On groups with all subgroups almost subnormal.

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Abstract

In this paper we consider groups in which every subgroup has finite index in the $n$th term of its normal closure series, for a fixed integer $n$. We prove that such a group is the extension of a finite normal subgroup by a nilpotent group, whose class is bounded in terms of $n$ only, provided it is either periodic or torsion-free.

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A subgroup $H$ of a group $G$ is said to be almost subnormal if it has finite index in some subnormal subgroup of $G$. This occurs when $H$ has finite index in some term $H^{G,n}$, $n \geq 0$, of its normal closure series in $G$; recall that $H^{G,0} = G$ and $H^{G,n} = H^{H^{G,n-1}}$.

A finite-by-nilpotent group has every subgroup almost subnormal, and for finitely generated groups the converse holds (see [8, 6.3.3]). Note that, if a group $G$ has a finite normal subgroup $N$ such that $G/N$ is nilpotent of class $n$, then each subgroup $H$ of $G$ has finite index in $H^{G,n}$. For $n = 1$ the converse is settled by a well-known Theorem by B. Neumann [10]: a group $G$, in which every subgroup $H$ has finite index in its normal closure $H^G$, is finite-by-abelian. Later, Lennox [7] considered the case in which $n$ is larger than 1 and there is also a bound on the indices. He proved that there exists a function $\mu$ such that if $|H^{G,n} : H| \leq c$ for every subgroup $H$ of a group $G$, where $n$ and $c$ are fixed integer, then the $\mu(n+c)$-th term $\gamma_{\mu(n+c)}(G)$ of the lower central series of $G$ is finite of order at
most e!. Recall that a Theorem by Roseblade states that a group $G$ in which $H = H^{G,n}$ for every subgroup $H$, is nilpotent and $\gamma_{\rho(n)+1}(G) = 1$, for a well-defined function $\rho$. Recently, Casolo and Mainardis in [2] and [3] gave a description of the structure of groups with all subgroups almost subnormal, proving, in particular, that such groups are finite-by-soluble.

In this paper we consider the class $A_n$, $n \geq 1$, of groups $G$ in which $|H^{G,n} : H|$ is finite for every subgroup $H$ of $G$, but no bound on the indices $|H^{G,n} : H|$ is assumed. In particular, we give a generalization of Neumann’s Theorem to periodic $A_n$-groups:

**Theorem 1.** There exists a function $\delta$ of $n$, such that if $G$ is a torsion group with the property that $|H^{G,n} : H| < \infty$ for every subgroup $H$ of $G$, then $\gamma_{\delta(n)}(G)$ is finite.

We then consider torsion-free groups. By a result due to Casolo and Mainardis [2], torsion-free $A_n$-groups have every subgroup subnormal and so they reveal to be nilpotent, by a recent result by Smith [14] (see also Casolo [1]). Here, we give a different proof of their nilpotency and, in particular, a bound on their nilpotency class, thus generalizing B. Neumann’s Theorem to torsion-free $A_n$-groups:

**Theorem 2.** There exists a function $\eta$ of $n$ such that each torsion-free group $G$ in which $|H^{G,n} : H| < \infty$ for every subgroup $H$, is nilpotent of class at most $\eta(n)$.

This also gives a different proof of Roseblade’s Theorem for torsion-free groups with all subgroups subnormal of bounded defect.

Finally, we observe that H. Smith in [13] gives examples of $A_2$-groups which are not finite-by-nilpotent. Thus, Theorems 1 and 2 are no longer true if we drop the assumptions that $G$ is either periodic or torsion-free. Also, Casolo and Mainardis, in [2], construct a non-hypercentral $A_2$-group. On the other hand, in Proposition 13 we shall prove that locally nilpotent $A_n$-groups are hypercentral, partially answering the question posed at the end of pag. 191 of [8]. To be mentioned that Heineken-Mohamed groups [6] are example of groups in which every subgroups is almost subnormal but they do not belong to any of the classes $A_n$. 
1 \( A_n^+ \)-Groups

To achieve our result on periodic \( A_n \)-groups, we find it convenient to study a largest class of groups. We denote by \( A_n^+ \) the class of all groups \( G \) in which there exists a finite subgroup \( F \) with the property that every subgroup \( H \) containing \( F \) has finite index in the \( n \)-th term \( H^{G,n} \) of its normal closure series. By abuse of notation, we shall denote the above by \((G,F) \in A_n^+ \). Note that \( A_n \subseteq A_n^+ \) but \( A_n \neq A_n^+ \). Indeed, the group described in [4, Proposition 4] is a periodic \( A_2^+ \)-group but it is not finite-by-nilpotent, and so, by Theorem 1, it does not belong to \( A_n \).

