\]

according to the one-to-one relation between the Cartesian coordinates \( x = (x_1, x_2) \) and the polar coordinates \( (t, \theta) \), as described above along with the parameterization of the lines \( t = 1 \) in (2). For basic details concerning the derivation of the filtered back projection formula, we refer to \([5]\), and for a more comprehensive mathematical treatment of the Radon transform and its inversion, we refer to the textbooks \([7, 10]\).

In practical application scenarios, however, only a finite set of Radon data,

\[
\mathcal{R}_c(f) = (\mathcal{R}f(t_k, \theta_k))_{k=1}^m,
\]

given as integrals of \( f \) over a finite set of \( m \) pairwise distinct lines,

\[
\mathcal{L} = \{t_{kn} : (t_k, \theta_k) \in \mathbb{R} \times [0, \infty) \text{ for } k = 1, \ldots, m\},
\]

is available. In this case, an approximate reconstruction of \( f \) from Radon data \( \mathcal{R}_c f \) is sought. In standard techniques of medical imaging, the reconstruction of \( f \) is accomplished by using a suitable discretization of the FBP in (3). For this class of Fourier-based reconstruction methods, the discrete lines in \( \mathcal{L} \), over which the line integrals of \( f \) are known, are usually required to be regularly spaced in the plane, e.g. by assuming parallel beam geometry or fan beam geometry (see \([5]\) for the geometric assumptions on \( \mathcal{L} \)).

In many clinical scenarios of data acquisition, however, we may face a limited range of angles or a limited dosage of X-ray expositions, so that the Radon data are partly corrupt or incomplete. In such relevant cases, the Radon data in (5) are scattered, i.e., the distribution of lines in \( \mathcal{L} \) is essentially not regular but scattered, in which case standard Fourier methods, such as the Fourier-based FBP discretization in (3), do no longer apply. This requires more flexible approximation methods which work for arbitrary geometries of (scattered) Radon lines \( \mathcal{L} \).

To approximate \( f \) from scattered Radon data \( \mathcal{R}_c f \), algebraic reconstruction techniques (ART) \([6]\) can be applied. The concept of ART is essentially different from that of Fourier-based reconstructions: in the setting of ART one fixes a set \( \mathcal{G} = \{g_j\}_{j=1}^n \) of basis functions beforehand to solve the reconstruction problem

\[
\mathcal{R}_c(g) = \mathcal{R}_c(f)
\]

by using a linear combination

\[
g = \sum_{j=1}^n c_j g_j
\]

of the basis functions in \( \mathcal{G} \). This ansatz amounts to solving the linear system

\[
Ac = b
\]

for the unknown coefficients \( c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n \) of \( g \), where the \( m \times n \) matrix \( A \) has the form

\[
A = [\mathcal{R}g_j(t_k, \theta_k)]_{j=1,...,n; k=1,...,m} \in \mathbb{R}^{m \times n}
\]

and the right hand side \( b \) is given by the \( m \) Radon samples coincides with the number \( n \) of coefficients, the linear system in (7) is either overdetermined, for \( m > n \), or underdetermined, for \( n > m \). In case of an overdetermined system, the classical method of linear least squares approximation \([2]\) is applied to minimize the residual (Euclidean) norm \( \|Ac - b\| \), whereas for an underdetermined system the iterative method of Kaczmarz \([5, \text{Section 9.3}]\) is a standard tool to compute an approximate solution \( c \) satisfying \( Ac \approx b \). We remark that in either case the linear system in (7) is not guaranteed to have a unique solution, not even in the rare case, where \( m = n \). In fact, the latter is due to the Mairhuber theorem \([4, \text{Chapter 1}]\) from multivariate approximation theory.

In this paper, we propose a kernel-based algebraic reconstruction method, whose solution is always unique. Our proposed reconstruction scheme relies on the concept of kernel-based multivariate approximation from generalized Hermite-Birkhoff data \([8]\), which we adapt to the special case of reconstruction from scattered Radon data. For the well-posedness of the reconstruction scheme we apply weighted Radon transforms for the purpose of regularization, as explained later in this paper. This way we obtain a very flexible reconstruction scheme, which works for arbitrary scattered Radon data.
2 Generalized Hermite-Birkhoff Interpolation

To solve the reconstruction problem (6), we consider applying Hermite-Birkhoff interpolation [8]. To explain the general framework of this particular method, let \( \Lambda = \{ \lambda_1, \ldots, \lambda_n \} \) denote a set of linearly independent linear functionals. Moreover, suppose we are given a vector \( f_k = (\lambda_1(f), \ldots, \lambda_n(f))^T \in \mathbb{R}^n \) of samples taken from an unknown function \( f \). Now the solution of the general Hermite-Birkhoff reconstruction problem requires finding a function \( g \) satisfying the interpolation conditions \( g_k = f_k \), i.e.,

\[
\lambda_k(g) = \lambda_k(f) \quad \text{for all } k = 1, \ldots, n. \tag{8}
\]

This general framework covers our reconstruction problem (6), when the linear functionals \( \lambda_k \) are defined as

\[
\lambda_k(f) := R f (t_k, \theta_k) \quad \text{for } k = 1, \ldots, n. \tag{9}
\]

By the interpolation conditions in (8), we obtain \( n \) linear equations of the form

\[
\sum_{j=1}^{n} c_j \lambda_j(g) = \lambda_k(f) \quad \text{for } k = 1, \ldots, n,
\]

corresponding to the linear system in (7), and by the choice of our particular ansatz, the number of data matches the number of basis functions, i.e., \( n = m \). To obtain a well-posed reconstruction scheme, we assume that the basis functions \( g_j \) are of the form

\[
g_j(x) = \lambda_j^T K(x, y) \quad \text{for } j = 1, \ldots, n, \tag{10}
\]

where \( \lambda_j^T K(x, y) \) denotes action of the functional \( \lambda_j \) to \( K \) w.r.t. variable \( y \in \mathbb{R}^2 \). Moreover, the kernel function \( K \equiv K(x, y) \) is required to be symmetric, i.e.,

\[
K(x, y) = K(y, x) \quad \text{for all } x, y \in \mathbb{R}^2,
\]

and positive definite.

