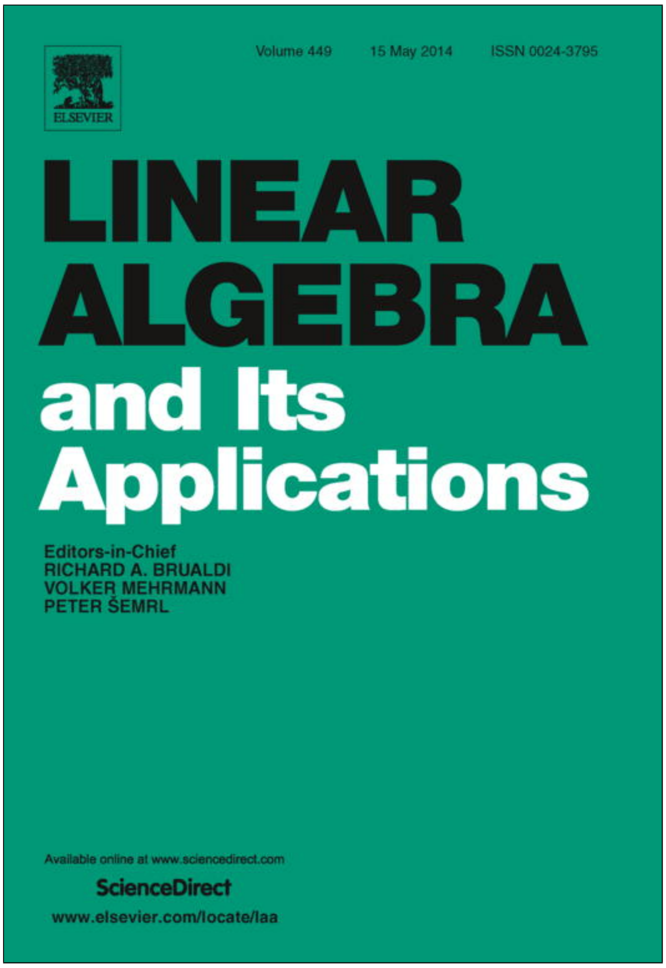
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On certain multivariate Vandermonde determinants whose variables separate

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A B S T R A C T

We prove that for almost square tensor product grids and certain sets of bivariate polynomials the Vandermonde determinant can be factored into a product of univariate Vandermonde determinants. This result generalizes the conjecture [3, Lemma 1]. As a special case, we apply the result to Padua and Padua-like points.

1. Introduction

This paper is mainly inspired by the results in [3] where the authors discussed the properties of the Vandermonde determinants associated to point sets on the square that

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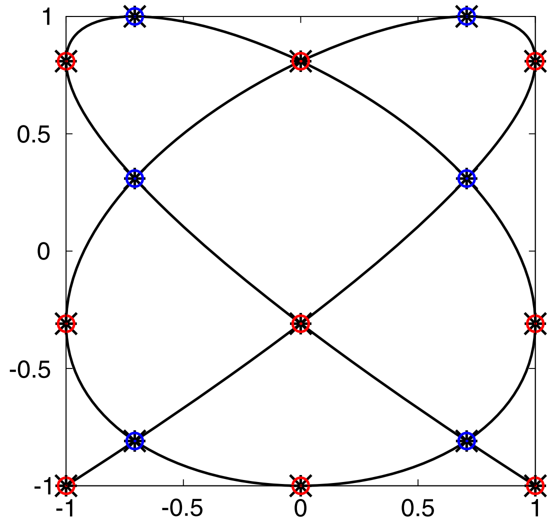

Fig. 1. Padua points and their generating curve for $n = 4$. The grids of odd and even indices are indicated with different colours and style. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

distribute as the Padua points. In order to understand the result of this article, we briefly recall the definition and the construction of the Padua points.

The Padua points are the first known near-optimal point set for bivariate polynomial interpolation of total degree in the square $[-1,1]^2$, whose Lebesgue constants have minimal order of growth of $\mathcal{O}((\log n)^2)$, n being the polynomial degree [1,2].

It has been observed that these points have the structure of the union of *two* (tensor product) grids of Chebyshev–Lobatto points, one square and the other rectangular. Actually there are four families of Padua points, obtainable one from the other by a suitable rotation of 90, 180 or 270 degrees. For the sake of simplicity, we consider here only the construction of the points belonging to the first family, displayed in Fig. 1.