Also, we denote by \( \mathcal{U}_n^+ \) the class of all groups \( G \) in which there exists a finite subgroup \( F \) such that every subgroup of \( G \) containing \( F \) is subnormal of defect at most \( n \) in \( G \). Clearly, \( \mathcal{U}_n^+ \subseteq A_n^+ \), but \( \mathcal{U}_n^+ \neq A_n^+ \), since Smith’s groups [13] are locally nilpotent \( A_2 \)-groups which are not finite-by-nilpotent while, for \( \mathcal{U}_n^+ \)-groups, the following holds:

**Theorem 3.** (Detomi [4]) There exists a function \( \beta(n) \) of \( n \), such that if \( G \) belongs to \( \mathcal{U}_n^+ \) and it is either a locally nilpotent group or a torsion group with \( \pi(G) \) finite, then \( \gamma_{\beta(n)}(G) \) is finite. In particular, if \( G \) is locally nilpotent, then \( G \) is nilpotent and its nilpotency class is bounded by a function depending on \( n \) and \(|F|\).

Here \( \pi(G) \) denotes the set of primes dividing the orders of the elements of \( G \).

The following are two known result which we include without proofs. If \( N \) is a subgroup (normal subgroup) with finite index in \( G \), then we write \( N \leq_f G \) \((N \leq_f G)\).

**Lemma 4.** Let \( G \) be a countable residually finite group and let \( H \) be a finite subgroup of \( G \). Then \( H = \bigcap_{N \leq_f G} HN \).

**Lemma 5.** Let \( G \) be a group and let \( F \) be a finitely generated subgroup of a subgroup \( H \) of \( G \). If \( |G_n V| \leq V \) for every finitely generated subgroup \( V \) of \( H \) such that \( F \leq V \), then \( |G_n H| \leq H \).

We set up an elementary property of periodic \( A_n^+ \)-groups:

**Lemma 6.** A periodic \( A_n^+ \)-group is locally finite and finite-by-soluble.
Proof. Let \((G, F) \in A_n^+\). Then \(F \leq F^{G,n}\) gives that \(F^{G,n}\) is finite and that every section \(F^{G,1}/F^{G,1+1}\) belongs to \(A_n\). Since, by the already mentioned result by Casolo-Mainardis, every \(A_n\)-group is finite-by-soluble, the group \(G\) has a finite series in which each factor is finite or soluble.

Let \(X\) be a finitely generated subgroup of \(G\). Clearly \(X\) has a finite series with finite or soluble factors. Hence, since a finitely generated torsion soluble group is finite and a subgroup with finite index in a finitely generated group is finitely generated, each factor in this series of \(X\) is finite, and so \(X\) is finite. This proves that \(G\) is locally finite.

Now, since \(G\) has a finite series with finite or soluble factors, to prove that \(G\) is finite-by-soluble it is sufficient to show that soluble-by-finite periodic \(A_n^+\)-groups are finite-by-soluble.

Let \((G, F) \in A_n^+\) be a torsion group and let \(A\) be a soluble normal subgroup with finite index in \(G\). We can assume that \(A \leq G\), since \(A_G\) has finite index in \(G\). Let \(\tau\) be a left transversal to \(A\) in \(G\) and set \(H = \langle \tau, F \rangle\). As \(H\) has finite index in \(K = H^{G,n}\), \(K\) is finitely generated and hence finite, by the local finiteness of \(G\). Note that \(G = AK\).

We proceed by induction on the defect \(d\) of subnormality of \(K\) in \(G\). If \(K\) is normal in \(G\), then \(G/K \cong A/A \cap K\) is soluble, and we are done. If \(d > 1\), then, as \(K\) has defect of subnormality bounded by \(d - 1\) in \(K^G\), we can apply the induction hypothesis to \(K^G\), obtaining that some terms of the derived series of \(K^G\) is finite (and normal in \(G\)). Therefore, as \(G/K^G \cong A/A \cap K^G\) is soluble, we get that \(G\) is finite-by-soluble, which is the desired conclusion.