Rather than dwelling much on explaining positive definite functions, we remark that for the purposes of this paper it is sufficient to say that a symmetric function \( K \equiv K(x, y) \) is positive definite, iff the matrix

\[
\Lambda_{\Lambda,K} = (\lambda_j^T \lambda_k^T K(x, y))_{1 \leq j, k \leq n} \in \mathbb{R}^{n \times n} \tag{11}
\]

is symmetric positive definite for any set \( \Lambda = \{ \lambda_j \}_{j=1}^n \) of linearly independent functionals \( \lambda_j \). For a comprehensive account to the construction and characterization of positive definite kernels, we refer the reader to [9, 13].

To make relevant examples for positive definite kernels, we resort to radially symmetric kernels, in which case \( K \) is assumed to have the form

\[
K(x, y) = \phi(||x - y||) \quad \text{for } x, y \in \mathbb{R}^2,
\]

where \( \phi(|| \cdot ||) \) is radial w.r.t. the Euclidean norm \( || \cdot || \). Classical choices for radial kernels are the Gaussians

\[
\phi_\varepsilon(||x||) = e^{-||x||^2} \quad \text{for } x \in \mathbb{R}^2 \text{ and } \varepsilon > 0,
\]

which are for any \( \varepsilon > 0 \) positive definite, i.e.,

\[
K(x, y) = \exp((-\varepsilon ||x - y||)^2) \tag{12}
\]

is positive definite. Other popular examples for radial positive definite kernels are the inverse multiquadrics

\[
\phi_\varepsilon(||x||) = (1 + ||x||)^{-1/2} \quad \text{for } x \in \mathbb{R}^2 \text{ and } \varepsilon > 0,
\]

and Askey's compactly supported radial characteristic functions [1],

\[
\phi_\varepsilon(||x||) = \begin{cases} 
(1 - \varepsilon ||x||)^p & \text{for } ||x|| < 1/\varepsilon \\
0 & \text{for } ||x|| \geq 1/\varepsilon
\end{cases}
\]

which are positive definite for any \( \beta > 3/2 \).

In the following discussion, we show that Gaussian kernels are suitable choices for \( \phi_\varepsilon \), especially for reconstructions from Radon data, where we need to assemble the reconstruction matrix \( A \equiv A_{\Lambda,K} \) in (11) by applying the Radon transform \( R \) on the chosen kernel \( K \).

Our preference for the Gaussian kernel is also supported by our extensive numerical experiments in [14], where we have observed that the approximation quality of Gaussian reconstruction is superior to that of other commonly used positive definite kernels. This complies with available error estimates for (plain) Lagrange interpolation by Gaussian kernels [15, Section 11.3], which are shown to provide spectral convergence orders.
3 Construction of the Basis Functions

We remark that the basis functions \( g_i \) in (10) are rather straightforward to compute. In the case of the inverse multiquadrics, however, the basis functions \( g_i \) are not well-defined. In fact, note that the Radon transform

\[
R^2 K(x, y)(t, \theta) = \int_{\mathbb{R}} K(x, (t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta))) ds
\]

(13)
of the basis functions \( g_i \) in (10) may be undefined for specific choices of \((t, \theta) \in \mathbb{R} \times [0, \pi)\) and \(x \in \mathbb{R}^2\). Indeed, in this case, we have

\[
K(x, (t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta))) = \mathcal{O}(s^{-1})
\]

for \( s \to \pm \infty \), so that the integral in (14) is singular for any \((t, \theta) \in \mathbb{R} \times [0, \pi)\) and \(x \in \mathbb{R}^2\).

To solve this problem, we apply a Gabor transform, with window function

\[
G_\sigma(r) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{for } \sigma > 0,
\]

to the kernel \( K(x, y) = \phi_\epsilon(\|x - y\|) \), i.e., we replace the inverse multiquadrics \( \phi_\epsilon \) by the windowed inverse multiquadrics kernel

\[
\phi_\epsilon,\sigma(r) = \phi_\epsilon(r) G_\sigma(r),
\]

where the width \(2\sigma > 0\) of the Gabor window \( G_\sigma \) is chosen to match the diameter of the computational domain, i.e., we let \( \sigma := \text{diam}(\Omega)/2 \). Note that \( \phi_\epsilon,\sigma \) is positive definite, since each factor, \( \phi_\epsilon \) and \( G_\sigma \), is positive definite, i.e., \( \phi_\epsilon,\sigma \) is for any \( \epsilon, \sigma > 0 \) a positive definite radial kernel function. In this way, we obtain a well-defined integral (14), since the integrand

\[
K(x, y) = \phi_\epsilon,\sigma(\|x - y\|) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\|x - y\|^2}{2\sigma^2}\right)
\]
is for any \( x \in \mathbb{R}^2 \) in \( L^1(\mathbb{R}^2) \) w.r.t. variable \( y \equiv y(s) = (t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta)) \).