Let start by taking the $n+1$ Chebyshev–Lobatto points on $[-1,1]$

$$C_{n+1} := \left\{ z_j^n = \cos\left(\frac{(j-1)\pi}{n}\right),\ j = 1, \ldots, n+1 \right\}.$$

We then consider two subsets of points with *odd* and *even* indices

$$C_{n+1}^{\mathrm{o}} := \left\{ z_j^n,\ j = 1, \ldots, n+1,\ j \text{ odd} \right\},$$
$$C_{n+1}^{\mathrm{e}} := \left\{ z_j^n,\ j = 1, \ldots, n+1,\ j \text{ even} \right\}.$$

Then, the Padua points of the first family are the set

$$\mathcal{P}_n := \left(C_{n+1}^{\mathrm{o}} \times C_{n+2}^{\mathrm{o}}\right) \cup \left(C_{n+1}^{\mathrm{e}} \times C_{n+2}^{\mathrm{e}}\right) \subset C_{n+1} \times C_{n+2}. \tag{1}$$

These points have cardinality of the space of bivariate polynomials of degree $\leqslant n$, i.e. $N = (n+1)(n+2)/2$.

There is another interesting geometric interpretation: Padua points are self-intersections and boundary contacts of the following (parametric and periodic) generating curve

$$\gamma(t) = \bigl(-\cos\bigl((n+1)t\bigr), -\cos(nt)\bigr), \quad t \in [0, \pi]$$

which turns out to be a *Lissajous curve* [1]. In Fig. 1, we show the two grids and the generating curve for $n = 4$. In this case, the square grid has 9 points while the rectangular one has 6 points.

For more details on Padua points, their properties and applications we refer the interested reader to the web page http://www.math.unipd.it/~marcov/CAApadua.html that also contains an up-to-date bibliography on the topic.

In [3] has been conjectured an interesting formula for the Vandermonde determinant of *any set of points* with exactly a similar structure like that of Padua points. Surprisingly, the Vandermonde determinant factors into the product of two univariate functions. The technical Lemma 1 [3, Lemma 1], very important in that paper, was conjectured to be true but up to now a correct proof has not been given. This article provides a general proof of this (special) factorization that applies to any set of points having a similar structure.

2. Notation

We denote by $\mathbb{R}^{m\times n}$ the space of $m \times n$ real matrices, by $\operatorname{diag}(V) \in \mathbb{R}^{n\times n}$ the diagonal matrix constructed from $V \in \mathbb{R}^n$, and by $I_n \in \mathbb{R}^{n\times n}$ the identity matrix. We denote by $\mathbb{R}^{m\times n}[\mathbf{z}]$ the space of $m \times n$ real polynomial matrices, in q variables $\mathbf{z} = (z_1, \dots, z_q)$. For a polynomial matrix $P \in \mathbb{R}^{m\times n}[\mathbf{z}]$, we denote by $P_{:,j} \in \mathbb{R}^{m\times 1}[\mathbf{z}]$ (resp. $P_{i,:} \in \mathbb{R}^{1\times n}[\mathbf{z}]$) the j-th column (resp. i-th row), and by $P_{:,j:k} \in \mathbb{R}^{m\times(k-j+1)}[\mathbf{z}]$ the submatrix constructed from the j-th to k-th columns of P. For two univariate polynomial matrices $P \in \mathbb{R}^{m\times n}[x]$ and $Q \in \mathbb{R}^{m\times n}[y]$, we denote their Hadamard (element-wise) product as $P \circ Q \in \mathbb{R}^{m\times n}[\mathbf{z}]$, $\mathbf{z} = (x, y)$.