With the same argument as in Lemma 9 of [4], it is easy to see that:

**Lemma 7.** Let \(G \in A_n^+\) be a locally finite group. If there exists a subgroup \(A\) with finite index in \(G\) such that \(\gamma_{m+1}(A)\) is finite, then \(\gamma_{nm+1}(G)\) is finite.

Roughly speaking, the next proposition says that periodic \(A_n^+\)-groups are near to being \(\Omega_n^+\)-groups.

**Proposition 8.** Let \(G\) be a countable residually finite torsion group and let \(G \in A_n^+\). Then there exists a subgroup \(A\) with finite index in \(G\) such that \(A \in \Omega_n^+\).
Proof. Assume that the lemma is false and let $G$ be a counterexample. Proceeding recursively we construct

(a) a descending chain $\{K_i \mid i \in \mathbb{N}\}$ of subgroups with finite index in $G$,

(b) an ascending chain $\{F_i \mid i \in \mathbb{N}\}$ of finitely generated subgroups of $\bigcap_{i=0}^{\infty} K_i$, and

(c) a sequence of elements $\{x_i \in [K_{i-1,n} F_i] \setminus K_i \mid 1 \leq i \in \mathbb{N}\}$.

Set $K_0 = G$ and let $F_0$ be a finite subgroup of $G$ such that $|H^{G,n} : H| < \infty$ whenever $F_0 \leq H \leq G$.

Suppose we have already defined $F_i$, $K_i$, and $x_i \in [K_{i-1,n} F_i] \setminus K_i$. As $F_i$ is a finitely generated subgroup of $K_i \leq_f G$, and as $G$ is a counterexample, there exists a subgroup $F_i \leq H \leq K_i$ which is not subnormal of defect less or equal to $n$ in $K_i$, that is $[K_i, n H] \not\leq H$. So, by Lemma 5, there exists a finitely generated subgroup $F_{i+1}$ of $H$ with $F_i \leq F_{i+1}$ and $[K_i, n F_{i+1}] \not\leq F_{i+1}$. Let us fix an element $x_{i+1} \in [K_i, n F_{i+1}] \setminus F_{i+1}$. Since, by Lemma 6, $G$ is locally finite, we can apply Lemma 4 to the finitely generated, hence finite subgroup $F_{i+1}$, and so we get that $x_{i+1} \not\in F_{i+1} N$ for a suitable subgroup $N \leq_f K_i$. Then we set $K_{i+1} = F_{i+1} N$, so that $F_{i+1} \leq K_{i+1} \leq_f G$ and $x_{i+1} \in [K_i, n F_{i+1}] \setminus K_{i+1}$. Note that $K_{i+1}$ contains all the subgroups $F_0, \ldots, F_{i+1}$.

Now we consider the subgroups

$$K = \bigcap_{i \in \mathbb{N}} K_i \quad \text{and} \quad H = \langle F_i \mid i \in \mathbb{N} \rangle.$$ 

Since $H \geq F_0$, by assumption we have that $H$ has finite index in $H^{G,n}$. So, the chain $\{H^{G,n} \cap K_i\}_{i \in \mathbb{N}}$, stretching from $H^{G,n}$ to $H$, is finite and there exists an integer $i$ such that $H^{G,n} \cap K_i = H^{G,n} \cap K_j$ for every $j \geq i$. But, since $[G, n H] \leq H^{G,n}$ and $F_{i+1} \leq H \cap K_i$, we get that

$$x_{i+1} \in [K_{i+n} F_{i+1}] \leq [K_{i+n} H \cap K_i] \leq [G, n H] \cap K_i \leq H^{G,n} \cap K_i = H^{G,n} \cap K_{i+1},$$

which proves the lemma.
that is \( x_{i+1} \in K_{i+1} \), in contradiction to our construction.

**Theorem 9.** There exists a function \( \delta(n) \) of \( n \), such that if \( G \) is a periodic \( A_n^+ \)-group and if either \( G \) is locally nilpotent or \( \pi(G) \) is finite, then \( \gamma_{\delta(n)}(G) \) is finite. In particular, if \( G \) is locally nilpotent then \( G \) is nilpotent.

