On the Relation Module of a Finite Group

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Definitions

Recalling some basics in group theory ...

Let G be a group.

- Let X be a set, and let $F(X)$ denote the free group over the set X. A surjective homomorphism $\pi \colon F(X) \to G$ is called a **presentation of** G.
- If X is finite, the presentation π will be called **finite**.
- Let $\pi \colon F(X) \to G$ be a presentation of G. The left $\mathbb{Z}[G]$-module

$$\mathfrak{R}_\pi := \ker(\pi)/[\ker(\pi), \ker(\pi)]$$

is called the **relation module** of π.

Elementary properties, I

Easy exercises . . .

Proposition (Schreier's formula)

Let $\pi\colon F(X) \to G$ be a finite presentation of the finite group G. Then $\mathfrak{R}_\pi$ is a free $\mathbb{Z}$-module of rank

$$\mathrm{rk}_{\mathbb{Z}}(\mathfrak{R}_\pi) = 1 + |G| \cdot (|X| - 1).$$

Elementary properties, II

Easy exercises ...

Definition (Stable equivalence)

Let G be a finite group. The left $\mathbb{Z}[G]$-modules $A, B \in ob({}_G\text{Mod})$ are called *stably-equivalent*, if there exists projective left $\mathbb{Z}[G]$-modules P and Q such that

$$A \oplus P \simeq B \oplus Q.$$

Proposition (K.W.Gruenberg [3])

Let $\omega(G) := \ker(\varepsilon)$ *denote the augmentation ideal of the integral group algebra* $\mathbb{Z}[G]$ *of* G*. The left* $\mathbb{Z}[G]$*-module* $\mathfrak{R}_\pi$ *is stably-equivalent to* $\omega(G) \otimes \omega(G)$*.*

An alternative description

Using homological algebra …

Proposition (folklore)

Let $\pi\colon F(X) \to G$ be a finite presentation of a finite group, and let $(P_\bullet, \partial_\bullet^P)$ be the chain complex of left $\mathbb{Z}[G]$-modules concentrated in degrees 1 and 0

$$\cdots \longrightarrow 0 \longrightarrow \coprod_{x\in X} \mathbb{Z}[G]\langle x\rangle \xrightarrow{\partial_1^P} \mathbb{Z}[G]\langle 1\rangle \longrightarrow 0 \longrightarrow \cdots,$$

where $\partial_1^P(\langle x\rangle) := (\pi(x) - 1)\langle 1\rangle$.
Then one has canonical isomorphisms

- $H_0(P_\bullet, \partial_\bullet^P) = \mathrm{coker}(\partial_1^P) \simeq \mathbb{Z}$,
- $H_1(P_\bullet, \partial_\bullet^P) = \ker(\partial_1^P) \simeq \mathfrak{R}_\pi$.

An example

Easy but not intrinsic …

Example (S_3 and the standard generators)

Let $s, t \in S_3$, $\mathrm{ord}(s) = 2$, $\mathrm{ord}(t) = 3$, and let $\pi \colon F(s,t) \to S_3$ denote the corresponding presentation. Then $\mathfrak{R}_\pi \leq \mathbb{Z}[S_3]\langle s\rangle \oplus \mathbb{Z}[S_3]\langle t\rangle$ is the submodule

$$\begin{aligned}\mathfrak{R}_\pi = \mathbb{Z}[S_3](s+1)\langle s\rangle + \mathbb{Z}[S_3](t^2+t+1)\langle t\rangle \\ + \mathbb{Z}[S_3]\big((st+1)\langle s\rangle + (sts+s)\langle t\rangle\big).\end{aligned}$$

The co-augmentation

A special feature of relation modules of finite presentations of finite groups …

Proposition and Definition (Co-augmentation)

Let $\pi\colon F(X) \to G$ be a finite presentation of finite groups, and let $N_G \in \mathbb{Z}[G]$ be the element $N_G := \sum_{g\in G} g$.