Note that for the other two examples of kernel functions, i.e., for the Gaussians and for the compactly supported radial kernels, this problem does not occur. Indeed, in either of these two cases, we have \( \phi_\epsilon(r) \in L^1([0, \infty)) \), so that the function \( K(x, \cdot) = \phi_\epsilon(\|x - \cdot\|) \) is, for any \( \epsilon > 0 \) and \( x \in \mathbb{R}^2 \), integrable over \( \mathbb{R}^2 \). In particular, the integral in (14) is in this case well-defined for any combination of \((t, \theta) \in \mathbb{R} \times [0, \pi)\) and \(x \in \mathbb{R}^2\), and so the basis functions \( g_i \) in (10) are well-defined.

For all of the three above-mentioned examples – Gaussians, inverse multiquadrics (with using Gabor transform) and compactly supported radial kernels – we have computed the basis functions \( g_i \) in (10). This is done in all details in [14]. For the compactly supported radial kernels, the basis functions are found in of [14, § 5.4.1], and for the inverse multiquadrics they are computed in [14, § 5.2]. For instance, the basis functions for the compactly supported kernel \( \phi_\epsilon(r) = (1 - \epsilon r)^2 \) are given as

\[
g_i(x) = (R^2 K(x, y))(t_j, \theta_j) = \begin{cases} h(t_j - x \cdot n_j) & \text{for } |t_j - x \cdot n_j| \leq 1/\epsilon, \\ 0 & \text{for } |t_j - x \cdot n_j| > 1/\epsilon, \end{cases}
\]

where \( n_j = (\cos(\theta_j), \sin(\theta_j)) \), and where

\[
h(t) = \begin{cases} \frac{\epsilon}{2} \left[ \frac{1}{2} + (2(\epsilon t)^2 - (\epsilon t)^2 \cosh(\epsilon^{-1}|t|^{-1})) \right] & \text{for } t \neq 0, \\ \frac{\epsilon}{2} & \text{for } t = 0. \end{cases}
\]

We decided to omit the tedious details of the required computations for the two cases of the inverse multiquadrics and compactly supported radial kernel, but rather refer to [14]. For the purposes of this paper, it is sufficient and quite instructive to show how the computations for the Gaussian kernels are done. This is discussed in the next section.

4 Construction of the Gaussian Basis Functions

Let \( K(x, y) = \phi_\epsilon(\|x - y\|) \) denote the Gaussian kernel in (12). To discuss the assembly of the reconstruction matrix \( A \) in (11), let us first compute the Gaussian basis functions \( g_i(x) = R^2 K(x, y)(t_j, \theta_j) \) in (10).

To this end, note that \( \phi_\epsilon(r) \in L^1([0, \infty)) \), and so the Radon transform

\[
R^2 K(x, y)(t, \theta) = \int_{\mathbb{R}} K(x, (y(s, t, \theta))) ds
\]

(14)

where \( y(s, t, \theta) = (t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta)) \), is for any combination of \((t, \theta) \in \mathbb{R} \times [0, \pi)\) and \(x \in \mathbb{R}^2 \) well-defined, i.e., the Gaussian basis functions \( g_i \) are well-defined. Note that this is in contrast to reconstruction by inverse multiquadrics, where the integral in (14) is singular for any \((t, \theta) \in \mathbb{R} \times [0, \pi)\) and \(x \in \mathbb{R}^2\).

For the following of our analysis it will be convenient to use the shift property of the Radon transform.
Lemma 4.1. Let $K(x, y) = \phi(||x - y||)$ with $\phi \in L^1(\mathbb{R})$. Then, for any $x \in \mathbb{R}^2$ the Radon transform $\mathcal{R}^y K(x, y)$ at $(t, \theta) \in \mathbb{R} \times [0, \pi)$ can be expressed as

$$\mathcal{R}^y K(x, y)(t, \theta) = (\mathcal{R}^y K(0, y))(t - x \cdot n_0, \theta),$$  

(15)

where $n_0 = (\cos \theta, \sin \theta)$.

Although this is a rather standard fact about Radon transforms [7], we provide a short proof of this property.

Proof. First note that identity (15) can be rewritten as

$$\int_{t_1, \theta} \phi(||y - x||) dy = \int_{t_1 - x \cdot n_0, \theta} \phi(||z||) dz.$$  

To see this is true, further note that $y = (y_1, y_2) \in \ell_{t, \theta}$ implies

$$t = y \cdot n_0 = y_1 \cos \theta + y_2 \sin \theta$$  

from (4). But this in turn yields $y - x \in \ell_{t - x \cdot n_0, \theta}$, since

$$t - x \cdot n_0 = y \cdot n_0 - x \cdot n_0 = (y - x) \cdot n_0.$$  

In other words, for $y \in \ell_{t, \theta}$, we see that $\ell_{t - x \cdot n_0, \theta}$ is, for fixed $\theta$, the unique straight line passing through $y - x$. \(\square\)

Theorem 4.2. For any $(t, \theta) \in \mathbb{R} \times [0, \pi)$, the Gaussian basis function $g$ is given as

$$g(x) = (\mathcal{R}^y K(x, y))(t, \theta) = \frac{\sqrt{\pi}}{e} e^{-\frac{1}{2}t^2 - x \cdot n_0},$$  

(16)

where $n_0 = (\cos \theta, \sin \theta)$.