**Proof.** Set \( \delta(1) = 2 \) and define recursively \( \delta(n) = 2\beta(n) - 1 + 2\delta(n - 1) + 1 \), where \( \beta \) is the function defined in Theorem 3.

Assume first that \( G \) is countable. We shall proceed by induction on \( n \). Let \( F \) be a finite subgroup of \( G \) such that every subgroup \( H \) containing \( F \) has finite index in \( H^G \).

If \( n = 1 \) then \( [F^G : F] < \infty \) and \( F^G \) is finite. Since \( G/F^G \in A_1 \), the quotient \( G'F^G/F^G \) is finite by B.H. Neumann’s Theorem. Hence \( G' = \gamma_2(G) \) is finite.

Let now \( n > 1 \) and let \( X \) be a finitely generated subgroup of \( G \) with \( X \geq F \). Because \( G \) is locally finite, \( X \) is finite. Observe that, for every subgroup \( H \) of \( X^G \) containing \( X \), we have \( H^G = X^G \) and so \( [H^{X^G} : H] < \infty \). Thus \( X^G \) belongs to \( A_{n-1}^+ \) and by the induction hypothesis we get that \( \gamma_{\delta(n-1)}(X^G) \) is finite. Now, by a Theorem of P. Hall it follows that \( \zeta_{2\delta(n-1)-2}(X^G) \) has finite index in \( X^G \). Thus, the index of \( C_G(X^G/\zeta_{2\delta(n-1)-2}(X^G)) \) in \( G \) is finite and, denoting by \( R = \bigcap_{N \in G} N \) the finite residual of \( G \), we obtain that \( [R, X^G] \leq \zeta_{2\delta(n-1)-2}(X^G) \). In particular

\[
[R, X^G] \leq [R, X^G^{\zeta_{2\delta(n-1)-2}}] = 1.
\]

Therefore, if we take \( s = 2\delta(n-1) \) elements in \( G \), say \( x_1, \ldots, x_s \), and we consider the finitely generated subgroup \( X = \langle x_1, \ldots, x_s, F \rangle \), then we get \( [R, x_1, \ldots, x_s] \leq [R, x_s X^G] = 1 \), which implies \( R \leq \zeta_s(G) \).

Now, as \( G/R \in A_n^+ \) is a countable residually finite torsion group, by Proposition 8 it follows that there exists a subgroup \( A \) with finite index in \( G \), such that \( A/R \in \mathfrak{U}_n^+ \). Also, \( A/R \) satisfies the assumptions of Theorem 3 and so \( \gamma_{\delta(n)}(A/R) \) is finite. By Lemma 7 it follows that \( \gamma_{n(\beta(n)-1)+1}(G/R) \) is finite and then Hall’s Theorem gives that \( \zeta_{2n(\beta(n)-1)}(G/R) \) has finite index in \( G/R \). Therefore, as \( R \leq \zeta_s(G) \), clearly \( \zeta_{2n(\beta(n)-1)+s}(G) \) has finite index in \( G \) and, by a Theorem of Baer (see [12, 14.5.1]), we conclude that
\[ \gamma_{2n(n-1)+s+1}(G) \] is finite. This proves that \( \gamma_{\delta(n)}(G) \) is finite, for every countable group \( G \) satisfying the assumption of the Theorem.

For the general case, we assume, contrary to our claim, that there exists a group \( G \), satisfying the assumption of the Theorem, such that \( \gamma_{\delta(n)}(G) \) is not finite. Let \( T \) be a countable and not finite subset of \( \gamma_{\delta(n)}(G) \). Then we can find a countable set of commutators \( x_i = [y_{1,i}, \ldots, y_{\delta(n),i}], \ i \in \mathbb{N}, \ y_{j,i} \in G, \) such that \( T \leq \langle x_i \mid i \in \mathbb{N} \rangle \).

Let \( Y = \langle F, y_{j,i} \mid j = 1, \ldots, \delta(n) \ i \in \mathbb{N} \rangle \). As \( Y \) is a countable \( A_n^+ \)-group, by the first part of the proof, \( \gamma_{\delta(n)}(Y) \) is finite. Thus \( T \leq \gamma_{\delta(n)}(Y) \) is finite, against our assumption.

