Generalized Vandermonde determinants, Toeplitz matrices and Schur functions

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Abstract

The aim of this paper is to present some results concerning computational methods for Vandermonde determinants of the form:

$$V_{s;\ell(\mu)}(x_1,...,x_s) = |x_i^{\mu_k}|\,, \quad i,k=1,...,s\ ,$$

where the x_i are distinct points belonging to an interval $[a,b]$ of the real line, the index s stands for the degree, the sequence μ consists of ordered integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_s$ and $\ell(\mu)$ is its length μ, that is the number of non-zero elements.

These generalized Vandermonde determinants can be factored as a product of the classical Vandermonde determinant and a homogeneous symmetric function of the points involved, that is a *Schur function*. Moreover, if n is the degree of the Schur function, then it can be written as $(x-x_1)\cdots(x-x_k)\cdot P_{n-k}(x)$. Algorithms for the computation of the coefficients of P_{n-k} are also presented.

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1 Introduction

Let $X_s = \{x_1,...,x_s\}$ be a set of $s \ge 2$ *distinct points* belonging to an interval $[a,b]$ of the real axis. Let $VDM(x_1,...,x_s)$ be the *classical* Vandermonde determinant of order s, that is:

$$VDM(x_1,...,x_s) = \begin{vmatrix} 1 & x_1 & \dots & x_1^{s-1} \\ 1 & x_2 & \dots & x_2^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_s & \dots & x_s^{s-1} \end{vmatrix}\ .$$

It is well-known that (cf. [3]):

$$VDM(x_1,...,x_s) = VDM(x_1,...,x_{s-1}) \left(\prod_{1\le i\le s-1} (x_s - x_i) \right)\ . \qquad (1)$$

Let $V_{s-1}(x) = VDM(x_1,...,x_{s-1},x)$ be the polynomial of degree $s-1$ obtained when $x_s = x$. From (1), $V_{s-1}(x)$ can be factored as the product of the classical Vandermonde of order $s-1$ times a polynomial of degree $s-1$.

This idea can be used to compute *generalized* Vandermonde determinants of the form:

$$V_{n;\ell(\mu)}(x_1, ..., x_n) = |x_i^{\mu_k}|\,, \quad i, k = 1, ..., n\,, \tag{2}$$

where the x_i are n distinct points belonging to an interval $[a, b]$ of the real line, the index n stands for the degree, the sequence μ consists of ordered integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_n$ and $\ell(\mu)$ is its length μ, that is the number of non-zero elements. This sequence can be seen as follows. The classical Vandermonde determinant of order n is based on the polynomial basis $\{1, x, ..., x^n\}$, the generalized ones are based on the basis $\{x^{\mu_1}, ..., x^{\mu_n}\}$ for some positive integers $0 \le \mu_1 < \mu_2 < \cdots < \mu_n$. This different basis *in some cases* is obtainable from the basis of bivariate polynomials of degree n restricted to some function $y = x^p$, $p \le n$. As an example, the bivariate polynomials of degree $n \ge 3$ restricted to $y = x^3$, reduce to the univariate polynomials of degree $3n$ where the power x^{3n-1} is missing (cf. [2]). This is a *geometrical* interpretation of generalized Vandermonde determinants.

Generalized Vandermonde determinants are strictly connected to Schur functions. In fact they can be factored as a product of the classical Vandermonde determinant and a *Schur function* (see Section 4, Theorem 4.2):

$$V_{n;\ell(\mu)}(x_1, ..., x_n) = VDM(x_1, ..., x_n) \cdot s_{m_{n-1}}(x_1, ..., x_n)\,,$$

where $m_{n-1} = \mu_n - \mu_1$. We shall prove that

$$s_{m_{n-1}}(x) = \prod_{i=1}^{n-1}(x - x_i) \cdot P_M(x)\,, \tag{3}$$

with $M = m_{n-1} - n + 1$ when x is taken instead of x_n.

Our main concern is the computation of the coefficients of the polynomial $P_M(x)$.

We have organized the paper so that we consider generalized Vandermonde determinants of ever more complicated types.

In Section 2 we present some results concerning the computation of a simple class of generalized Vandermonde determinants, based on the computation of the inverse of a lower triangular Toeplitz matrix. For more general Vandermonde determinants, in Section 3 we prove that the coefficients of the polynomial $P_M(x)$ are a *complete elementary symmetric* function (see Theorem 3.2). We also propose an algorithm to compute them with the discussion of its complexity. In Section 4 we introduce Schur functions and by using some of their properties we are able to compute the coefficients of the polynomial $P_M(x)$ for the most general Vandermonde determinants of the type (2). An algorithm to accomplish this task will be also presented.