- *The element $r := \sum_{x\in X} N_G.\langle x\rangle$ is contained in $\ker(\partial_1^P)$ and thus in $\mathfrak{R}_\pi$.*
- *The mapping $\eta\colon \mathbb{Z} \to \mathfrak{R}_\pi$, $\eta(1) := r$, will be called the* **co-augmentation**.
- *The left $\mathbb{Z}[G]$-module $\bar{\mathfrak{R}}_\pi := \operatorname{coker}(\eta)$ will be called the* **reduced relation module** *of the presentation π.*

Projectivity

Folklore . . .

Proposition (folklore)

Let $\pi: F(X) \to G$ be a finite presentation of the finite group G. Then $\bar{\mathfrak{R}}_\pi$ is projective, if and only if G is cyclic.

Definition (Euclidean $\mathbb{Z}[G]$-modules)

Let G be a finite group. A tuple (M, μ) is called an **euclidean $\mathbb{Z}[G]$-module**, if

- M is a finitely generated projective left $\mathbb{Z}[G]$-module,
- $\mu\colon M \otimes M \to \mathbb{Z}$ is a symmetric map of $\mathbb{Z}[G]$-modules,
- the induced map

$$\mu_\circ\colon M \longrightarrow M^*, \quad \mu_\circ(x)(y) := \mu(x \otimes y)$$

  is an isomorphism.

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[1] $\otimes = \otimes_{\mathbb{Z}}$,
[2] $\_^* = \mathrm{Hom}(\_, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z})$.

Maximally isotropic submodules

Like in geometry ...

Definition (Maximally isotropic submodules)

Let (M, μ) be an euclidean $\mathbb{Z}[G]$-module. The $\mathbb{Z}[G]$-submodule $R \leq M$ is called **maximally isotropic**, if

- $\mu|_{R \otimes R} = 0$,
- the canonical map $\mu_R \colon M/R \longrightarrow R^*$ induced by $\mu_\circ$ is an isomorphism.

Remark

The $\mathbb{Z}[G]$-submodule $R \leq M$ is maximally isotropic, if and only if the canonical sequence

$$0 \longrightarrow R \xrightarrow{\iota} M \xrightarrow{\bar{\mu}_R} R^* \longrightarrow 0$$

is exact.

The relation cone, I

An euclidean $\mathbb{Z}[G]$-module associated to π

Let $\pi\colon F(X) \longrightarrow G$ be a finite presentation of the finite group G.

- Let $X := \{x_0, \ldots, x_{r-1}\}$.
- Put

$$Q_\pi := \coprod_{1 \le i \le r-1} \mathbb{Z}[G]\langle x_i \rangle \oplus \coprod_{1 \le i \le r-1} \mathbb{Z}[G]\langle x_i^* \rangle.$$

Hence Q_π is a free left $\mathbb{Z}[G]$-module of rank $2(|X| - 1)$.

The relation cone, II

An euclidean $\mathbb{Z}[G]$-module associated to π

- Let $\xi_\pi \colon Q_\pi \otimes Q_\pi \to \mathbb{Z}[G]$ be the mapping of left $\mathbb{Z}[G] \otimes \mathbb{Z}[G]$-modules given by

$$\xi_\pi(x_i \otimes x_j) := -\delta_{i,j} - \pi(x_i)\pi(x_j)^{-1}$$
$$\xi_\pi(x_i \otimes x_j^*) := -\delta_{i,j} + \pi(x_i)(1 - \pi(x_j)^{-1})$$
$$\xi_\pi(x_i^* \otimes x_j) := -\delta_{i,j} + (1 - \pi(x_i))\pi(x_j)^{-1}$$
$$\xi_\pi(x_i^* \otimes x_j^*) := -(1 - \pi(x_i))(1 - \pi(x_j)^{-1})$$

- Put $\mu_\pi := \varepsilon \circ \xi_\pi$. Then (Q_π, μ_π) is an euclidean $\mathbb{Z}[G]$-module which will be called **the relation cone of** π.