Finally, if \( G \) is locally nilpotent, since every finite normal subgroup is contained in some term of the upper central series (by a theorem of Mal’cev and McLain [12, 12.1.6]), it follows that \( G \) is nilpotent, and the proof is complete.

As a consequence, we get the announced result on periodic \( A_n \)-groups:

**Proof of Theorem 1.** Let \( G \) be a periodic \( A_n \)-group. By a result of Casolo and Mainardis [3], there exists a finite normal subgroup \( N \) of \( G \) such that \( G/N \) has every subgroup subnormal. In particular, \( G/N \) is locally nilpotent. Now Theorem 9 gives that \( \gamma_{\delta(n)}(G/N) \) is finite and, as \( N \) is finite, the result follows.

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2 **Torsion-free \( A_n \)-groups**

First we observe some basic properties of isolators in locally nilpotent groups. Recall that the **isolator** of a subgroup \( H \) in a group \( G \) is defined to be the set \( I_G(H) = \{ x \in G \mid x^n \in H \text{ for some } 1 \leq n \in \mathbb{N} \} \). If \( G \) is a locally nilpotent group then \( I_G(H) \) is a subgroup of \( G \) and if \( G \) is also torsion-free then \( \gamma_n(I_G(H)) \leq I_G(\gamma_n(H)) \) (see for example [5] and [9]).

**Lemma 10.** Let \( G \) be a locally nilpotent group and let \( H \leq G \). Then:

1. if \( I_G(H) \) is finitely generated, then \( |I_G(H) : H| < \infty \);

2. if \( G \) is torsion-free and \( H \) is cyclic, then \( I_G(H) \) is locally cyclic.
Proof.

1. As $K = I_G(H)$ is a finitely generated nilpotent group, $H$ is subnormal in $K$, say $H = H^{K,n}$ for an integer $n$, and every section $H^{K,i}/H^{K,i+1}$ is finitely generated and nilpotent, for $i = 1, \ldots, n-1$. Furthermore, by definition of $I_G(H)$, each $H^{K,i}/H^{K,i+1}$ is periodic and hence finite. Thus, $H$ has finite index in $K$.

2. Let $K$ be a finitely generated subgroup of $I_G(H)$. As $H$ is cyclic, we can assume that $H \leq K$. Since $K$ is torsion-free and nilpotent, it has a central series with infinite cyclic factors (see [12, 5.2.20]). So, if $K$ is not cyclic, there is a cyclic normal subgroup $N$ of $K$ with infinite index in $K$. Now, since, by point 1, $H$ has finite index in $K$, then $H \cap N \neq 1$. Therefore, as $H$ is cyclic, $|K/N| \leq |NH/N| = |H/H \cap N|$ is finite, a contradiction. □

We state now a consequence of a well-known argument by D.J.S. Robinson (see [12, 5.2.5]). Recall that the Hirsch length of a polycyclic group $G$ is the number of infinite factors in a series of $G$ with cyclic factors.

Lemma 11. Let $H$ be a nilpotent group of class $c$. If $H/H'$ can be generated by $r$ elements, then the Hirsch length $h$ of $H$ is bounded by a function $g(c,r)$ of $c$ and $r$.

An already mentioned Theorem of Mal’cev and McLain [12, 12.1.6] states that each principal factor of a locally nilpotent group is central. The following consequence is well known, but we include the easy proof for the convenience of the reader:

Lemma 12. Let $G$ be a locally nilpotent group and let $N$ be a finitely generated normal subgroup of $G$. Then there exists an integer $n$ such that $N \leq \zeta_n(G)$. Moreover, if $N$ is torsion-free with Hirsch length $h$, then $N \leq \zeta_h(G)$.

Proof. The Theorem of Mal’cev and McLain implies that if $N$ is finite then it is contained in $\zeta_m(G)$ for an integer $m$ bounded by the composition length of $N$. Also, when $N$ is torsion-free with Hirsch length $h$, we get that $N/N^p$ is finite and so $N/N^p \leq \zeta_h(G/N^p)$ for every prime $p$; therefore $[N, N^p G] \leq \bigcap_p N^p = 1$ by a residual property of torsion-free finitely
generated nilpotent groups (see for example [11, pag 170]). Since the torsion subgroup of a finitely generated normal subgroup of $G$ is finite, the lemma follows.