In every Section we provide some examples in order to render the presentation as clearly as possible.

2 The inverse of a lower triangular Toeplitz matrix and the polynomial division algorithm

Let us consider for the sake of simplicity the polynomial associated to a simple 3×3 *generalized* Vandermonde determinant

$$V_{n;2}(x) = \begin{vmatrix} 1 & x_1 & x_1^n \\ 1 & x_2 & x_2^n \\ 1 & x & x^n \end{vmatrix} , \tag{4}$$

which corresponds to the Vandermonde determinant for the polynomial basis $\{1, x, x^n\}$, $n \geq 2$ on the points x_1, x_2, x. This basis can also be seen as the restriction of the basis $\{1, x, y\}$, of bivariate linear polynomials, to the curve $y = x^n$. This determinant vanishes on x_1 and x_2 and therefore it can be written as

$$V_{n;2}(x) = (x - x_1)(x - x_2)P^*_{n-2}(x) , \tag{5}$$

for some non monic polynomial $P^*_{n-2}(x)$ of degree $\leq n - 2$. Then, in order to complitely determine $V_{n;2}(x)$ we should find the coefficients of the polynomial $P^*_{n-2}(x)$.

Proposition 2.1 *Letting*

$$P^*_{n-2}(x) = a^*_{n-2}x^{n-2} + a^*_{n-3}x^{n-3} + \ldots + a^*_0 . \tag{6}$$

The coefficients a^*_i of the $P^*_{n-2}(x)$ in (6) for the determinant (5) are given by :

$$a^*_i = x_2^{n-i-1} - x_1^{n-i-1} , \; i = 0, ..., n - 2 .$$

Proof.

$$V_{n;2}(x) = \begin{vmatrix} 1 & x_1 & x_1^n \\ 1 & x_2 & x_2^n \\ 1 & x & x^n \end{vmatrix} = \begin{vmatrix} 1 & x - x_1 & x^n - x_1^n \\ 1 & x - x_2 & x^n - x_2^n \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= (x - x_1)(x - x_2) \begin{vmatrix} 1 & 1 & x^{n-1} + x^{n-2}x_1 + \cdots + x_1^{n-1} \\ 1 & 1 & x^{n-1} + x^{n-2}x_2 + \cdots + x_2^{n-1} \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= (x - x_1)(x - x_2) \left[x^{n-2}(x_2 - x_1) + x^{n-3}(x_2^2 - x_1^2) + \cdots + (x_2^{n-1} - x_1^{n-1}) \right] .$$

Then identifying the coefficients we conclude. □

As a remark, we notice that we can factor out of the coefficients the common factor $(x_2 - x_1)$ that is $VDM(x_1, x_2)$. Finally,

$$V_{n;2}(x) = VDM(x_1, x_2) \; (x - x_1)(x - x_2) \; P_{n-2}(x) , \tag{7}$$

with $P_{n-2}(x) = \sum_{i=0}^{n-2} a_i x^i$ a *monic* polynomial.

As proved in [4], the coefficents of $P_{n-2}(x)$ can be found as the solution of a linear system whose matrix is a *lower triangular Toeplitz* matrix whose elements are the *elementary symmetric functions* for the points involved. For this example the elementary symmetric functions up to degree 2 for the points x_1, x_2 are $e_0^{(2)} = 1$, $e_1^{(2)} = x_1 + x_2$, $e_2^{(2)} = x_1 x_2$.

Thus, our problem reduces to the computation of the *inverse* of a lower triangular Toeplitz matrix.

Definition 2.1 (**cf. [1]**) *$T = (t_{i,j})_{1 \le i,j \le n}$ is a lower triangular Toeplitz matrix if it can be written as*

$$T = \sum_{i=1}^{n} a_i S^{i-1} \ ,$$

where $\mathbf{a} = (a_1, ..., a_n)$ is a real or complex vector and the matrix $S = S^1$ is the **down-shift** *matrix, that is the matrix*

$$S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Thus a lower triangular Toeplitz matrix is *completely determined by its first column or its last row.* Moreover, this set of matrices is an *algebra* generated by the down-shift matrix S. If a lower triangular Toeplitz is invertible, then the inverse is still a lower triangular Toeplitz matrix. Hence, the inverse of an invertible lower triangular Toeplitz matrix is completely defined by the knowledge for example of its first column.