For a set of polynomials $\mathcal{P} = \{p_1(\mathbf{z}), \dots, p_n(\mathbf{z})\}$ and a set of points $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$, we denote by $\operatorname{VDM}(\mathcal{A}, \mathcal{P})$ the Vandermonde matrix $\operatorname{VDM}(\mathcal{A}, \mathcal{P}) = [p_j(\mathbf{a}_i)]_{i,j=1}^{n,n}$ and by $\operatorname{vdm}(\mathcal{A}, \mathcal{P}) = \det \operatorname{VDM}(\mathcal{A}, \mathcal{P})$ its determinant. We should note that $\operatorname{VDM}(\mathcal{A}, \mathcal{P})$ is defined uniquely only if a specific order of the elements of $\mathcal{A}$ and $\mathcal{P}$ is fixed. However, we are mainly interested in the absolute value of $\operatorname{vdm}(\mathcal{A}, \mathcal{P})$, and therefore, the particular order does not matter. For convenience, we also use the notation $\operatorname{vdm}(\mathcal{A}, P) = \operatorname{vdm}(\mathcal{A}, \{P_{i,j}(\mathbf{z})\}_{i,j=1}^{m,n})$ and $\operatorname{VDM}(\mathcal{A}, P) = \operatorname{VDM}(\mathcal{A}, \{P_{i,j}(\mathbf{z})\}_{i,j=1}^{m,n})$ for a polynomial matrix $P \in \mathbb{R}^{m\times n}[\mathbf{z}]$.

3. Vandermonde determinants whose variables separate

3.1. The main result

Proposition 1. *Assume that $m, n \geqslant 1$ are integers such that $n = m$ or $n = m + 1$. Let $P \in \mathbb{R}^{m\times n}[x]$, $Q \in \mathbb{R}^{m\times n}[y]$ be two polynomial matrices of the form*

$$Q(y) = \begin{bmatrix} 1 & * & \dots & \dots & * \\ 1 & q_1(y) & * & \dots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & q_1(y) & \dots & q_{n-1}(y) & * \end{bmatrix},$$

$$P(x) = \begin{bmatrix} * & 1 & 1 & \dots & 1 \\ * & * & p_1(x) & \dots & p_1(x) \\ \vdots & & \ddots & \ddots & \vdots \\ * & * & \dots & * & p_{n-1}(x) \end{bmatrix}, \tag{2}$$

i.e., the lower triangular block of Q (including the main diagonal) has constant columns, and the upper triangular block of P (excluding the main diagonal) has constant rows, and $*$ denote arbitrary (not necessarily equal to each other) polynomials. Also assume that

$$\textit{all } p_j, q_j \textit{ are monic}, \quad \deg(p_j) = \deg(q_j) = j.$$

Then for $\mathcal{X} = \{x_1, \dots, x_m\}$ and $\mathcal{Y} = \{y_1, \dots, y_n\}$, the following equality holds:

$$\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) = \pm \left(\prod_{j=1}^{n} \operatorname{vdm}(\mathcal{X}, P_{:,j}) \right) \cdot \left(\prod_{i=1}^{m} \operatorname{vdm}(\mathcal{Y}, Q_{i,:}) \right). \tag{3}$$

Remark. In (2) we show the matrices for $n = m + 1$. The matrices for $n = m$ can be obtained by deleting the last column of each matrix in (2).

Proof. We prove the proposition by induction. First we consider the case $m = 1$. For $n = 1$, (3) is trivial. For $n = m + 1 = 2$, consider $P(x) = [a_1(x)\ a_2(x)]$, $Q(y) = [b_1(y)\ b_2(y)]$, $\mathcal{X} = \{x_1\}$ and $\mathcal{Y} = \{y_1, y_2\}$. Then we have that

$$\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) = \pm a_1(x_1) a_2(x_1) \det \begin{bmatrix} b_1(y_1) & b_2(y_1) \\ b_1(y_2) & b_2(y_2) \end{bmatrix}.$$

Now we assume that (3) holds for $(m, n) = (k, k)$ and we prove it for $(m, n) = (k, k+1)$. Although we prove only the induction step $(k, k) \to (k, k + 1)$, the same derivations (almost without changes) hold for the step $(k, k + 1) \to (k + 1, k + 1)$.

Denote $N_j = \operatorname{VDM}(\mathcal{X}, P_{:,j})$. Then $\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q)$ can be written as

$$\begin{aligned} &\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) \\ &= \pm \det \left[\begin{array}{c|ccc} N_1 & N_2 \operatorname{diag}(Q_{:,2}(y_1)) & \cdots & N_m \operatorname{diag}(Q_{:,n}(y_1)) \\ \hline N_1 & N_2 \operatorname{diag}(Q_{:,2}(y_2)) & \cdots & N_m \operatorname{diag}(Q_{:,n}(y_2)) \\ \vdots & \vdots & & \vdots \\ N_1 & N_2 \operatorname{diag}(Q_{:,2}(y_n)) & \cdots & N_m \operatorname{diag}(Q_{:,n}(y_n)) \end{array} \right] \end{aligned}$$

