Recall that a nilpotent self-normalizing subgroup is called a *Carter subgroup*. In 1961 R. Carter proved that in each finite solvable group Carter subgroups exist and are conjugate. Moreover, a homomorphic image of a Carter subgroup (of a solvable group) under any homomorphism is a Carter subgroup.

Note that the intersection of a Carter subgroup $K$ of $G$ with a normal subgroup $H$ of $G$ might not be a Carter subgroup of $H$. Consider $G = \mathrm{Sym}_3$ and its normal subgroup $H = \mathrm{Alt}_3$. It is clear that $\mathrm{Sym}_3$ is solvable and its Carter subgroup $K$ is a Sylow 2-subgroup. On the other hand, the group $\mathrm{Alt}_3$ is cyclic, hence it coincides with its Carter subgroup and it is a 3-group.

If a group is not assumed to be finite, then Carter subgroups may be even non-isomorphic. A free product of two non-isomorphic nilpotent groups $K_1$ and $K_2$ gives such example. Further, arbitrary finite group may not contain Carter subgroups. The minimal example is $\mathrm{Alt}_5$. However, there is still not known any example of finite groups containing non-conjugate Carter subgroups. Thus the following problem (which we refer as a conjugacy problem) appears.

*Problem.* Are Carter subgroups of a finite group conjugate?

During last 30 years, many authors investigated Carter subgroups in some classes of finite groups close to be simple. In particular, the structure of Carter subgroups was obtained in the following groups:

symmetric and alternating (L. Di Martino, M.C. Tamburini, 1976);

in each $G$ with $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$ (N.A. Vavilov, 1979; L. Di Martino, M.C. Tamburini, 1987);

in $\mathrm{Sp}_{2n}(q)$, $\mathrm{GU}_n(q)$, and $\mathrm{GO}_n^{\pm}(q)$, in the last case $q$ is odd, (L. Di Martino, A.E. Zalesski, M.C. Tamburini, 1997). Later Carter subgroups were classified in each $G$ with $O^{p'}(S) \leq G \leq S$, where $S \in \{\mathrm{Sp}_{2n}(q), \mathrm{GU}_n(q), \mathrm{GO}_n^{\pm}(q)\}$ (A. Previtali, M.C. Tamburini, E.P. Vdovin, 2004).

In 1998 F. Dalla Volta, A. Lucchini, and M.C. Tamburini proved that a counter example of minimal order to the conjugacy problem should be an almost simple group. This result allows to solve the conjugacy problem by using the classification of finite simple groups.

It is easy to see that if $G$ is a counter example of minimal order and $K$ is a Carter subgroup of $G$, then none elements $z_1, z_2 \in Z(K)$ can be conjugate in $G$. In particular, none element $z \in Z(K)$ of prime order can be conjugate to its nontrivial power in $G$

Indeed, assume that $z, z^g \in Z(K)$ for some $g \in G$. Consider $\langle K, K^g\rangle$. It has a nontrivial center (since $z^g \in Z(K) \cap Z(K^g)$), so , $|\langle K, K^g\rangle| < |G|$, i. e., Carter subgroups of $\langle K, K^g\rangle$ are conjugate. Hence there exists $x \in \langle K, K^g\rangle$ such that $K^x = K^g$, in particular $z^x = z^g$. But $z^g \in Z(K) \cap Z(K^g) \leq Z(\langle K, K^g\rangle)$, i. e., $z^x = z$.

By using this simple fact and considering conjugate elements of prime order in finite groups, M.C. Tamburini and E.P. Vdovin in 2002 have shown that a wide class of almost simple groups does not contain a counter example of minimal order. In particular, this class contains all simple groups.