Given the lower triangular $n \times n$ Toeplitz matrix generated by the vector $\mathbf{a} = (1, a_1, a_2, ..., a_{n-1})^T$, that is the matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} , \tag{8}$$

the *first* column of the matrix $B = A^{-1}$, that is the vector $b_{i,1}, \ i = 1, ..., n$, can be computed by solving the system

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix} \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{n-1,1} \\ b_{n,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} .$$

The solution provides the following useful recurrence:

$$\begin{aligned} b_{1,1} &= 1 \ ; \\ b_{i,1} &= -\sum_{j=1}^{i-1} b_{j,1} a_{i-j}, \quad i = 2, ..., n \ . \end{aligned} \tag{9}$$

We point out that the previous recursion for the computation of the inverse matrix B requires $\frac{n(n-1)}{2}$ multiplications which is of order $\mathcal{O}(\frac{n^2}{2})$.

◇◇

The general case corresponding to equation (4), is the one in which we have to find a monic polynomial $P_{n-k}(x)$ of degree $n-k$ such that

$$V_{n;k}(x) = P_{n-k}(x) \cdot Q_k(x) = \left(x^{n-k} + \sum_{i=1}^{n-k} b_{i+1,1} x^{n-k-i} \right) \left(\prod_{i=1}^{k} (x - x_i) \right) , \tag{10}$$

where as usual $V_{n;k}(x)$ is the determinant in (4) for general n and k.

Since $V_{n;k}(x)$ is a polynomial of degree n then the following equation holds

$$Q_k(x) \cdot P_{n-k}(x) = x^n - R_{k-1}(x) , \tag{11}$$

where $R_{k-1}(x)$ is a polynomial of degree $\leq k-1$. Moving $R_{k-1}(x)$ to the left side, the equation says that *the polynomial $P_{n-k}(x)$ is the quotient of the division of x^n by $Q_k(x)$.*

Then, by means of the *division algorithm* for polynomials in which the dividend is simply x^n and the divisor is $Q_k(x)$, we can determine the coeffcents of $P_{n-k}(x)$.

As it is well-known (see for example [1, Ch.2, §5]) this reduces to the problem of finding the first column of the inverse of the lower triangular Toeplitz matrix of order $n-k+1$ generated by the vector whose first $k+1$ elements are the coefficients of the polynomial $Q_k(x)$ and the remaining $n-2k$ are set to zero.

Hence the recurrence (9) applied to the polynomial division $x^n : (x-x_1)\cdots(x-x_k)$ gives the coefficients $b_{j,1}$ in (10) of the polynomial $P_{n-k}(x)$ in terms of the coefficients $a_p, p = 0, 1, .., k$ of the polynomial $Q_k(x)$:

$$\begin{aligned} b_{1,1} &= 1 ; \\ b_{i,1} &= -\sum_{j=1}^{i-1} b_{j,1} a_{i-j}, \quad i = 2, ..., n-k+1 . \end{aligned} \tag{12}$$

Furthermore, if $n-k+1 > k$ then $a_s = 0,\ s = k+1, ..., n-k$.

In what follows we can avoid the double indexing for the array $\mathbf{b}$ and refer to it as the *array* of the coefficients of the polynomial $P_{n-k}(x)$.

To conclude this argument we give a simple example.

Example 2.1 Let us take $n = 5$ and $k = 2$, so that we consider the division of x^5 by $Q_2(x) = x^2 + a_1 x + a_2$. The quotient, is $P_3(x) = x^3 - a_1 x^2 + (a_1^2 - a_2)x + (-a_1^3 + 2a_1a_2 - a_3)$.

Now let us consider the lower triangular Toeplitz matrix A of dimension 4:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_1 & 1 & 0 \\ a_3 & a_2 & a_1 & 1 \end{pmatrix} . \tag{13}$$

The first column of the inverse B is determined by using the recurrence (12) applied to the vector $\mathbf{a} = (1,\ a_1,\ a_2, 0)^T$, obtaining $b_{1,1} = 1;\ b_{2,1} = -a_1;\ b_{3,1} = a_1^2 - a_2;\ b_{4,1} = -a_1^3 + 2a_1a_2 - a_3;$.

The above steps are then equivalent to find the expression of the generalized Vandermonde determinant

$$V_{5;2}(x) = \begin{vmatrix} 1 & x_1 & x_1^5 \\ 1 & x_2 & x_2^5 \\ 1 & x & x^5 \end{vmatrix} . \tag{14}$$

Since $Q_2(x) = x^2 - (x_1 + x_2)x + x_1x_2$ and $n - k + 1 > k$ then $\mathbf{a} = (1,\ -(x_1 + x_2),\ x_1x_2,\ 0)^T$ and by means of (12), $P_3(x)$ is:

$$P_3(x) = x^3 + (x_1 + x_2)x^2 + (x_1^2 + x_1x_2 + x_2^2)x + (x_2^3 + x_1x_2^2 + x_1^2x_2 + x_1^3) .$$

The conclusion is now easy since $V_{5;2}(x) = P_3(x) \cdot (x - x_1)(x - x_2) \cdot VDM(x_1, x_2)$.