**Proposition 13.** Let $G$ be a locally nilpotent $A_n$-group. Then $G$ is hypercentral.

**Proof.** By an already cited result of Casolo and Mainardis, $A_n$-groups are finite-by-soluble and so $G$ is soluble. It is sufficient to prove that $G$ has a non trivial centre. We proceed by induction on the derived length of $G$. Let $A$ be the centre of $G^0$; by inductive assumption, $A \neq 1$. Let $H$ be a finitely generated subgroup of $G$. As $|H^{G,n} : H|$ is finite, $H^{G,n}$ is finitely generated and so nilpotent; in particular $[A, n H]$ is finitely generated. Since $A = \zeta(G')$, $[A, n H]^g = [A, n H^g] \leq \langle A, n [H, (g)] \rangle = [A, n H]$ for $g \in G$, and so $[A, n H]$ is normal in $G$. Thus Proposition 12 gives that $[A, n H] \leq \zeta_k(G)$ for some $k \geq 1$. So, if $[A, n H] \neq 1$, then $\zeta(G) \neq 1$. Otherwise, $[A, n H] = 1$ for any finitely generated subgroup of $G$; thus $A \leq \zeta_n(G)$ and we again conclude that $\zeta(G) \neq 1$.

A group $G$ is said $n$-Engel if $[x, y] = 1$ for all $x, y \in G$. We recall that a torsion-free soluble $n$-Engel group $G$ with positive derived length $d$ is nilpotent of class at most $n^{d-1}$ (see [11, 7.36]).

Our interest on Engel groups is motivated by the following fact:

**Lemma 14.** A torsion-free $A_n$-group is $(n + 1)$-Engel.

**Proof.** Let $G$ be a torsion-free $A_n$-group and let $1 \neq x \in G$. By the definition of the class $A_n$, $\langle x \rangle$ has finite index in $\langle x \rangle^{G,n}$, so that $\langle x \rangle^{G,n}$ is a finitely generated subgroup of $I_G(\langle x \rangle)$. By the already mentioned result in [2], every subgroup of $G$ is subnormal, so that $G$ is locally nilpotent. Thus, by Lemma 10, $\langle x \rangle^{G,n}$ is cyclic, so that $\langle x \rangle \text{char} \langle x \rangle^{G,n}$, and hence $\langle x \rangle$ is subnormal of defect at most $n$ in $G$, that is $[G, n x] \leq \langle x \rangle$. Therefore, $[G_{n+1} x] = [G, n x, x] = 1$, as claimed.

Now we are in a condition to prove the announced result on torsion-free $A_n$-groups.
Proof of Theorem 2. Let $G \in A_n$ be a torsion-free group. As already noted, by a result in [2], $G$ is locally nilpotent.

Note that, if there exists a function $\eta(n)$ such that $\gamma_{\eta(n)+1}(H) = 1$, for every finitely generated subgroup $H$ of $G$, then $\gamma_{\eta(n)+1}(G) = 1$. Hence, without loss of generality, we can assume that $G$ is a finitely generated group. In particular, we get that $G$ is nilpotent and every subgroup of $G$ is finitely generated.

Proceeding by induction on $n$, we prove that there exists a function $\eta(n)$ such that every torsion-free finitely generated $A_n$-group has nilpotency class at most $\eta(n)$.

If $n = 1$, then B. Neumann’s Theorem gives that $G'$ is finite. Hence, since $G$ is torsion-free, $G$ is abelian, and so we can set $\eta(1) = 1$.

Let now $n > 1$ and let $H$ be a subgroup of $G$. Set $H^{G,i} = H_i$ for every $i$, so that, by the definition of the class $A_n$, we have

$$H \leq_f H_n \leq H_{n-1} \leq \ldots \leq H_2 \leq H_1 \leq G.$$  

Note that, for every subgroup $K$ such that $H \leq K \leq H_1$, we get $K^G = H^G = H_1$ and $K \leq_f K^{G,n-1}$. Hence $H_1/H_2 \in A_{n-1}$. With the same argument it is easy to see that the factor $H_i/H_{i+1}$, for $i = 1, \ldots, n-1$, belongs to $A_{n-i}$. By the induction hypothesis, the factor $H_i/I_{H_i}(H_{i+1})$, being a finitely generated torsion-free $A_{n-i}$-group, has nilpotency class at most $\eta(n-i)$; hence,