| $\mathrm{Soc}(A) = G$ | Conditions for $A$ |
|---|---|
| alternating, sporadic; $A_1(r^t),\ B_\ell(r^t),\ C_\ell(r^t),\ t$ even if $r = 3$; ${}^2B_2(2^{2n+1}),\ G_2(r^t),\ F_4(r^t),\ {}^2F_4(2^{2n+1})$; $E_7(r^t),\ r \neq 3;\ E_8(r^t),\ r \neq 3, 5$ | none |
| $D_{2\ell}(r^t),\ {}^3D_4(r^{3t}),\ {}^2D_{2\ell}(r^{2t})$, $t$ even if $r = 3$ and, if $G = D_4(r^t)$, $\lvert(\mathrm{Field}(G) \cap A) : (\widehat{G} \cap A)\rvert_{2'} > 1$ | $A/(A \cap \widehat{G})$ 2 − group or $\lvert\widehat{G} : (A \cap \widehat{G})\rvert = 2^k,\ k \geqslant 0$ |
| $B_\ell(3^t),\ C_\ell(3^t),\ D_{2\ell}(3^t),\ {}^3D_4(3^{3t}),\ {}^2D_{2\ell}(3^{2t})$, $D_{2\ell+1}(r^t),\ {}^2D_{2\ell+1}(r^{2t}),\ {}^2G_2(3^{2n+1})$, $E_6(r^t),\ {}^2E_6(r^{2t}),\ E_7(3^t),\ E_8(3^t),\ E_8(5^t)$ | $A = G$ |
| $A_\ell(r^t),\ {}^2A_\ell(r^{2t}),\ \ell > 1$ | $G \leq A \leq \widehat{G}$, |

If $G$ is a group, $A, B, H$ are subgroups of $G$ and $B$ is normal in $A$ ($B \trianglelefteq A$), then $N_H(A/B) := N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then $x$ induces an automorphism $Ba \mapsto Bx^{-1}ax$ of $A/B$. Thus there exists a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. The image of this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$. In particular, if $S$ is a composition factor of $G$, then for each $H \leq G$ the group $\mathrm{Aut}_H(S)$ is defined. A finite group $G$ said to satisfy **(C)** if for each its (non-Abelian) composition factor $S$ and each nilpotent subgroup $N$ of $\mathrm{Aut}_G(S)$, Carter subgroups of $\langle N, S\rangle$ are conjugate. Then the following theorem is true.

**Theorem 1.** (E.P. Vdovin, 2006) *If a finite group $G$ satisfies* **(C)***, then its Carter subgroups are conjugate.*

**Lemma 2.** *Let $G$ be a finite group, $K$ a Carter subgroup of $G$ with the center $Z(K)$. Assume also that $e \neq z \in (K)$ and $C_G(z)$ satisfies* **(C)**.
(1) *Each subgroup $Y$ containing $K$ and satisfying* **(C)** *is self-normalizing in $G$.*
(2) *None conjugate of $z$ in $G$, except $z$, is not in $Z(G)$.*
(3) *If $H$ is a Carter subgroup of $G$, which is non-conjugate with $K$, then $z$ is not conjugate to any element from the center of $H$.*
*In particular, the centralizer $C_G(z)$ is self-normalizing in $G$, and $z$ is not conjugate to a nontrivial power $z^k \neq z$.*

**Lemma 3.** (A criterion of existence of a Carter subgroup containing a Sylow 2-subgroup) *Let $G$ be a finite group and $S$ a Sylow 2-subgroup of $G$.*

*Then $G$ contains a Carter subgroup $K \geq S$ if and only if $N_G(S) = SC_G(S)$.*

In view of Lemma 3, we shall say that a finite group $G$ satisfies **(ESyl2)** if for a Sylow 2-subgroup $S$ of $G$ the equality $N_G(S) = SC_G(S)$ holds.

**Lemma 4.** (Descending Lemma) *Let $G$ be a finite group and $H$ a Carter subgroup of $G$. Assume that there exists a normal subgroup $B = T_1 \times \ldots \times T_k$ of $G$ such that $T_1 \simeq \ldots \simeq T_k \simeq T$, $Z(T_i) = \{1\}$ for all $i$, and $G = H(T_1 \times \ldots \times T_k)$.*

*Then $\mathrm{Aut}_H(T_i)$ is a Carter subgroup of $\langle \mathrm{Aut}_H(T_i), T_i \rangle$.*

**Lemma 5.** (First ascending lemma) *Let $G$ be a finite group, $S$ a Sylow 2-subgroup of $G$ and $x \in N_G(S)$ be of odd order. Assume that there exist normal subgroups $G_1, \ldots, G_k$ of $G$ such that $G_1 \cap \ldots \cap G_k \cap S \leq Z(N_G(S))$. If $\varphi_i : G \to G/G_i$ is a natural homomorphism, assume also that $x^{\varphi_i}$ centralizes $SG_i/G_i$. Then $x$ centralizes $S$.*

**Lemma 6.** (Second ascending lemma) *Let $G$ be a finite group and $H$ be a normal subgroup of $G$ such that $|G : H| = 2^t$. Let $S, T$ be Sylow 2-subgroups of $G, H$ respectively and $N_H(T) = TC_H(T)$.*