3 More general Vandermonde determinants

Let us now consider determinants of the form

$$V_{n;k}(x) = \begin{vmatrix} 1 & x_1 & \dots & x_1^{k-1} & x_1^n \\ 1 & x_2 & \dots & x_2^{k-1} & x_2^n \\ \vdots & & \vdots & & \vdots \\ 1 & x_k & \dots & x_k^{k-1} & x_k^n \\ 1 & x & \dots & x^{k-1} & x^n \end{vmatrix} , \tag{15}$$

where $2 < k < n - 1$, that is the $n - k$ powers $k, ..., n - 1$ are missing.

From the previous Section we have seen that we can factor it as

$$V_{n;k}(x) = VDM(x_1, ..., x_k)\ (x - x_1) \cdots (x - x_k)\ P_{n-k}(x) . \tag{16}$$

In order to compute the coefficients of the polynomial $P_{n-k}(x)$ we first divide by the constant term $VDM(x_1, ..., x_k)$ and then we can make use of the recurrence (12).

We firstly observe that the coefficients of the divisor $Q_k(x) = \prod_{i=1}^{k}(x - x_i)$ are, up to the sign, the *elementary symmetric functions* of the points $x_i,\ i = 1, ..., k$, that is the points on which the determinant (15) annihilates. This can help us in going into our investigations, but firstly we need some extra notation.

NOTATION 3.1 *Let λ be any partition of length $\leq r$ a multivariate polynomial will be denoted by*

$$m_\lambda(x_1, ..., x_r) = \sum x_1^{\alpha_1} \cdots x_r^{\alpha_r} ,$$

where α varies on all **distinct** *permutations of λ.*

Moreover, $\mathbf{1}^r = (1, ..., 1)$ will denote the multi-index with r positions equal to 1.

With the symbol $e_r^{(k)}$ we indicate the r-th elementary symmetric function built on the k distinct points $x_1, ..., x_k$, that is

$$\begin{aligned} e_0^{(k)} &= 1 \\ e_r^{(k)} &= \sum_{i_1 < \ldots < i_k} x_{i_1} \cdots x_{i_k} = m_{\mathbf{1}^r}, \quad 1 \leq r \leq k \\ e_r^{(k)} &= 0, \quad r < 0, \quad \text{or} \quad r > k . \end{aligned}$$

the latter is normally assumed *by definition.* Thus, $e_r^{(k)}$ is simply the sum of all products of length $\leq r$ of k distinct variables $x_1, ..., x_k$.

An algorithm that allows us to compute the coefficients of the quotient polynomial $P_{n-k}(x)$ can be formulated as follows.

Algorithm 1

`Inputs:` $n,\ k$ and the points $x_i,\ i=1,...,k$.

1. Form the $n-k+1$ dimensional array $\mathbf{a}$, that is

$$\mathbf{a} = (\underbrace{e_0^{(k)}}_{=1}, -e_1^{(k)}, e_2^{(k)}, ..., (-1)^k\ e_k^{(k)}, \underbrace{e_{k+1}^{(k)}, \dots, e_{n-k}^{(k)}}_{=0}) \ ;$$

2. By means of the recurrence (12) compute the array $\mathbf{b}$.

`Output:` the array $\mathbf{b}$ of the coefficients of $P_{n-k}(x)$.

This algorithm is a *simple* recipe for computing a general Vandermonde determinant of the form given above. It is simple because it does not make use of complicated multidimensional notations such as in [7] and it provides an explicit expression for the coefficients of the quotient polynomial, moreover it is easy to implement in a symbolic based language.