Proof. For the Radon transform $\mathcal{R} K_0$ of $K_0(y) = K(0, y)$ at $(t, \theta) \in \mathbb{R} \times [0, \pi)$ we get

$$\mathcal{R} K_0(t, \theta) = \int_{t_1, \theta} K_0(y) dy = \int_{\mathbb{R}} K_0(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) ds$$

$$= \int_{\mathbb{R}} e^{-\frac{1}{2}t^2(t \cos \theta - s \sin \theta)^2 - x^2(t \sin \theta + s \cos \theta)^2} ds = \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} e^{-\frac{1}{2}x^2} ds = e^{-\frac{1}{2}t^2} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} ds$$

$$= \frac{\sqrt{\pi}}{e} e^{-\frac{1}{2}t^2},$$

which in turn yields the Gaussian basis function

$$(\mathcal{R} K_0(t, \theta))(t, \theta) = (\mathcal{R} K_0)(t - x \cdot n_0, \theta) = \frac{\sqrt{\pi}}{e} e^{-\frac{1}{2}t^2 - x \cdot n_0}$$

from the Radon transform's shift property, Lemma 4.1. \(\square\)

Now let us turn to the entries of the reconstruction matrix $A = (a_{ij})_{1 \leq j \leq \mathbb{N}}$ in (11). Note that the matrix entry $a_{ij}$ is given by the application of the Radon transform $\mathcal{R} \equiv \mathcal{R}(t_k, \theta_j)$ to the basis function $g_j = \mathcal{R}^y K(\cdot, y)(t_j, \theta_j)$, so that

$$a_{ij} = \mathcal{R}^y(\mathcal{R}^y K(x, y)(t_j, \theta_j))(t_k, \theta_j).$$  

(17)

We can compute the entries of matrix $A$ as follows.

Theorem 4.3. For the Gaussian kernel $\phi_\epsilon$, the entries (17) of the reconstruction matrix $A$ are given as

$$a_{ij} = \frac{\pi}{e^{\frac{1}{2}x \cdot n_0}}.$$  

(18)

Proof. We let $(t, \theta) := (t_j, \theta_j)$ and $(r, \psi) := (t_k, \theta_k)$ for the sake of notational simplicity. Recalling the form of the Gaussian basis function in (16), we can compute the entry $a_{ij}$ by

$$\mathcal{R}^y\left[e^{-(t \cdot x n_0)^2}\right](r, \psi) = \int_{\mathbb{R}} e^{-\frac{1}{2}(r \cos \psi - s \sin \psi)^2 - \frac{1}{2}(r \sin \psi + s \cos \psi)^2} ds$$

$$= \int_{\mathbb{R}} e^{-\frac{1}{2}(r \cos \psi - s \sin \psi)^2 - \frac{1}{2}(r \sin \psi + s \cos \psi)^2} ds$$

$$= \int_{\mathbb{R}} e^{-\frac{1}{2}(r \cos \psi - s \sin \psi)^2} ds$$

$$= \int_{\mathbb{R}} e^{-(r \cos \psi - s \sin \psi)^2} ds,$$
where we let $a := \sin(\varphi - \theta)$ and $b := t - r \cos(\varphi - \theta)$. Hence, for $a \neq 0$, we get

$$a_{kj} = R^4 \left[ \frac{\sqrt{\pi}}{e} e^{-\frac{r^2 (t - r \cos(\varphi - \theta))^2}{e^2}} \right] (r, \varphi) = \frac{\sqrt{\pi}}{e^2 a^2} = \frac{\pi}{e^2 a} = \frac{\pi}{e^2 \sin(\varphi - \theta)}$$

as stated.

Note that the above entries $a_{kj}$ do not depend on the radial parameters, $t_j$ and $t_k$, but only on the angular parameters, $\theta_j$ and $\theta_k$. This is due to the radial symmetry of the Gaussian kernel $K$.

For different angles $\theta_j, \theta_k \in [0, \pi)$, $\theta_j \neq \theta_k$, the entry $a_{kj}$ of $A$ in (18) is well-defined. For coincident angles $\theta_j = \theta_k$, however, the entry $a_{kj}$ is singular, i.e., the Radon transform $a_{kj} = (R(g))(t_j, \theta_k)$ is singular, iff the angular parameters of the Radon lines $\ell_{t_j, \theta_j}$ and $\ell_{t_k, \theta_k}$ coincide, i.e., iff $\theta_j = \theta_k$. So, in particular, are the diagonal entries of $A$ in (11) singular.

### 5 Regularization by Gaussian Weights

To obtain non-singular entries in the reconstruction matrix $A$ in (11), we require suitable regularizations for the Radon transform $R$. To this end, we work with weighted Radon transforms of the form

$$R_w f(t, \theta) := R(f w)(t, \theta) = \int_{t_j \in \ell} f(x) w(x) dx$$

where $w \equiv w(x)$ is a fixed weight function. This yields a standard regularization for the Radon transform $R$, which also arises in relevant applications of single photon emission computed tomography (SPECT), where $R_w$ is called attenuated Radon transform [11].

As for the choice of the weight $w$, we prefer to use the radially symmetric Gaussian function

$$w_{\nu}(x) = \exp\left(-\frac{\nu \|x\|^2}{2}\right) \quad \text{for } \nu > 0. \quad (20)$$

This particular choice is well-motivated, since the product $\phi \cdot w_{\nu}$ of the two positive definite Gaussians, $\phi$, and $w_{\nu}$, is positive definite, due to the classical Bohm theorem [3]. Indeed, according to Bohner’s theorem, any function $f \in L^2$ with positive Fourier transform $\hat{f}$ is positive definite. Hence, for any pair of functions $f, g \in L^1$ with positive Fourier transforms $\hat{f}, \hat{g}$, their product $f \cdot g$ is positive definite, since their convolution $f * g$ (giving their Fourier transform) is positive.

In this way, the resulting reconstruction problem

$$R_w g(t_k, \theta_k) = R f(t_k, \theta_k) \quad \text{for all } k = 1, \ldots, n,$$

assuming (10) for the form of the Gaussian basis functions, leads to a well-posed reconstruction scheme with reconstruction matrix

$$A \equiv A_w = \left( R_w g(t_k, \theta_k) \right)_{1 \leq j, k \leq n} \in \mathbb{R}^{n \times n}. \quad (21)$$

We compute the entries of $A$ analytically as follows.