*Then $N_G(S) = SC_G(S)$. In particular, the groups $G, H$ contain Carter subgroups $K, L$ with $S \leq K$ and $T \leq L$.*

**Theorem 7.** *Let $G$ be either $D_4(q)$, or ${}^3D_4(q^3)$. Assume that $\tau$ is a graph automorphism of $G$ of order 3 (in case of ${}^3D_4(q^3)$ this is an automorphism, which has the set of stable points isomorphic to $G_2(q)$). Denote by $A_1$ the subgroup of $\mathrm{Aut}(D_4(q))$ generated by inner-diagonal and field automorphisms, and also by a graph automorphism of order 2. Let $A \leq \mathrm{Aut}(G)$ be such that $A \not\leq A_1$ (if $G \simeq D_4(q)$), and let $K$ be a Carter subgroup of $A$.*

*If $G \simeq {}^3D_4(q^3)$ and $(|A : G|, 3) = 1$, then $q$ is odd and $K$ contains a Sylow 2-subgroup of $A$. Otherwise, up to conjugation in $G$, we have that $\tau \in K$, if $q$ is odd, then $K \cap A_1$ contains a Sylow 2-subgroup of $C_A(\tau) \simeq \Gamma G_2(q)$, if $q = 2^t$ is even, then $K \cap A_1$ contains a Sylow 2-subgroup of $G_2(2^{t_{2'}})$. Note that if $q$ is odd, then in $\Gamma G_2(q)$ Carter subgroups always exist, while for $q = 2^t$ even Carter subgroups exist if and only if $|\Gamma G_2(q) : G_2(q)| = t$.*

*In particular, Carter subgroups of $A$ are conjugate.*

**Theorem 8.** *Let $G$ be a finite group of Lie type ($G$ is not necessary simple) over a field of characteristic $p$ and $\overline{G}$, $\sigma$ are chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Assume also that $G \not\simeq {}^3D_4(q^3)$. Choose a subgroup $A$ of $\mathrm{Aut}(O^{p'}(\overline{G}_\sigma))$ with $A \cap \overline{G}_\sigma = G$ and assume that $A$ is contained in $A_1$, defined in Theorem 7, if $G = D_4(q)$. Let $K$ be a Carter subgroup of $A$ and assume that $A = KG$.*

*Then one of the following statements hold.*

1. $A = \Gamma G$ *and either* $\Gamma G = {}^2A_2(2^{2t}) \rtimes \langle \zeta \rangle$, *or* $\Gamma G = \widehat{{}^2A_2(2^{2t})} \rtimes \langle \zeta \rangle$; $|\zeta| = t$ *is odd,* $C_G(\zeta) \simeq \widehat{{}^2A_2(2)}$ *(note that* $C_G(\zeta) \simeq \widehat{{}^2A_2(2)}$ *if* $G = \widehat{{}^2A_2(2^{2t})}$ *or* $t$ *is divisible by 3),* $K \cap G$ *has order* $2 \cdot 3^k$, *where* $3^{k-1} = t_3$..
2. $G$ *is defined over* $GF(2^t)$, *a field automorphism* $\zeta$ *is in* $A$, $|\zeta| = t$, *and denoting by* $\tau$ *a graph automorphism of order* $\leqslant 2$ *with* $\tau \in A$, *up to conjugation in* $G$, *we have* $K = S \rtimes \langle \zeta, \tau \rangle$, *where* $S$ *is a Sylow 2-subgroup of* $G_{\zeta_{2'}}$.
3. $G \simeq \mathrm{PSL}_2(3^t)$, $\zeta \in A = \mathrm{Aut}(G)$, $|\zeta| = t$ *and, up to conjugation in* $G$, *we have* $K = S \rtimes \langle \zeta \rangle$, *where* $S$ *is a Sylow 3-subgroup of* $G_{\zeta_{3'}}$.
4. $A = \Gamma G = {}^2G_2(3^{2n+1}) \rtimes \langle \zeta \rangle$, $|\zeta| = 2n+1$, *and, up to conjugation in* $G$, *we have* $K \cap {}^2G_2(3^{2n+1}) = S \times P$, *where* $S$ *is of order 2 and* $|P| = 3^{|\zeta|_3}$.
5. $p$ *does not divide* $|K \cap G|$ *and* $K$ *contains a Sylow 2-subgroup of* $A$. *Note that the group* $A$ *satisfies* **(ESyl2)** *if and only if* $G$ *satisfies* **(ESyl2)**.