$$= \pm \det(N_1) \cdot \det \left[\begin{array}{c|ccc} I_m & N_2 \operatorname{diag}(Q_{:,2}(y_1)) & \cdots & N_n \operatorname{diag}(Q_{:,n}(y_1)) \\ \hline I_m & N_2 \operatorname{diag}(Q_{:,2}(y_2)) & \cdots & N_n \operatorname{diag}(Q_{:,n}(y_2)) \\ \vdots & \vdots & & \vdots \\ I_m & N_2 \operatorname{diag}(Q_{:,2}(y_n)) & \cdots & N_n \operatorname{diag}(Q_{:,n}(y_n)) \end{array}\right]. \tag{4}$$

Note that in [3] a different block representation was used. That different representation was an obstacle to derive the proof of [3, Lemma 1].

By applying the Schur complement formula to the block matrix in (4), we have that

$$\begin{aligned}
&\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) \\
&= \pm \det(N_1) \det \begin{bmatrix} N_2 \operatorname{diag}(Q_{:,2}(y_2) - Q_{:,2}(y_1)) & \cdots & N_n \operatorname{diag}(Q_{:,n}(y_2) - Q_{:,n}(y_1)) \\ \vdots & & \vdots \\ N_2 \operatorname{diag}(Q_{:,2}(y_n) - Q_{:,2}(y_1)) & \cdots & N_n \operatorname{diag}(Q_{:,n}(y_n) - Q_{:,n}(y_1)) \end{bmatrix} \\
&= \pm \det(N_1) \left(\prod_{j=2}^{n} (y_j - y_1) \right)^m \det \begin{bmatrix} N_2 \operatorname{diag}(\widetilde{Q}_{:,1}(y_2)) & \cdots & N_n \operatorname{diag}(\widetilde{Q}_{:,n-1}(y_2)) \\ \vdots & & \vdots \\ N_2 \operatorname{diag}(\widetilde{Q}_{:,1}(y_n)) & \cdots & N_n \operatorname{diag}(\widetilde{Q}_{:,n-1}(y_n)) \end{bmatrix} \\
&= \pm \operatorname{vdm}(\mathcal{X}, P_{:,1}) \left(\prod_{j=2}^{n} (y_j - y_1) \right)^m \operatorname{vdm}(\mathcal{X} \times \widetilde{\mathcal{Y}}, P_{:,2:n} \circ \widetilde{Q}), \qquad (5)
\end{aligned}$$

where $\widetilde{\mathcal{Y}} := \{y_2, \dots, y_n\}$ and $\widetilde{Q} \in \mathbb{R}^{m \times (n-1)}[y]$ is the polynomial matrix defined as

$$\widetilde{Q}(y) := \frac{Q_{:,2:n}(y) - Q_{:,2:n}(y_1)}{y - y_1}.$$

Note that the matrix $\widetilde{Q}$ has the form

$$\widetilde{Q}(y) = \begin{bmatrix} \widetilde{*} & \widetilde{*} & \dots & \dots & \widetilde{*} \\ 1 & \widetilde{*} & \dots & \dots & \widetilde{*} \\ 1 & \widetilde{q}_1(y) & \ddots & & \widetilde{*} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \widetilde{q}_1(y) & \dots & \widetilde{q}_{n-2}(y) & \widetilde{*} \end{bmatrix},$$

where $\widetilde{q}_j(y) := \frac{q_{j+1}(y) - q_{j+1}(y_1)}{y - y_1}$, and hence $\deg(\widetilde{q}_j) = j$ and $\widetilde{q}_j$ is monic. Therefore, we can interchange the variables x and y, transpose the matrices $P_{:,2:n}$ and $\widetilde{Q}$ and apply the induction assumption to (5). Formally, for

$$P'(x') := \left(\widetilde{Q}(x')\right)^{\top} \in \mathbb{R}^{k \times k}[x'], \qquad Q'(y') := \left(P_{:,2:n}(y')\right)^{\top} \in \mathbb{R}^{k \times k}[y'],$$
$$\mathcal{X}' := \widetilde{\mathcal{Y}}, \qquad \mathcal{Y}' := \mathcal{X},$$

the equality (3) takes place by the induction assumption, and we have that

$$\mathrm{vdm}(\mathcal{X} \times \widetilde{\mathcal{Y}}, P_{:,2:n} \circ \widetilde{Q}) = \pm \left(\prod_{j=2}^{n} \mathrm{vdm}(\mathcal{X}, P_{:,j}) \right) \left(\prod_{i=1}^{m} \mathrm{vdm}(\widetilde{\mathcal{Y}}, \widetilde{Q}_{i,:}) \right).$$