**Theorem 5.1.** For the Gaussian kernel $\phi, \nu > 0$, and the Gaussian weight $w_{\nu}, \nu > 0$, the entries (21) of the reconstruction matrix $A_w = (a_{kj})_{1 \leq j, k \leq n}$ are given as

$$a_{kj} = \frac{\sqrt{\pi}}{e^{\frac{1}{2}} \alpha_{kj}^{\nu}} \exp\left[-\nu^2 \left( t_k^2 + \frac{\beta_{kj}^2}{\alpha_{kj}^{\nu} + \nu^2} \right) \right],$$

where $\beta_{kj} = t_j - t_k \cos(\theta_j - \theta_k)$ and $\alpha_{kj} = \sin(\theta_j - \theta_k)$.

**Proof.** For notational convenience, we let $(t, \theta) = (t_1, \theta_1)$ and $(r, \varphi) = (t_k, \theta_k)$. Recalling the representation of the Gaussian basis function $g \equiv g(t, \theta)$ in Theorem 4.2, and the weighted Radon transform $R_w g$ by following along the lines of our proof in Theorem 4.3, which yields

$$(R_w g)(r, \varphi) = \int_R \frac{g(x(s)) e^{-\frac{r^2 |x(s)|^2}{\nu^2}} ds}{\nu^2} = \frac{\sqrt{\pi}}{e} \int_R e^{-\frac{r^2 (a + b)^2}{\nu^2}} e^{-\frac{r^2 (t - r \cos(\varphi - \theta))^2}{\nu^2}} ds,$$

where we let $a = \sin(\varphi - \theta)$ and $b = t - r \cos(\varphi - \theta)$, and where we have used the identity

$$|x(s)|^2 = r^2 + s^2 \quad \text{for } x(s) = (r \cos \varphi - s \sin \varphi, r \sin \varphi + s \cos \varphi).$$

This immediately yields the entries of $A_w$ by

$$(R_w g)(r, \varphi) = \frac{\sqrt{\pi}}{e} \int_R \frac{e^{-\frac{r^2 (a + b)^2}{\nu^2}} ds}{\nu^2} = \frac{\sqrt{\pi}}{e} \int_R e^{-\frac{r^2 (a + b)^2}{\nu^2}} \exp\left[-\nu^2 \left( t_k^2 + \frac{\beta_{kj}^2}{\alpha_{kj}^{\nu} + \nu^2} \right) \right] ds$$

$$= \frac{\pi}{e^{\frac{1}{2}} \alpha_{kj}^{\nu}} \exp\left[-\nu^2 \left( t_k^2 + \frac{\beta_{kj}^2}{\alpha_{kj}^{\nu} + \nu^2} \right) \right].$$
We have implemented the proposed kernel-based reconstruction scheme for the three kernels $\phi$, — Gaussians, inverse multiquadrics, and compactly supported radial kernels — in combination with the Radon transform’s regularization and by using the Gaussian weight function $w$. We have performed quite comprehensive numerical experiments for various possible combinations of kernels $\phi$, and weights $w$, a selection of which we present here for the purpose of illustration.

In our series of numerical experiments concerning the popular phantoms *bull’s eye*, the *crescent-shaped phantom*, and the Shepp-Logan phantom, we found that the Gaussian kernel performs much better than the inverse multiquadrics and the compactly supported kernel $\phi_s(r) = (1 - \epsilon r^2)^+\epsilon$, where the performance of the latter two kernels is comparable. This is supported by our numerical example in Figure 3. Indeed, we obtained the best numerical results — in terms of the reconstructions’ visual quality and the root mean square error (RMSE) in (22) — for the Gaussian kernel $\phi(r) = \exp(-(\epsilon r)^2)$ in combination with the Gaussian weight function $w_s(x) = \exp(-(|x|/\nu)^2)$, provided that the shape parameters $\epsilon$ and $\nu$ were properly chosen.

In fact, we found that the performance of the Gaussian reconstruction method is quite sensitive to the choice of the method parameters $\epsilon$ and $\nu$. Therefore, we decided to perform a "fine-tuning" of the Gaussian parameters $\epsilon$ and $\nu$, whose numerical results are presented in Subsection 6.2. Moreover, in Subsection 6.4 we provide numerical comparisons with Fourier-based reconstructions, relying on the filtered back projection formula, for Radon data on parallel beam geometry. In Subsection 6.5, we present numerical results concerning Gaussian reconstruction from scattered Radon data, and in Subsection 6.3 from regular Radon data (on parallel beam geometry), where we also provide a comparison between different kernels. Comparisons concerning Gaussian reconstruction on regular Radon data vs scattered Radon data are provided in the following Subsection 6.1. For a more comprehensive documentation on our numerical experiments, we refer to [14].

6 Numerical Results

We have implemented the proposed kernel-based reconstruction scheme for the three kernels $\phi$, — Gaussians, inverse multiquadrics, and compactly supported radial kernels — in combination with the Radon transform’s regularization and by using the Gaussian weight function $w$. We have performed quite comprehensive numerical experiments for various possible combinations of kernels $\phi$, and weights $w$, a selection of which we present here for the purpose of illustration.