*In particular, Carter subgroups of $A$ are conjugate.*

## Proof scheme.

1. We construct in $A$ parabolic subgroups and reductive subgroups of maximal rank and transfer to $A$ the structure results on the normalizes of $p$-subgroups and centralizes of semisimple elements. It is easy to see that $K \trianglerighteq K \cap G \neq \{e\}$, so $Z(K) \cap G \neq \{e\}$ and there exists an element $x \in Z(K) \cap G$ of prime order.

2. If there exists $x \in Z(K) \cap G$ with $|x| = p$, then $K$ is contained in a proper parabolic subgroup of $A$. The order of a Carter subgroup of its Levi factor is not divisible by $p$. By using descending lemma, we obtain that for each non-Abelian composition factor of the Levi factor point (5) of Theorem 8 holds. By using ascending lemmas we obtain that $K$ contains a Sylow 2-subgroup of the parabolic subgroup. But a Sylow 2-subgroup of the Levi factor does not centralizes elements of order $p$ if $G \not\simeq C_n(q)$. If $G \simeq C_n(q)$, then more accurate computations show that in this case $K$ satisfies to one of the statements of the theorem.

3. Now $p$ does not divide $|K \cap G|$. Thus $|x| \neq p$, then $K$ is contained in $C_A(x)$, that contains a normal reductive subgroup of maximal rank $R$ of $A$. By using descending lemma, we obtain that for each non-Abelian composition factor of $R$ point (5) of Theorem 8 is true. Thus we may assume that $|x| = 2$.

4. Now $x$ is an involution. By using descending lemma, we obtain that $K$ contains Sylow 2-subgroups of all non-Abelian composition factors of $R$. By ascending lemmas it follows that $K$ contains a Sylow 2-subgroup of $C_A(x)$. Hence $K$ contains a Sylow 2-subgroup of $A$.

As a corollary of Theorems 7 and 8 we obtain the following theorem. It can be stated without using of the classification of finite simple groups, if all composition factors of $G$ are assumed to be known simple groups.

**Theorem 9.** *Let $G$ be a finite group. Then the Carter subgroups of $G$ are conjugate.*

Moreover, as a corollary of Theorem 9 we obtain immediately that a homomorphic image of a Carter subgroup is again a Carter subgroup.

**Theorem 10.** *Let $G$ be a finite group, $K$ its Carter subgroup and $\varphi$ a homomorphism of $G$. Then $K^\varphi$ is a Carter subgroup of $G^\varphi$.*

In Theorems 7 and 8 there is a condition $A = KG$, that is used during the proofs of the theorems. However, direct computations show that it can be removed without lost of correctness of the theorems. Moreover, the following theorem is true.

**Theorem 11.** *Let $G$ be a finite simple group and $G \leq A \leq \mathrm{Aut}(G)$ an almost simple group. Assume that $A$ contains a subgroup $S$ such that $G \leq S$ and $S$ contains a Carter subgroup.*
*Then $A$ contains a Carter subgroup.*

Let $G = G_0 \geq G_1 \geq \ldots \geq G_n = \{e\}$ be a chief normal series of $G$. Then $G_i/G_{i+1} = T_{i,1} \times \ldots \times T_{i,k_i}$, where $T_{i,1} \simeq \ldots \simeq T_{i,k_i} \simeq T_i$ and $T_i$ is a simple group. If $i \geqslant 1$, then denote by $\overline{K}_i$ a Carter subgroup of $G/G_i$ (if it exists) and by $K_i$ its complete preimage in $G/G_{i+1}$. If $i = 0$, then $\overline{K}_0 = \{e\}$ and $K_0 = G/G_1$ (note that $\overline{K}_0$ always exists). A finite group $G$ is said to satisfy **(E)**, if, for every $i$ and $j$, either $\overline{K}_i$ does not exist, or $\mathrm{Aut}_{K_i}(T_{i,j})$ contains a Carter subgroup. The following theorem gives a criterion of existence of a Carter subgroup.

**Theorem 12.** (Existence Criterion) *Let $G$ be a finite group. Then $G$ contains a Carter subgroup if and only if $G$ satisfies* **(E)**.