The computational cost necessary to form the array $\mathbf{a}$ in the previous Algorithm 1 can be computed as follows. For a given set of k points, there are $k+1$ symmetric elementary functions non zeros, i.e $e_0^{(k)}, e_1^{(k)}, e_2^{(k)}, ..., e_k^{(k)}$. Each of them consists of the sum of $\binom{k}{r}$, $r = 0,...,k$ terms. Each such a term is made by $r-1$ multiplications. Thus, to compute these $k+1$ symmetric elementary functions the multiplications involved are

$$\sum_{r=1}^{k}(r-1)\binom{k}{r} = 2^{k-1}(k-2)+1 \ . \tag{17}$$

To determine the total complexity of the Algorithm 1, one has to add the number of multiplications necessary to compute the array $\mathbf{b}$. As outlined in the previous section this is of $\mathcal{O}(\frac{(n-k+1)^2}{2})$ or more precisely

$$\sum_{i=1}^{n-k+1}(i-2) = \frac{1}{2}(n-k-2)(n-k-1) \ . \tag{18}$$

Therefore, the exact complexity for the Algorithm 1 is

$$\frac{n}{2}(n-2k-1) + \frac{k}{2}(2^k+k+1) - 2^k \ . \tag{19}$$

$\diamond\diamond$

We now introduce the *complete elementary symmetric functions* of the set of q points, that is the homogeneous symmetric polynomials so defined:

$$\begin{aligned} h_s^{(q)} &= \sum_{|\lambda|=s} m_\lambda \\ h_s^{(q)} &= 0, \quad s<0, \quad \text{or} \quad s>q \ . \end{aligned}$$

that is $h_s^{(q)}$ is the sum of all monomials of total degree s in the variables $x_1, ..., x_q$. In particular $h_0^{(q)} = e_0^{(q)}$, $h_1^{(q)} = e_1^{(q)}$.

Lemma 3.1

$$h_s^{(q)} = det\,(e^{(q)}_{\lambda_i - i + j}), \qquad 1 \le i, j \le s\ ,$$

and $\lambda = \underbrace{(1, ..., 1)}_{s}$.

Proof. The proof is based on [6, p. 21, formula (2.6')] by solving for h_n (cf. also [6, p. 28]). □

Finally, we can state the following result.

Theorem 3.2 *Given k distinct points, then the coefficients of the quotient polynomial $P_{n-k}(x)$ in the factorization (16), i.e. the elements of the array* $\mathbf{b}$*, are the complete elementary symmetric functions:*

$$\mathbf{b}_i = h_{i-1}^{(k)}\ , \quad i = 1, ..., n-k+1.$$

Proof. Let us consider the array $\mathbf{a}$ as above, made up of the elementary symmetric functions of the k points $x_1, ..., x_k$. From the recurrence (12) we recognize that

$$\mathbf{b}_i = \det \begin{pmatrix} e_1^{(k)} & e_2^{(k)} & \dots & \dots & e_{i-1}^{(k)} \\ e_0^{(k)} & e_1^{(k)} & \dots & \dots & e_{i-2}^{(k)} \\ e_{-1}^{(k)} & e_0^{(k)} & e_1^{(k)} & \dots & e_{i-3}^{(k)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ e_{-i-2}^{(k)} & \dots & \dots & e_0^{(k)} & e_1^{(k)} \end{pmatrix},$$

that by the previous lemma, is exactly $h_{i-1}^{(k)}$. □

Corollary 3.3 *The quotient polynomial $P_{n-k}(x)$ can be written in determinantal form, that is*

$$P_{n-k}(x) = \begin{vmatrix} e_0^{(k)} & -e_1^{(k)} & \dots & \dots & (-1)^{n-k} e_{n-k}^{(k)} \\ 0 & e_0^{(k)} & -e_1^{(k)} & \dots & (-1)^{n-k-1} e_{n-k-1}^{(k)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & e_0^{(k)} & -e_1^{(k)} \\ 1 & x & \dots & x^{n-k-1} & x^{n-k} \end{vmatrix}. \qquad (20)$$

Example 3.1 Given the determinant:

$$V_{7;4}(x)=\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^7 \\ \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^7 \\ \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^7 \\ \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^7 \\ \\ 1 & x & x^2 & x^3 & x^7 \end{vmatrix}. \tag{21}$$

Thanks to our results we can say:

(i) $V_{7;4}(x)=VDM(x_1,x_2,x_3,x_4)\ (x-x_1)(x-x_2)(x-x_3)(x-x_4)\ P_3(x)$,

(ii) and $P_3(x)=x^3+h_1^{(4)}x^2+h_2^{(4)}x+h_3^{(4)}$.

4 Schur functions

A *Schur function* is defined as follows (cf. [6]).

Definition 4.1 *Given a partition $\lambda=(\lambda_1,...,\lambda_n)$ of length $|\lambda|\leq n$, the Schur function s_λ, defined on $I\!R^n$, is the following ratio*

$$s_\lambda(x_1,...,x_n)=\frac{det\ (x_i^{\lambda_j+n-j})}{det\ (x_i^{n-j})},\quad 1\leq i,j\leq n. \tag{22}$$

The denominator is nothing else than the classical Vandermonde determinant $VDM(x_1,...,x_n)$.