In our series of numerical experiments concerning the popular phantoms *bull’s eye*, the *crescent-shaped phantom*, and the Shepp-Logan phantom, we found that the Gaussian kernel performs much better than the inverse multiquadrics and the compactly supported kernel $\phi_s(r) = (1 - \epsilon r^2)^+\epsilon$, where the performance of the latter two kernels is comparable. This is supported by our numerical example in Figure 3. Indeed, we obtained the best numerical results — in terms of the reconstructions’ visual quality and the root mean square error (RMSE) in (22) — for the Gaussian kernel $\phi(r) = \exp(-(\epsilon r)^2)$ in combination with the Gaussian weight function $w_s(x) = \exp(-(|x|/\nu)^2)$, provided that the shape parameters $\epsilon$ and $\nu$ were properly chosen.

In fact, we found that the performance of the Gaussian reconstruction method is quite sensitive to the choice of the method parameters $\epsilon$ and $\nu$. Therefore, we decided to perform a "fine-tuning" of the Gaussian parameters $\epsilon$ and $\nu$, whose numerical results are presented in Subsection 6.2. Moreover, in Subsection 6.4 we provide numerical comparisons with Fourier-based reconstructions, relying on the filtered back projection formula, for Radon data on parallel beam geometry. In Subsection 6.5, we present numerical results concerning Gaussian reconstruction from scattered Radon data, and in Subsection 6.3 from regular Radon data (on parallel beam geometry), where we also provide a comparison between different kernels. Comparisons concerning Gaussian reconstruction on regular Radon data vs scattered Radon data are provided in the following Subsection 6.1. For a more comprehensive documentation on our numerical experiments, we refer to [14].

6.1 Reconstruction of the Crescent-Shaped Phantom

Let us first regard the popular test case *crescent-shaped phantom*, given by the function

\[ f(x, y) = \begin{cases} 
1.0 & \text{for } (x, y) \in \Omega, \\
0.5 & \text{for } (x, y) \in D_1, \\
0.0 & \text{for } (x, y) \in \mathbb{R}^2 \setminus D_0, 
\end{cases} \]

on the unit square $\Omega = [0, 1]^2$, where

\[ D_1 = \{(x, y) \in \mathbb{R}^2 : (x - 1/8)^2 + y^2 \leq 9/64\} \]
\[ D_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/4\}, \]

as shown in Figure 1 (a). Figures 1 (b), (c) show the reconstruction of the crescent-shaped phantom obtained with the Gaussian kernel $\phi_s$ and the Gaussian weight function $w$, for parameters $\epsilon = 40$ and $\nu = 0.7$, and for $n = 3888$ different Radon lines. The Radon samples were taken on a set of $n$ regularly distributed lines generated by *parallel beam geometry*, and on a set of $n$ scattered Radon lines. In either case, the crescent-shaped phantom $f$ is localized and reconstructed fairly well by the proposed kernel-based reconstruction scheme. But the quality of the reconstruction on parallel beams, Figure 1 (b), is superior to that on scattered lines, Figure 1 (c). This is due to the uniform coverage of the Radon lines over the computational domain $\Omega$. The geometry of the Radon lines are visible in the display of the reconstructions in Figures 1 (b), (c), especially between the boundary of the domain $\Omega$ and the phantom’s boundary.

Figure 1: Crescent-shaped phantom. (a) Original phantom; (b)-(c) reconstruction of crescent-shaped phantom using the Gaussian kernel $\phi_s$ with $\epsilon = 40$ and the Gaussian weight function $w$, with $\nu = 0.7$ from $n = 3888$ Radon samples taken on (b) regularly spaced Radon lines (parallel beam geometry), (c) scattered Radon lines.
6.2 Selection of the Gaussian Shape Parameters

In this subsection, we discuss the selection of the method parameters $\epsilon$ (for the Gaussian kernel $\phi_{\epsilon}$) and $\nu$ (for the Gaussian weight function $w_{\nu}$), here applied to the phantom bull’s eye, defined as

$$f(x, y) = \begin{cases} 
1/2 & \text{iff } x^2 + y^2 \leq 1/16; \\
1/4 & \text{iff } 1/16 < x^2 + y^2 \leq 1/4; \\
1 & \text{iff } 1/4 < x^2 + y^2 \leq 9/16; \\
0 & \text{iff } 9/16 < x^2 + y^2.
\end{cases}$$

The phantom bull’s eye is displayed in Figure 3, second row, first column. In our numerical experiments, we have analyzed the behaviour of the Gaussian reconstruction error as a bivariate function of the parameters $\epsilon$ and $\nu$. For the error measure, we consider using the standard root mean square error

$$\text{RMSE} = \sqrt{\frac{1}{J} \sum_{j=1}^{J} (f_j - \hat{f}_j)^2}, \quad (22)$$

where $J$ is the size of the image (i.e., the number of pixels) and $\{f_j\}_{j=1}^{J}$, $\{\hat{f}_j\}_{j=1}^{J}$ are the greyscale values at the pixels of the original image bull’s eye and of the reconstructed image, respectively.