Hence, from (5) we have that

$$\mathrm{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) = \pm \left(\prod_{j=1}^{n} \mathrm{vdm}(\mathcal{X}, P_{:,j}) \right) \left(\prod_{j=2}^{n} (y_j - y_1) \right)^m \left(\prod_{i=1}^{m} \mathrm{vdm}(\widetilde{\mathcal{Y}}, \widetilde{Q}_{i,:}) \right).$$

Then the equality (3) will hold if for all $i = 1, \ldots, m$ the following equality holds:

$$\pm \left(\prod_{j=2}^{n} (y_j - y_1) \right) \mathrm{vdm}(\widetilde{\mathcal{Y}}, \widetilde{Q}_{i,:}) = \mathrm{vdm}(\mathcal{Y}, Q_{i,:}). \tag{6}$$

Consider a row vector polynomial $A(y) = [1 \; a_1(y) \; \cdots \; a_{n-1}(y)]$. Then,

$$\begin{aligned} \mathrm{vdm}(\mathcal{Y}, A) &= \pm \det \left[\begin{array}{c|ccc} 1 & a_1(y_1) & \cdots & a_{n-1}(y_1) \\ \hline 1 & a_1(y_2) & \cdots & a_{n-1}(y_2) \\ \vdots & \vdots & & \vdots \\ 1 & a_1(y_n) & \cdots & a_{n-1}(y_n) \end{array} \right] \\ &= \pm \det \begin{bmatrix} a_1(y_2) - a_1(y_1) & \cdots & a_{n-1}(y_2) - a_{n-1}(y_1) \\ \vdots & & \vdots \\ a_1(y_n) - a_1(y_1) & \cdots & a_{n-1}(y_n) - a_{n-1}(y_1) \end{bmatrix} \\ &= \pm \left(\prod_{j=2}^{n} (y_j - y_1) \right) \mathrm{vdm}(\widetilde{\mathcal{Y}}, \widetilde{A}), \end{aligned}$$

where

$$\widetilde{A}(y) := \frac{[\, a_1(y) - a_1(y_1) \quad \cdots \quad a_{n-1}(y) - a_{n-1}(y_1) \,]}{y - y_1}. \tag{7}$$

This proves (6). □

3.2. Discussion

Consider a case which is simpler than that of Proposition 1. Let $\mathcal{X} = \{x_1, \ldots, x_m\}$ and $\mathcal{Y} = \{y_1, \ldots, y_n\}$. Let $P(x) \in \mathbb{R}^{m \times n}[x]$ and $Q(y) \in \mathbb{R}^{m \times n}$ be given by

$$P(x) = \begin{bmatrix} p_1(x) & p_1(x) & \dots & p_1(x) \\ p_2(x) & p_2(x) & \dots & p_2(x) \\ \vdots & \vdots & & \vdots \\ p_m(x) & p_m(x) & \dots & p_m(x) \end{bmatrix}, \qquad Q(y) = \begin{bmatrix} q_1(y) & q_2(y) & \dots & q_n(y) \\ q_1(y) & q_2(y) & \dots & q_n(y) \\ \vdots & \vdots & & \vdots \\ q_1(y) & q_2(y) & \dots & q_n(y) \end{bmatrix}. \quad (8)$$

Then we have that

$$(P \circ Q)(x, y) = \begin{bmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_m(x) \end{bmatrix} \begin{bmatrix} q_1(y) & q_2(y) & \dots & q_n(y) \end{bmatrix},$$

and therefore

$$\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) = \pm \det\bigl(\operatorname{VDM}\bigl(\mathcal{X}, \{p_1, \dots, p_m\}\bigr) \otimes \operatorname{VDM}\bigl(\mathcal{Y}, \{q_1, \dots, q_n\}\bigr)\bigr). \quad (9)$$