Schur functions provides a very useful tool to study and compute generalized Vandermonde determinants. To fit our setting in this framework, one has simply to substitute the last variable involved, say x_n, with x.

Example 4.1 Let us consider the 3×3 matrix of the generalized Vandermonde determinant when 2 powers are missing, that is

$$V_{4;2}=\begin{vmatrix} 1 & x_1 & x_1^4 \\ 1 & x_2 & x_2^4 \\ 1 & x_3 & x_3^4 \end{vmatrix}. \tag{23}$$

The determinant can be also written as

$$V_{4;2}=(x_1-x_3)(x_2-x_3)\begin{vmatrix} 0 & x_1-x_3 & x_1^4-x_3^4 \\ 0 & x_2-x_3 & x_2^4-x_3^4 \\ 1 & x_3 & x_3^4 \end{vmatrix}.$$

this makes it easy to understand that we can factor out $(x_1-x_3)(x_2-x_3)$. Then, after a few calculations,

$$\begin{aligned} V_{4;2} &= (x_1 - x_3)(x_2 - x_3)[(x_2^3 - x_1^3) + (x_2^2 - x_1^2)x_3 + (x_2 - x_1)x_3^2] \\ &= (x_1 - x_3)(x_2 - x_3)(x_2 - x_1)[x_2^2 + x_1x_2 + x_1^2 + x_2x_3 + x_1x_3 + x_3^2] \ . \end{aligned}$$

which proves again than in general we can factor the classical Vandermonde $VDM(x_1, x_2, x_3)$ out of the determinant.

Hence, let $\alpha = (4, 1, 0)$ and $\delta = (2, 1, 0)$ be the partitions consisting of the degrees of the monomials involved considered in *reverse order*, then letting $\lambda = \alpha - \delta = (2, 0, 0)$, the above determinant divided by $VDM(x_1, x_2, x_3)$ is the following Schur function:

$$\frac{\begin{vmatrix} 1 & x_1 & x_1^4 \\ 1 & x_2 & x_2^4 \\ 1 & x_3 & x_3^4 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}} = x_2^2 + x_1x_2 + x_1^2 + x_2x_3 + x_1x_3 + x_3^2 = s_\lambda(x_1, x_2, x_3) \ . \tag{24}$$

Remark. We recognize that the above Schur function is $h_2^{(3)}$. This again is a general setting for these determinants that we shall prove in the next Section.

4.1 Schur functions for the case of a group of missing powers

In the case of a group of k missing powers, let $i^{(n-k)} = \underbrace{(i, ..., i)}_{n-k}$ be a multi-index, we can state the following connection between Schur function and generalized Vandermonde determinants.

Proposition 4.1 *Let $\lambda = (k, 0^{(n-k)})$, then the generalized Vandermonde determinant,*

$$V_{n;n-k} = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-k-1} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-k-1} & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-k} & x_{n-k}^2 & \dots & x_{n-k}^{n-k-1} & x_{n-k}^n \\ 1 & x_{n-k+1} & x_{n-k+1}^2 & \dots & x_{n-k+1}^{n-k-1} & x_{n-k+1}^n \end{vmatrix} .$$

when the k powers $n-k, ..., n-1$ are missed, can be written

$$V_{n;n-k} = s_\lambda \ \cdot \ VDM(x_1, ..., x_k) = h_k^{(n-k+1)}(x) \ \cdot \ VDM(x_1, ..., x_{n-k+1})$$

where $h_k^{(n-k+1)}$ is the k-th complete symmetric function of the $n-k+1$ variables $x_1, ..., x_{n-k+1}$.

Proof. From the `Jacobi-Trudi identity` (cf. [6, formula (3.4), p. 41]) :

$$s_\lambda = \det(h_{\lambda_i - i + j}^{(n-k+1)}(x)) \ , \qquad 1 \le i, j \le n - k + 1 \ . \tag{25}$$

Then, given the partitions $\alpha = (n, n-k-1, n-k-2, ..., 1, 0)$ and $\delta = (n-k, n-k-1, n-k-2, ..., 1, 0)$ having both $n-k+1$ positions then $\lambda = \alpha - \delta = (k, 0^{(n-k)})$. Therefore,

$$\frac{V_{n;n-k}}{VDM(x_1, ..., x_{n-k+1})} = s_\lambda = h_k^{(n-k+1)} \; . \tag{26}$$