A structure theorem

on the reduced relation module . . .

Theorem (T.W. [4])

Let $\pi : F(X) \to G$ *be a finite presentation of the finite group G.*
Let (Q_π, μ_π) *be the euclidean* $\mathbb{Z}[G]$*-module as described before.*
Then $\bar{\mathfrak{R}}_\pi$ *is a maximally isotropic submodule of* (Q_π, μ_π).

The standard complex

A well-known object . . .

Definition (standard complex for free groups)

Let $F(X)$ be a free group over the set X. Let $(P_\bullet, \partial^P_\bullet)$ be the chain complex of left $\mathbb{Z}[F(X)]$-modules concentrated in degrees 1 and 0

$$\longrightarrow 0 \longrightarrow \coprod_{x \in X} \mathbb{Z}[F(X)]\langle x \rangle \xrightarrow{\partial^P_1} \mathbb{Z}[F(X)]\langle 1 \rangle \longrightarrow 0 \longrightarrow$$

where $\partial^P_1(\langle x \rangle) := (x - 1).\langle 1 \rangle$.
Then $P := (P_\bullet, \partial^P_\bullet)$ is a projective resolution of the trivial left $\mathbb{Z}[F(X)]$-module $\mathbb{Z}$ called the **standard complex of F(X)**.

The dual of the standard complex, I

Another well-known object . . .

Proposition (folklore)

Let X be a finite set, and let $\_^{\circledast} := {}^{\times}\mathrm{Hom}_{F(X)}(\_, \mathbb{Z}[F(X)])$.

(a) $P^{\circledast} = (P^{\circledast}_{\bullet}, \partial^{P^{\circledast}}_{\bullet})$ *is a chain complex concentrated in degrees* 0 *and* -1 *given by*

$$\longrightarrow \mathbb{Z}[F(X)]\langle 1^* \rangle \xrightarrow{\partial_0^{P^{\circledast}}} \coprod_{x \in X} \mathbb{Z}[F(X)]\langle x^* \rangle \longrightarrow$$

where $\partial_0^{P^{\circledast}}(\langle 1^* \rangle) = \sum_{x \in X}(x^{-1} - 1)\langle x^* \rangle$.

The dual of the standard complex, II

Another well-known object . . .

Proposition (folklore)

(b)

- *$H_{-k}(P^{\circledast}) = H^k(F(X), \mathbb{Z}[F(X)]) = 0$ for $k \in \mathbb{Z}$, $k \neq 1$.*
- *$H_{-1}(P^{\circledast}) = H^1(F(X), \mathbb{Z}[F(X)]) = {}^{\times}D_{F(X)}$ is isomorphic to the (left) dualizing module of $F(X)$.*
- *$F(X)$ is a duality group - in the sense of R.Bieri and B.Eckmann [2] - of cohomological dimension 1.*

By $P^{\circledast}[1]$ one denotes the chain complex $P^{\circledast}$ shifted one position to the left with $\partial_1^{P^{\circledast}[1]} = -\partial_0^{P^{\circledast}}$.

A distinguished triangle, I

A less well-known object . . .

Theorem (T.W. [4])

(a) *The mapping* $\zeta\colon P^{\circledast}[1] \to P$

$$\begin{array}{ccccccc}
\longrightarrow & \mathbb{Z}[F(X)]\langle 1^* \rangle & \xrightarrow{\partial_1^{P^{\circledast}[1]}} & \coprod_{x\in X} \mathbb{Z}[F(X)]\langle x^* \rangle & \longrightarrow \\
& \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_0} & \\
\longrightarrow & \coprod_{x\in X} \mathbb{Z}[F(X)]\langle x \rangle & \xrightarrow{\partial_1^{P}} & \mathbb{Z}[F(X)]\langle 1 \rangle & \longrightarrow
\end{array}$$

where

$$\zeta_0(\langle x^* \rangle) := x.\langle 1 \rangle,$$

$$\zeta_1(\langle 1^* \rangle) := \sum_{x\in X} \langle x \rangle,$$

is a mapping of chain complexes.