Note that in condition **(E)** the chief series can not be substituted by a composition series. The following simple example shows this fact. This example shows also that an extension of a group containing a Carter subgroup by a group containing a Carter subgroup may fail to contain a Carter subgroup. Consider $L = \Gamma\mathrm{SL}_2(3^3) = \mathrm{PSL}_2(3^3) \leftthreetimes \langle\, \varphi \,\rangle$, where $\varphi$ is a field automorphism of $\mathrm{PSL}_2(3^3)$. Let $X = (L_1 \times L_2) \leftthreetimes \mathrm{Sym}_2$, where $L_1 \simeq L_2 \simeq L$ and, if $\sigma = (1,2) \in \mathrm{Sym}_2 \setminus \{e\}$, $(x,y) \in L_1 \times L_2$, then $\sigma(x,y)\sigma = (y,x)$ (permutational wreath product of $L$ and $\mathrm{Sym}_2$). Denote by $H = \mathrm{PSL}_2(3^3) \times \mathrm{PSL}_2(3^3)$ the minimal normal subgroup of $X$ and by $M = L_1 \times L_2$. Let $G = (H \leftthreetimes \langle (\varphi, \varphi^{-1})\, \rangle) \leftthreetimes \mathrm{Sym}_2$ be a subgroup of $X$. Then the following statements hold:

1. For each composition factor $S$ of $G$, $\mathrm{Aut}_G(S)$ contains a Carter subgroup.
2. $G \cap M \trianglelefteq G$ contains a Carter subgroup.
3. $G/(G \cap L)$ is nilpotent.
4. $G$ does not contain a Carter subgroup.

Tables given below are organized in the following way. In the first column is given a simple group $S$ such that Carter subgroups of $\mathrm{Aut}(S)$ are classified. In the second column we give conditions for a subgroup $A \leq \mathrm{Aut}(S)$ to contain a Carter subgroup. In the third column we give the structure of a Carter subgroup $K$. In every subgroup of $\mathrm{Aut}(S)$ lying between $S$ and $A$ Carter subgroups does not exist. By $P_r(G)$ a Sylow $r$-subgroup of $G$ is denoted. By $\varphi$ we denote a field automorphism of a group of Lie type $S$, by $\tau$ we denote a graph automorphism of a group of Lie type $S$ contained in $K$ (since graph automorphisms of order 2 and 3 of $D_4(q)$ does not commute, only one of them can be in $K$). If $A$ does not contains a graph automorphism, then we suppose $\tau = e$. By $\psi$ we denote a field automorphism of $S$ of maximal order contained in $A$ (it is a power of $\varphi$, but $\langle\psi\rangle$ can be a proper subgroup of $\langle\varphi\rangle$). If $G$ is a solvable group, by $K(G)$ we denote a Carter subgroup of $G$. In Table 15 by $\chi$ the $2'$-part of a field automorphism $\varphi$ of ${}^2A_2(2^{2t})$ is denoted.

**Table 13.** Groups of automorphisms of alternating groups, containing Carter subgroups.

| Group $S$ | Conditions on $A$ | Structure of $K$ |
|---|---|---|
| $\mathrm{Alt}_5$ | $A = \mathrm{Sym}_5$ | $K = P_2(\mathrm{Sym}_5)$ |
| $\mathrm{Alt}_n$, $n \geqslant 6$ | none | $K = N_A(P_2(S))$ |

**Table 14.** Groups of automorphisms of sporadic groups, containing Carter subgroups.

| Group $S$ | Conditions on $A$ | Structure of $K$ |
|---|---|---|
| $J_2, J_3, Suz, HN$ | $A = \mathrm{Aut}(S)$ | $K = P_2(A)$ |
| $\not\simeq J_1, J_2, J_3, Suz, HN$ | none | $K = P_2(A)$ |