The last equality can be proved by the Jacobi-Trudi identity. In fact

$$s_\lambda = \begin{vmatrix} h_k^{(n-k+1)} & h_{k+1}^{(n-k+1)} & h_{k+2}^{(n-k+1)} & \dots & h_n^{(n-k+1)} \\ h_{-1}^{(n-k+1)} & h_0^{(n-k+1)} & h_1^{(n-k+1)} & \dots & h_{n-k-1}^{(n-k+1)} \\ h_{-2}^{(n-k+1)} & h_{-1}^{(n-k+1)} & h_0^{(n-k+1)} & \dots & h_{n-k-2}^{(n-k+1)} \\ \vdots & & \ddots & & \vdots \\ h_{-n+k}^{(n-k+1)} & h_{-n+k+1}^{(n-k+1)} & \dots & \dots & h_0^{(n-k+1)} \end{vmatrix} ,$$

and since $h_p^{(n-k+1)} = 0, \; p \leq -1$ we get the claimed result. □

Remark. From lemma (3.1) we observe that $h_k^{(n-k+1)}$ is a function of the elementary symmetric functions that is,

$$h_k^{(n-k+1)} = \det\left(e_{\lambda_i^*-i+j}^{(n-k+1)}\right), \;\; 1 \leq i, j \leq n-k+1 \; , \tag{27}$$

where here $\lambda^* = \underbrace{(1, ..., 1)}_{n-k+1}$.

Therefore, once we compute the determinant (27), whose matrix is in Hessenberg form, which is a special instance of a Toeplitz matrix, we may express h_k as a function of the elementary symmetric functions $e_0, ..., e_k$.

4.2 The general setting

To conclude our investigations we extend our results to general Vandermonde determinants. To be precise, given a sequence of integers, $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$, and a set of n distinct points, $x_1, ..., x_n$, we can consider determinants of the form

$$V_{\mu_n;\ell(\mu)} = \left| x_i^{\mu_j} \right|_{1 \leq i,j \leq n} \; . \tag{28}$$

It is obvious that these determinants can be rewritten

$$V_{\mu_n;\ell(\mu)} = \left(\prod_{i=1}^n x_i^{\mu_1}\right) \det\left(x_i^{\mu_j-\mu_1}\right)_{1 \leq i,j \leq n} \; , \tag{29}$$

and so for our purposes *it is enough* to consider polynomials of the type

$$V_{m_{n-1};\ell(m)}(x) = \begin{vmatrix} 1 & x_1^{m_1} & \dots & x_1^{m_{n-1}} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n-1}^{m_1} & \dots & x_{n-1}^{m_{n-1}} \\ 1 & x^{m_1} & \dots & x^{m_{n-1}} \end{vmatrix} , \tag{30}$$

with obvious meaning for the symbols used.

We shall use another concept related to partitions, the so-called *diagram* (cf. [6, p. 2]).

DEFINITION 4.2 *Given a partition λ, its diagram is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$.*

DEFINITION 4.3 *The* **conjugate** *of a partition λ, is the partition λ' whose diagram is the transpose of the diagram of λ obtained by reflection in the main diagonal.*

For example, if $\lambda = (3,2,0)$, then $\lambda' = (2,2,1)$. The definition of conjugate of a partition allows to introduce an equivalent relation to the Jacobi-Trudi identity for Schur functions. That is, for any partition λ of n,

$$s_\lambda = \det \left(e_{\lambda'_i - i + j} \right)_{1 \leq i,j \leq n} , \tag{31}$$

with λ' its conjugate partition.

We are now ready to generalize the Theorem 3.2 and the Corollary 3.3.

Theorem 4.2 *Let $\alpha = (m_{n-1}, m_{n-2}, ..., m_1, 0)$ and $\delta = (n-1, n-2, ..., 1, 0)$ be the partitions of length n associated to the determinant (30). Let $\lambda = \alpha - \delta$ and λ' be its conjugate partition.*

$$V_{m_{n-1};\ell(m)}(x) = VDM(x_1, ..., x_{n-1}) \left(\prod_{i=1}^{n-1} (x - x_i) \right) P_M(x) . \tag{32}$$

where $M = m_{n-1} - n + 1$. Moreover, if $\lambda_1 \geq n$, then

$$P_M(x) = \begin{vmatrix} (-1)^{\lambda'_1 - 1} e_{\lambda'_1 - 1} & (-1)^{\lambda'_1} e_{\lambda'_1} & \cdots & (-1)^{\lambda'_1 + M - 1} e_{\lambda'_1 + M - 1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{\lambda'_n - M} e_{\lambda'_n - M} & (-1)^{\lambda'_n - M + 1} e_{\lambda'_n - M + 1} & \cdots & (-1)^{\lambda'_n} e_{\lambda'_n} \\ 1 & x & \cdots & x^M \end{vmatrix} . \tag{33}$$

otherwise λ' has some zeros, that correspondingly reduce the degree M.