Let us first consider the variation of the shape parameter $\epsilon$ of the Gaussian kernel $\phi_{\epsilon}$. In our numerical experiments, we have fixed the weight parameter at $\nu = 0.7$. For different choices of $N$ and $M$, we have used parallel beam geometry with $N$ angles and $2M + 1$ parallel lines per angle, resulting in a number of $n = N \times (2M + 1)$ Radon samples for each numerical experiment. With the variation of the shape parameter $\epsilon$, we have recorded an “optimal” value $\epsilon^*$, where, according to our numerical observations, the RMSE is minimal. The following Table 1 reflects our results. Note that the “optimal” value $\epsilon^*$ increases with the number $n$ of Radon data taken. This is not too surprising insofar as in plain kernel-based interpolation, the shape parameter of the Gaussian kernel is recommended to be proportional to the density of the sample points: the higher the density of points, the larger $\epsilon$ should be chosen. In our numerical experiments, the sample density increases with the number $n$ of lines, which explains the monotonicity of $\epsilon^*$ in Table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
$N$ & 36 & 54 & 72 & 90 & 108 & 126 & 144 & 162 & 180 \\
$M$ & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \\
$n$ & 1,476 & 3,294 & 5,832 & 9,090 & 13,068 & 17,766 & 23,184 & 29,322 & 36,180 \\
$\epsilon^*$ & 21.05 & 28.42 & 37.36 & 48.42 & 59.47 & 72.10 & 84.21 & 90.52 & 97.37 \\
\hline
\end{tabular}
\caption{Parameter-tuning for Gaussian kernel $\phi_{\epsilon}$. Reconstruction of phantom bull’s eye by Gaussian kernel $\phi_{\epsilon}$, with variation of the kernel parameter $\epsilon$. For fixed weight parameter $\nu = 0.7$, and for different choices of $N$ angles and $2M + 1$ lines per angle in a parallel beam geometry. For each experiment, the “optimal” shape parameter $\epsilon^*$, whose resulting RMSE is minimal, is shown.}
\end{table}

Next we consider the variation of the shape parameter $\nu$ for the Gaussian weight function $w_{\nu}$ in (20). Figure 2 shows the RMSE as a function of $\nu$, for constant kernel shape parameter $\epsilon = 30$ and for a fixed number $n = 1230$ of Radon lines (on a parallel beam geometry). We have observed that the minimal RMSE is attained at $\nu^* = 0.4522$ (cf. Figure 2). Note that the graph in Figure 2 illustrates the dependence between the RMSE and the weight parameter $\nu$ quite well.

We have further performed a series of rather extensive numerical experiments concerning the phantoms bull’s eye, the crescent-shaped phantom, and the Shepp-Logan phantom, where our reconstructions were computed on a parallel beam geometry.
with \( N = 30 \) angles and \( 2M + 1 = 41 \) lines per angle, so that the number of Radon samples is \( n = 1230 \). Table 2 shows "optimal" combinations of values \( \epsilon^* \) for the Gaussian kernel and \( \nu^* \) for the Gaussian weight, whose resulting \( \text{RMSE} = \text{RMSE}(\epsilon, \nu) \) is minimal.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>kernel ( \phi_{\epsilon} )</th>
<th>weight ( w_{\nu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped phantom</td>
<td>( \epsilon^* = 19.66 )</td>
<td>( \nu^* = 0.51 )</td>
</tr>
<tr>
<td>bull’s eye</td>
<td>( \epsilon^* = 15.52 )</td>
<td>( \nu^* = 0.45 )</td>
</tr>
<tr>
<td>Shepp-Logan phantom</td>
<td>( \epsilon^* = 18.28 )</td>
<td>( \nu^* = 2.06 )</td>
</tr>
</tbody>
</table>

Table 2: "Optimal" shape parameters for Gaussian kernel and Gaussian weight. Fine-tuning of shape parameter \( \epsilon^* \) and weight parameter \( \nu^* \) for three different phantoms by Gaussian reconstruction on parallel beam geometry (for \( N = 30 \) angles and \( 2M + 1 = 41 \) lines per angle).

6.3 Comparison of Kernel-based Reconstructions on Parallel Beam Geometry

In Figure 3 we show a comparison between reconstructions from the three different kernels, where for each kernel, we have performed a fine-tuning of the method parameters \( \epsilon \) and \( \nu \). For the Gaussian kernel, the 'optimal' parameters \( \epsilon^* \) and \( \nu^* \) are in Table 2.

On our numerical results of Figure 3 we can conclude that the reconstruction quality obtained from the Gaussian kernel is clearly superior to those obtained from the compactly supported and the inverse multiquadric kernel, provided that the Gaussian method parameters \( \epsilon \) and \( \nu \) are well-chosen. Note that the performance of the compactly supported kernel and the inverse multiquadric kernel is comparable. This complies with our numerical results, as documented in [14].
6.4 Kernel-based vs Fourier-based Reconstruction

We compare the proposed kernel-based Gaussian reconstruction method with Fourier-based reconstructions, relying on the filtered back projection formula (3), where we considered using the standard Shepp-Logan low pass filter, given as

\[
A(\omega) = |\omega| \cdot \left( \frac{\sin(\pi \omega/(2L))}{\pi \omega/(2L)} \right) \cdot \chi_{[-L,L]}(\omega) = \begin{cases} 
\frac{2}{\pi} & |\sin(\pi \omega/(2L))| \quad \text{iff} \ |\omega| \leq L; \\
0 & \text{iff} \ |\omega| > L.
\end{cases}
\]

so that the filter \(|S|\) in (3) is being replaced by \(A(S)\). Note that \(A(S) \approx |S|\) for small frequencies \(S\).

![Crescent-shaped phantom](image1)
![Bull’s eye](image2)
![Shepp-Logan phantom](image3)

Figure 4: Three popular phantoms. Crescent-shaped phantom, bull’s eye, and Shepp-Logan (each of size 256 \(\times\) 256).

We remark that the standard implementation of the Fourier-based filtered back projection formula (3) relies on a regular distribution of Radon lines. To this end, we work with a parallel beam geometry. In this case, by the discretization \(\theta_k = k\pi/N\), for \(k = 0, \ldots, N-1\), of the angular variable \(\theta \in [0, \pi]\) and \(t_j = jd\) (for some fixed sampling spacing \(d > 0\), where \(j = -M, \ldots, M\), for the radial variable \(t \in \mathbb{R}\), the data acquisition relies on \((2M + 1) \times N\) regularly distributed Radon lines \(\ell_{t_j, \theta_k}\). Note that for any fixed angle \(\theta_k\), the Radon lines \(\ell_{t_j, \theta_k}, -M \leq j \leq M\), are parallel (at uniform distance \(d\) and symmetric about the origin), which explains the naming parallel beam geometry. For illustration, Figure 5 (a) shows a set of \((2M + 1) \times N = 110\) Radon lines \(\ell_{t_j, \theta_k}\) on parallel beam geometry, where we let \(N = 10\), \(M = 5\), and \(d = 0.2\).