**Table 15.** Groups of automorphisms of classical groups, containing Carter subgroups.

| Group $S$ | Conditions on $A$ | Structure of $K$ |
|---|---|---|
| $A_1(q)$, $q \equiv \pm 1 \pmod 8$ | none | $K = N_A(P_2(S))$ |
| $A_1(q)$, $q \equiv \pm 3 \pmod 8$ | $\widehat{S} \leq A$ | $K = N_A(P_2(\widehat{S}))$ |
| $A_n(2^t)$, $t \geqslant 2$, if $n = 1$ | $\varphi \in A$ | $K = \langle \varphi, \tau \rangle \leftthreetimes S_{\varphi_{2'}}$ |
| $A_n(q)$, $q$ is odd, $n \geqslant 2$ | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| ${}^2A_2(2^{2t})$, $t$ is odd, $t \not\equiv 0 \pmod 3$ | $A = \langle \chi \rangle \leftthreetimes \widehat{S}$ | $K = \langle \chi \rangle \times K(U_3(2))$ |
| ${}^2A_2(2^{2t})$, $t$ is odd, $t_3 = 3^{k-1}$, $k \geqslant 2$ | $\langle \chi \rangle \leftthreetimes S \leq A \leq \langle \chi \rangle \leftthreetimes \widehat{S}$ | $K = \langle \chi \rangle \leftthreetimes (2 \times 3^k)$ |
| ${}^2A_2(2^{2t})$ | $A = \mathrm{Aut}(S)$ | $K = \langle \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$ |
| ${}^2A_n(q^2)$, $q$ is odd | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| ${}^2A_n(2^{2t})$, $n \geqslant 3$ | $A = \mathrm{Aut}(S)$ | $K = \langle \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$ |
| $B_2(q)$, $q \equiv \pm 1 \pmod 8$ | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| $B_2(2^t)$, $t \geqslant 2$ | $\varphi \in A$ | $K = \langle \varphi, \tau \rangle \leftthreetimes P_2((S_\tau)_\varphi)$ |
| $B_2(q)$, $q \equiv \pm 3 \pmod 8$ | $\widehat{S} \leq A$ | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| $B_n(q)$, $q$ is odd, $n \geqslant 3$ | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| $C_n(q)$, $q \equiv \pm 1 \pmod 8$ | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| $C_n(q)$, $q \equiv \pm 3 \pmod 8$ | $\widehat{S} \leq A$ | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| $C_n(2^t)$, $n \geqslant 3$ | $A = \mathrm{Aut}(S)$ | $K = \langle \varphi \rangle \times P_2(S_{\varphi_{2'}})$ |
| $D_4(q)$, $q$ is odd | none | if $\lvert\tau\rvert \leqslant 2$, then $K = P_2(A) \times K(O(N_A(P_2(A))))$; if $\lvert\tau\rvert = 3$, then $K = \langle \tau, \psi \rangle \leftthreetimes P_2(S_\tau)$ |
| $D_4(2^t)$ | $\varphi \in A$ | if $\lvert\tau\rvert \leqslant 2$, then $K = \langle \tau, \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$; if $\lvert\tau\rvert = 3$, then $K = \langle \tau, \varphi \rangle \leftthreetimes P_2((S_\tau)_{\varphi_{2'}})$ |
| $D_n(q)$, $q$ is odd, $n \geqslant 5$ | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| $D_n(2^t)$, $n \geqslant 5$ | $\varphi \in A$ | $K = \langle \tau, \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$ |
| ${}^2D_n(q^2)$, $q$ is odd | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| ${}^2D_n(2^{2t})$ | $A = \mathrm{Aut}(S)$ | $K = \langle \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$ |

**Table 16.** Groups of automorphisms of exceptional groups of Lie type, containing Carter subgroups.

| Group $S$ | Conditions on $A$ | Structure of $K$ |
|---|---|---|
| ${}^2B_2(2^{2n+1})$, $n \geqslant 1$ | $A = \mathrm{Aut}(S)$ | $K = \langle \varphi \rangle \times P_2({}^2B_2(2))$ |
| $({}^2F_4(2))'$ | none | $K = P_2(A)$ |
| ${}^2F_4(2^{2n+1})$, $n \geqslant 1$ | $A = \mathrm{Aut}(S)$ | $K = \langle \varphi \rangle \times P_2({}^2F_4(2))$ |
| ${}^2G_3(3^{2n+1})$ | $A = \mathrm{Aut}(G)$ | $\langle \varphi \rangle \leftthreetimes (2 \times P)$, where $\lvert P\rvert = 3^{\lvert\varphi\rvert_3}$ |
| ${}^3D_4(q^3)$, $q = 2^t$ is even | $A = \mathrm{Aut}(G) = G \rightthreetimes \langle \zeta \rangle$ | $K = P_2(G_{\zeta_{2'}}) \rightthreetimes \langle \zeta \rangle$ |
| ${}^3D_4(q^3)$, $q$ is odd | $(\lvert A : G\rvert, 3) = 1$ | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| ${}^3D_4(q^3)$, $q$ is odd | $\lvert\tau\rvert = 2$ and $A = G \rightthreetimes \langle \psi, \tau \rangle$ | $K = P_2(G_\tau) \rightthreetimes \langle \tau, \psi \rangle$ |
| others, $q$ is odd | none | $K = P_2(A) \times K(O(N_A(P_2(A))))$ |
| others, $q = 2^t$ | $\varphi \in A$ | $\langle \tau, \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$ |