By properties of the Kronecker product [4], (9) can be rewritten as

$$\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) = \pm\bigl(\operatorname{vdm}\bigl(\mathcal{X}, \{p_1, \dots, p_m\}\bigr)\bigr)^n \bigl(\operatorname{vdm}\bigl(\mathcal{Y}, \{q_1, \dots, q_n\}\bigr)\bigr)^m. \quad (10)$$

An example of (8) is

$$P(x) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x & x & \dots & x \\ \vdots & \vdots & & \vdots \\ x^{m-1} & x^{m-1} & \dots & x^{m-1} \end{bmatrix}, \qquad Q(y) = \begin{bmatrix} 1 & y & \dots & y^{n-1} \\ 1 & y & \dots & y^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & y & \dots & y^{n-1} \end{bmatrix},$$

where

$$\operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q) = \pm\Bigl(\prod_{1 \leqslant i < j \leqslant m} (x_i - x_j)\Bigr)^n \Bigl(\prod_{1 \leqslant i < j \leqslant n} (y_i - y_j)\Bigr)^m.$$

It is easy to see, that (10) is an equivalent of (3) for matrices of the form (8). Thus, Proposition 1 can be interpreted as extension of the factorization property from a special case of rank-one matrices (8) to a more general class (2).

For the class of matrices (2), any submatrix that is contained in the upper triangle of $P(x)$ has rank 1. The same holds for any submatrix contained in the lower triangle of $Q(y)$. Therefore, the matrix $P \circ Q$ is the Hadamard product of a *lower semiseparable* matrix $Q(y)$ and an *upper semiseparable* matrix $P(x)$ (these matrices are also called Hessenberg-like matrices in [5, Ch. 8]). However, the relation to semiseparable matrices was not used in the proof of Proposition 1.

4. Application to Padua and Padua-like points

4.1. Main definitions

We define a class of points that distribute as the n-degree Padua points (1). For simplicity, we consider n even (the case n odd is similar). The n-degree *Padua-like points* $\mathcal{A}_n$ are defined as a union of two grids

$$\mathcal{A}_n := \mathcal{A}_n^o \cup \mathcal{A}_n^e,$$

where

$$\mathcal{A}_n^o := \left\{ (x_{2i+1}, y_{2j+1}) \;\middle|\; 0 \leqslant i \leqslant \frac{n}{2},\, 0 \leqslant j \leqslant \frac{n}{2} \right\}$$

and

$$\mathcal{A}_n^e := \left\{ (x_{2i}, y_{2j}) \;\middle|\; 1 \leqslant i \leqslant \frac{n}{2},\, 1 \leqslant j \leqslant \frac{n}{2} + 1 \right\},$$

and $\{x_i\}_{i=1}^{n+1}$, $\{y_j\}_{j=1}^{n+2}$ are distinct sets of points.

The Padua points (1) are a special case of Padua-like points, with

$$\mathcal{A}_n^o = C_{n+1}^{\mathrm{o}} \times C_{n+2}^{\mathrm{o}}, \qquad \mathcal{A}_n^e = C_{n+1}^{\mathrm{e}} \times C_{n+2}^{\mathrm{e}},$$

and $\{x_i\}_{i=1}^{n+1} = C_{n+1}$, $\{y_j\}_{j=1}^{n+2} = C_{n+2}$.

We are interested in expressing the Vandermonde determinant $\operatorname{vdm}(\mathcal{A}_n, \mathcal{B}_n)$, where

$$\mathcal{B}_n := \left\{ x^\alpha y^\beta \;\middle|\; \alpha + \beta \leqslant n;\, \alpha, \beta \geqslant 0 \right\}$$

is the set of all monomials of degree $\leqslant n$. We also define the square set of monomials

$$\mathcal{T}_n := \left\{ x^\alpha y^\beta \;\middle|\; 0 \leqslant \alpha,\, \beta \leqslant n \right\}.$$

4.2. Vandermonde determinant for Padua-like points

It is easy to show (see [3]) that

$$\operatorname{vdm}(\mathcal{A}_n, \mathcal{B}_n) = \pm \operatorname{vdm}\big(\mathcal{A}_n, \mathcal{T}_{\frac{n}{2}} \cup \mathcal{T}_{\frac{n}{2}}^e\big),$$

where

$$\mathcal{T}_{\frac{n}{2}}^e := \big(a(x)\mathcal{B}_{\frac{n}{2}-1}\big) \cup \big(b(y)\mathcal{B}_{\frac{n}{2}-1}\big),$$

and $a(x)$, $b(y)$ are the annihilating polynomials of the points $\mathcal{A}_n^o$, that is

$$a(x) = \prod_{i=0}^{\frac{n}{2}} (x - x_{2i+1}), \qquad b(y) = \prod_{j=0}^{\frac{n}{2}} (y - y_{2j+1}).$$