![Radon lines on parallel beam geometry](image4)

Figure 5: (a) Radon lines on parallel beam geometry. A set of 110 Radon lines \(\ell_{t_j, \theta_k}\) is shown, using increments of \(\pi/10\) for the radial variable \(\theta_k\), i.e., \(N = 10\), with \(2M + 1 = 11\) different Radon lines per angle, letting \(t_j \neq jd\), for \(j = -5, \ldots, 5\), at sampling spacing \(d = 0.2\). (b) Scattered Radon lines. A set of 170 randomly distributed Radon lines \(\ell_{t_j, \theta_k}\) is shown, \(j = 1, \ldots, 170\).

Now let us turn to our numerical comparisons between Fourier-based and kernel-based reconstructions. In each test case, we let \(N = 45\) and \(M = 40\), so that \((2M + 1) \times N = 3645\) Radon samples are taken. In the kernel-based Gaussian reconstruction, we have used the “optimal” parameters shown in Table 2. The resulting reconstructions obtained from the two methods are displayed in Figure 6. Given our numerical results, we can conclude that our kernel-based reconstruction method is competitive to the Fourier-based reconstruction method, by the visual quality of their reconstructions and by the RMSE.

6.5 Reconstruction from Scattered Radon Data

Let us turn to the reconstruction from scattered Radon data. In our numerical experiments, we have generated sets of unstructured Radon lines by random, so that the geometry of the sample lines is unstructured. Figure 5 (b) shows one example for a set of 170 randomly distributed Radon lines in the domain \([-1, 1]^2\).
Figure 6: Kernel-based vs Fourier-based reconstruction. For the phantoms crescent-shaped, bull's eye, and Shepp-Logan, their Fourier-based reconstructions (1st row) and kernel-based reconstructions (2nd row) are shown. The six reconstructions are computed from \((2M + 1) \times N = 3645\) Radon samples (where we let \(N = 45\) and \(M = 40\)) on parallel beam geometry, respectively. The corresponding root mean square errors (RMSE) are displayed in Table 3.

Table 3: Kernel-based vs Fourier-based reconstruction. For the reconstructions in Figure 6, their RMSE are shown.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>Kernel-based</th>
<th>Fourier-based</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped</td>
<td>0.10</td>
<td>0.12</td>
</tr>
<tr>
<td>bull's eye</td>
<td>0.14</td>
<td>0.13</td>
</tr>
<tr>
<td>Shepp-Logan</td>
<td>0.16</td>
<td>0.18</td>
</tr>
</tbody>
</table>

We applied our kernel-based reconstruction method to a sequence of \(n = 2000, 5000, 10000, 20000\) randomly chosen scattered Radon lines, taken from the phantoms bull's eye and crescent-shaped.

Figure 7 shows the obtained sequences of reconstructions, where in each test case we have used "optimal" shape parameters from Table 2. The corresponding RMSE values are shown in Table 4.

Although the geometry of the scattered Radon lines are visible in the displays of Figure 7, the basic features of the two phantoms are captured very well, where the reconstruction quality is improved as the number \(n\) of Radon samples increases.

Note that the RMSE values (in Table 4) are not monotonically decreasing. This is due to the random distribution of the scattered Radon lines, cf. Figure 7. Nevertheless, the global trend of the RMSE values is, at increasing \(n\), monotonically decreasing.

Table 4: Scattered Radon lines. RMSE for kernel-based reconstructions in Figure 7, for \(n = 2000, \ldots, 20000\) scattered Radon lines.

<table>
<thead>
<tr>
<th>Phantom</th>
<th>2000</th>
<th>5000</th>
<th>10000</th>
<th>20000</th>
</tr>
</thead>
<tbody>
<tr>
<td>crescent-shaped</td>
<td>0.15</td>
<td>0.14</td>
<td>0.14</td>
<td>0.12</td>
</tr>
<tr>
<td>bull's eye</td>
<td>0.19</td>
<td>0.17</td>
<td>0.21</td>
<td>0.19</td>
</tr>
</tbody>
</table>

7 Final Remarks and Conclusion

We developed a kernel-based image reconstruction method for scattered Radon data. The proposed reconstruction method relies on Hermite-Birkhoff interpolation by positive definite Gaussian kernels in combination with a regularization of the Radon transform by Gaussian weights. Our kernel-based reconstruction method is shown to be competitive with Fourier-based reconstructions from Radon data on parallel beam geometry. The main advantage of the proposed kernel-based reconstruction method is its enhanced flexibility. In fact, the kernel-based method can unconditionally be applied to arbitrary distributions of Radon lines. This is in contrast to classical Fourier-based reconstructions whose implementation relies on the filtered back projection formula.
Figure 7: **Kernel-based reconstruction from scattered Radon data.** For the test cases bull's eye (first row) and crescent-shaped phantom (second row), a sequence of scattered Radon data is taken. The figure shows the corresponding reconstructions for an increasing number $n = 2000, 5000, 10000, 20000$ of scattered Radon samples (from left to right), obtained from the proposed kernel-based reconstruction method. The corresponding root mean square errors (RMSE) are displayed in Table 4.

References