$$\gamma_{\eta(n-i)+1}(H_i) \leq I_{H_i}(H_{i+1}) \leq I_G(H_{i+1}).$$

Thus,

$$\gamma_{\eta(n-i)+1}(I_G(H_i)) \leq I_G(\gamma_{\eta(n-i)+1}(H_i)) \leq I_G(I_G(H_{i+1})) = I_G(H_{i+1}),$$

for every $i$, so that

$$\gamma_{\eta(n-1)+1}(I_G(H_1)) \leq I_G(H_2),$$

$$\gamma_{\eta(n-2)+1} \left( \gamma_{\eta(n-1)+1}(I_G(H_1)) \right) \leq \gamma_{\eta(n-2)+1}(I_G(H_2)) \leq I_G(H_3),$$

$$\ldots$$

$$\gamma_{\eta(1)+1} \left( \gamma_{\eta(2)+1} \left( \ldots \left( \gamma_{\eta(n-1)+1}(I_G(H_1)) \right) \ldots \right) \right) \leq \gamma_{\eta(1)+1}(I_G(H_{n-1}))$$

$$\leq I_G(H_n) = I_G(H),$$
where the last equality is due to the fact that $H \leq f H_n \leq I_G(H)$.

In particular, for $k = k(n) = \sum_{i=1}^{n-1} (\eta(i) + 1)$, the $k$-th term $H^{(k)}_n$ of the derived series of $H_1$ is a subgroup of $I_G(H)$, so that $I_G(H^{(k)}_1) \leq I_G(H)$. Now, by Lemma 14, $H_1/I_G(H^{(k)}_1)$ is a soluble torsion-free $(n+1)$-Engel group and so $H_1/I_G(H^{(k)}_1)$ is nilpotent of class at most $(n + 1)^{k-1}$. Thus, for $c = c(n) = (n + 1)^{k-1} + 1$, we get that $\gamma_c(H_1) \leq I_G(H^{(k)}_1) \leq I_G(H)$. This proves that

$$
\gamma_c(H^G) \leq I_G(H),
$$

for every subgroup $H$ of $G$.

Now take $c$ elements of $G$, say $x_1, \ldots, x_c$, and consider the subgroup $H = \langle x_1, \ldots, x_c \rangle$. Clearly we can write $H_1 = H^G$ as a product of the $c$ normal subgroups $\langle x_i \rangle^G$. Since $\gamma_c(\langle x_i \rangle^G) \leq I_G(\langle x_i \rangle)$ and, by Lemma 10, $I_G(\langle x_i \rangle)$ is a cyclic group, then $[\gamma_c(\langle x_i \rangle^G), x_i] = 1$. Moreover $[\gamma_c(\langle x_i \rangle^G), x_i^g] = 1$ for every $g \in G$. Thus $\gamma_c(\langle x_i \rangle^G) \leq \zeta(\langle x_i \rangle^G)$ and $\langle x_i \rangle^G$ has nilpotency class at most $c$. Therefore $H_1$ is generated by $c$ normal nilpotent subgroups of class at most $c$, and by Fitting’s theorem it follows that $H_1$ is nilpotent with class $cl(H_1) \leq c^2$.

Now, since $H$ is a $c$-generated torsion-free nilpotent group of class $cl(H) \leq cl(H_1) \leq c^2$, Lemma 11 implies that the Hirsch length $h(H)$ of $H$ is bounded by $g_1 = g(c^2, c) = \frac{c^2+1}{c-1}$. Also, by Lemma 10, $|I_G(H) : H| < \infty$, so that $h(I_G(H)) = h(H) \leq g_1$.

Therefore $\gamma_c(H_1)$ is a finitely generated normal subgroup of $G$ with Hirsch length $h(\gamma_c(H_1)) \leq h(I_G(H)) \leq g_1$ and so, by Proposition 12,

$$
\gamma_c(H_1) \leq \zeta_{g_1}(G).
$$

In particular

$$
[x_1, \ldots, x_c, y_1, \ldots, y_{g_1}] = 1,
$$

for every $y_1, \ldots, y_{g_1}$ in $G$, so that

$$
\gamma_{c+g_1}(G) = 1.
$$

Finally, since $c = c(n)$ and $g_1 = g_1(n)$ depend only on $n$, the result follows on defining $\eta(n) = c + g_1 - 1$. ■
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