By construction, all the elements of $\mathcal{T}^e_{\frac{n}{2}}$ vanish on $\mathcal{A}^o_n$, which allows us to split the Vandermonde determinant into a product of two determinants:

$$\operatorname{vdm}(\mathcal{A}_n, \mathcal{B}_n) = \pm \det \begin{bmatrix} \operatorname{VDM}(\mathcal{A}^o_n, \mathcal{T}_{\frac{n}{2}}) & 0 \\ \operatorname{VDM}(\mathcal{A}^e_n, \mathcal{T}_{\frac{n}{2}}) & \operatorname{VDM}(\mathcal{A}^e_n, \mathcal{T}^e_{\frac{n}{2}}) \end{bmatrix}$$
$$= \pm \operatorname{vdm}\bigl(\mathcal{A}^o_n, \mathcal{T}_{\frac{n}{2}}\bigr) \times \operatorname{vdm}\bigl(\mathcal{A}^e_n, \mathcal{T}^e_{\frac{n}{2}}\bigr). \tag{11}$$

We consider the computation of the second factor in (11). For this, we define two polynomial matrices of size $(\frac{n}{2}) \times (\frac{n}{2}+1)$

$$Q(y) = \begin{bmatrix} 1 & b(y) & yb(y) & \dots & y^{\frac{n}{2}-1}b(y) \\ 1 & y & b(y) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & yb(y) \\ 1 & y & \dots & y^{\frac{n}{2}-1} & b(y) \end{bmatrix},$$
$$P(x) = \begin{bmatrix} a(x) & 1 & 1 & \dots & 1 \\ xa(x) & a(x) & x & \dots & x \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x^{\frac{n}{2}-1}a(x) & \dots & xa(x) & a(x) & x^{\frac{n}{2}-1} \end{bmatrix}. \tag{12}$$

It is easy to see that $\mathcal{T}^e_{\frac{n}{2}}$ consists of the entries of $P \circ Q$, and therefore

$$\operatorname{vdm}\bigl(\mathcal{A}^e_n, \mathcal{T}^e_{\frac{n}{2}}\bigr) = \operatorname{vdm}(\mathcal{X} \times \mathcal{Y}, P \circ Q),$$

for $\mathcal{X} = \{x_{2i} \mid 1 \leqslant i \leqslant \frac{n}{2}\}$ and $\mathcal{Y} = \{y_{2i} \mid 1 \leqslant i \leqslant \frac{n}{2}+1\}$. Since the matrices $P(x)$ and $Q(y)$ satisfy the condition (2), Proposition 1 can be applied.

Corollary 1.

$$\operatorname{vdm}\bigl(\mathcal{A}^e_n, \mathcal{T}^e_{\frac{n}{2}}\bigr) = \pm \left(\prod_{j=1}^{\frac{n}{2}+1} \operatorname{vdm}(\mathcal{X}, P_{:,j}) \right) \cdot \left(\prod_{i=1}^{\frac{n}{2}} \operatorname{vdm}(\mathcal{Y}, Q_{i,:}) \right). \tag{13}$$

4.3. Possible simplifications for inner determinants

Note that we still need to compute the inner determinants in (13), which have the form $\operatorname{vdm}(\mathcal{X}, P_{:,j})$, with $\mathcal{X} = \{x_1, \dots, x_m\}$, $1 \leqslant j \leqslant m+1$, and

$$P_{:,j}(x) := \begin{bmatrix} 1 & x & \dots & x^{j-2} & a(x) & xa(x) & \dots & x^{m-j}a(x) \end{bmatrix}^\top.$$