Proof. The equality (32) is simply proved by resorting to the definition of Schur functions when $x = x_n$.

Since $\lambda_1 \geq n$, then λ' has no zeros. To get the determinant (33) we observe that $P_M(x)$ is the determinant of the matrix obtained by bordering the matrix

$$\left(e_{\lambda'_i - i + j} \right)_{1 \leq i,\, j \leq M} , \tag{34}$$

with the column consisting of the elementary symmetric functions translated by 1 to the left starting from the first column of the matrix (34), i.e

$$\left((-1)^{\lambda'_1 - 1} e_{\lambda'_1 - 1},\ ..., (-1)^{\lambda'_n - M} e_{\lambda'_n - M}, 1 \right)^T$$

and with the last row consisting of the vector of the powers $(1, x, \ ..., x^M)$.

If $\lambda_1 < n$, then λ' has $n - \lambda_1$ zeros. Therefore in the matrix (34) there are not the corresponding rows. ▯

We now give two examples.

Example 4.2 Let us consider this determinant:

$$V_{7;3}(x) = \begin{vmatrix} 1 & x_1^3 & x_1^4 & x_1^7 \\ \\ 1 & x_2^3 & x_2^4 & x_2^7 \\ \\ 1 & x_3^3 & x_3^4 & x_3^7 \\ \\ 1 & x^3 & x^4 & x^7 \end{vmatrix}.$$

By the Theorem 4.2, $V_{7;3}(x) = VDM(x_1, x_2, x_3)\ (x - x_1)(x - x_2)(x - x_3)\ P_4(x)$. Since $\alpha = (7, 4, 3, 0)$, $\delta = (3, 2, 1, 0)$, $\lambda = (4, 2, 2, 0)$ and $\lambda' = (3, 3, 1, 1)$, then the determinantal form for $P_4(x)$ is:

$$P_4(x) = \begin{vmatrix} e_2 & -e_3 & e_4 & -e_5 & e_6 \\ -e_1 & e_2 & -e_3 & e_4 & -e_5 \\ e_{-2} & -e_{-1} & e_0 & -e_1 & e_2 \\ -e_{-3} & e_{-2} & -e_{-1} & e_0 & -e_1 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix}.$$

Finally, since $e_j = 0, j > 3$ and $j < 0$ then

$$P_4(x) = \begin{vmatrix} e_2 & -e_3 & 0 & 0 & 0 \\ -e_1 & e_2 & -e_3 & 0 & 0 \\ 0 & 0 & 1 & -e_1 & e_2 \\ 0 & 0 & 0 & 1 & -e_1 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix}.$$

Example 4.3 Let us consider the determinant:

$$V_{4;2}(x) = \begin{vmatrix} 1 & x_1 & x_1^4 \\ \\ 1 & x_2 & x_2^4 \\ \\ 1 & x & x^4 \end{vmatrix}.$$

By the Theorem 4.2, $V_{4;2}(x) = VDM(x_1, x_2)\ (x - x_1)(x - x_2)\ P_2(x)$. Since $\alpha = (4, 1, 0)$, $\delta = (2, 1, 0)$, $\lambda = (2, 0, 0)$ and $\lambda' = (1, 1, 0)$, then the determinantal form for $P_2(x)$ is:

$$P_2(x) = \begin{vmatrix} e_0 & -e_1 & e_2 \\ -e_{-1} & e_0 & -e_1 \\ 1 & x & x^2 \end{vmatrix}.$$

Finally we can write down an algorithm for computing the coefficients of $P_M(x)$ based on the Theorem 4.2.

Algorithm 2

`Inputs`: the partitions α, δ of M and the points $x_i,\ i = 1, ..., M$.

1. Let $\lambda = \alpha - \delta$ and λ' its conjugate.

Build the matrix

$$E = \left(e_{\lambda'_i - i + j}\right)_{1 \le i, j \le M}$$

where e_p is the p-th elementary symmetric function of M points.

2. Build the matrix E_1 by bordering the matrix E:

add as first column the array

$$\left((-1)^{\lambda'_1 - 1} e_{\lambda'_1 - 1},\ ...,\ (-1)^{\lambda'_n - M} e_{\lambda'_n - M}, 1\right)^T$$

and as last row $(1, x,\ ..., x^M)$.

`Output`: the coefficients of $P_M(x)$, which are the minors of order $M - 1$

of the matrix E_1.

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