In the extreme cases

$$j = m+1: \quad P_{:,m+1}(x) := \begin{bmatrix} 1 & x & \dots & x^{m-1} \end{bmatrix}^\top,$$
$$j = 1: \quad P_{:,1}(x) := \begin{bmatrix} a(x) & xa(x) & \dots & x^{m-1}a(x) \end{bmatrix}^\top = a(x)P_{:,m+1}(x),$$

the determinant $\mathrm{vdm}(\mathcal{X}, P_{:,j}(x))$ can be computed explicitly, as it was done in [3, Lemma 3]. We will try to exploit (6) to simplify the expression. First, we note that the successive application of the operation (7) leads to

$$\frac{\frac{a(x)-a(x_1)}{x-x_1} - \frac{a(x_2)-a(x_1)}{x_2-x_1}}{x - x_2} = \frac{a(x) - \frac{x(a(x_2)-a(x_1))-x_1 a(x_2)+a(x_1)x_2}{x_2-x_1}}{(x-x_2)(x-x_1)}.$$

Therefore, the result of ℓ successive applications of (7) to a polynomial $a(x)$ is equal to

$$\widetilde{a}^{(\ell)}(x) = \frac{a(x) - p(x; x_1, \ldots, x_\ell)}{\prod_{i=1}^{\ell}(x - x_i)},$$

where $p(x; x_1, \ldots, x_\ell)$ is the $(\ell - 1)$-th degree interpolating polynomial of $a(x)$ at $\{x_1, \ldots, x_\ell\}$. Then the following proposition holds.

Proposition 2. *For $2 \leqslant j < (m-1)/2$ we have that*

$$\mathrm{vdm}(\mathcal{X}, P_{:,j}) = \pm \left(\prod_{i=1}^{j-1} \prod_{k=i+1}^{m} (x_i - x_k) \right) \mathrm{vdm}\big(\{x_j, \ldots, x_m\}, \mathcal{P}^{(j)}\big),$$

where

$$\mathcal{P}^{(j)} = \big\{\widetilde{a}^{(j-1)}(x), \ldots, \widetilde{a}^{(1)}(x), a(x), xa(x), \ldots, x^{m-2j+1}a(x)\big\}.$$

Proof. Consider a polynomial vector

$$A(x) = \begin{bmatrix} \widetilde{a}^{(p)}(x) & \ldots & \widetilde{a}^{(1)}(x) & 1 & \cdots & x^s & a(x) & xa(x) & \ldots & x^q a(x) \end{bmatrix} \in \mathbb{R}^{1 \times M}[x],$$

and apply the operation (7) to it (for the set of points $\mathcal{X} = \{x_1, \ldots, x_M\}$):

$$\widetilde{A}(x) = \begin{bmatrix} \widetilde{a}^{(p+1)}(x) & \ldots & \widetilde{a}^{(2)}(x) & \widetilde{x} & \cdots & \widetilde{x^s} & \widetilde{a(x)} & \widetilde{xa(x)} & \cdots & \widetilde{x^q a(x)} \end{bmatrix}.$$

Therefore,

$$\widetilde{x^k} = \frac{x^k - x_1^k}{x - x_1} = x^{k-1} + x_1 x^{k-2} + \cdots + x_1^{k-1},$$

and for $k \geqslant 1$,

$$\widetilde{x^k a(x)} = \frac{x^k a(x) - x_1^k a(x_1)}{x - x_1} = a(x)\frac{x^k - x_1^k}{x - x_1} + x_1^k \widetilde{a}(x).$$

Hence,

$$\widetilde{A}(x) = B(x) \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

where

$$B(x) = \begin{bmatrix} \widetilde{a}^{(p+1)}(x) & \ldots & \widetilde{a}^{(2)}(x) & \widetilde{a}^{(1)}(x) & 1 & \cdots & x^{s-1} & a(x) & \cdots & x^{q-1}a(x) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \operatorname{vdm}\bigl(\{x_1,\ldots,x_M\},A\bigr) &= \pm\left(\prod_{k=2}^{M}(x_1-x_k)\right)\operatorname{vdm}\bigl(\{x_2,\ldots,x_M\},\widetilde{A}\bigr) \\ &= \pm\left(\prod_{k=2}^{M}(x_1-x_k)\right)\operatorname{vdm}\bigl(\{x_2,\ldots,x_M\},B\bigr). \end{aligned}$$

The rest of the proof follows by applying recursively the same argument, since the polynomial vector B is of the same form as the polynomial vector A. □

We note that Proposition 2 can be also extended to handle the case $j \geqslant (m-1)/2$. We also note that Proposition 2 probably does not simplify much the expressions for the determinants of Padua-like points, since we still need to compute $\operatorname{vdm}(\{x_j,\ldots,x_m\},\mathcal{P}^{(j)})$.

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