A distinguished triangle, II

A less well-known object ...

Theorem (T.W. [4])

(b) *There exists a distinguished triangle (in $\mathfrak{D}_{(b)}({}_G\mathrm{Mod})$)*

$$\Delta: \qquad P[-1] \xrightarrow{a} C \xrightarrow{b} P^{\circledast}[1] \xrightarrow{\zeta} P\,, \qquad (\dagger)$$

where C is a chain complex concentrated in degree 0, and $C_0 = H_0(C)$ is a free $\mathbb{Z}[F(X)]$-module of rank $2(|X|-1)$. In particular, one has a short exact sequence

$$0 \longrightarrow C_0 \xrightarrow{H_0(b)} {}^{\times}D_{F(X)} \xrightarrow{H_0(\zeta)} \mathbb{Z} \longrightarrow 0\,.$$

A distinguished triangle, III

A less well-known object . . .

Theorem (T.W. [4])

(c) *There exists an isomorphism of distinguished triangles*

$$\begin{array}{ccccccc} P[-1] & \xrightarrow{a} & C & \xrightarrow{b} & P^{\circledast}[1] & \xrightarrow{\zeta} & P \\ \| & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle -id} & & \| \\ P[-1] & \xrightarrow{-b^{\circledast}_{\omega}} & C^{\circledast} & \xrightarrow{a^{\circledast}} & P^{\circledast}[1] & \xrightarrow{{}_{\omega}\zeta^{\circledast}[1]} & P \end{array} \qquad (\ddagger)$$

where

- $(\_^{\circledast}[1], \omega)$ *denotes the standard duality in* $\mathfrak{D}(\mathbb{Z}[F(X)], \sigma)[1]$ *(cf. P.Balmer [1]),*
- $(\_^{\circledast}, \varpi)$ *denotes the standard duality in* $\mathfrak{D}(\mathbb{Z}[F(X)], \sigma)[0]$*,*
- $c = c^{\circledast}_{\varpi}$*.*

Deflation on module categories

Another piece of abstract nonsense ...

Definition (Deflation)

Let $\phi\colon A \to B$ be a surjective homomorphism of groups. Then

$$\mathrm{def}^{\phi}(\_)\colon {}_A\mathrm{Mod} \longrightarrow {}_B\mathrm{Mod},$$
$$\mathrm{def}^{\phi}(M) := \mathbb{Z}[B] \otimes_{\mathbb{Z}[A]} M, \quad M \in ob({}_A\mathrm{Mod})$$

is called the **deflation functor** corresponding to ϕ.

Proposition

The functor $\mathrm{def}^{\phi}(\_)$ *has the following properties:*

- $\mathrm{def}^{\phi}(M) = M_{\mathrm{ker}(\phi)}$.
- $\mathrm{def}^{\phi}(\_)$ *is covariant, additive and right exact.*
- $\mathrm{def}^{\phi}(\_)$ *is mapping projectives to projectives.*

Deflation as derived functor, I

More abstract nonsense ...

Proposition

Let $\phi\colon A \to B$ be a surjective homomorphism of groups. The deflation functor induces a derived functor

$$\mathfrak{def}^{\phi}(\_)\colon \mathfrak{D}_{(b)}(\mathbb{Z}[A]) \longrightarrow \mathfrak{D}_{(b)}(\mathbb{Z}[B]).$$

In particular, $\mathfrak{def}^{\phi}(\_)$ maps distinguished triangles to distinguished triangles.

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Deflation as derived functor, II

More abstract nonsense ...

Remark (Eckmann-Shapiro-type lemma)

Let $\phi\colon A \to B$ *be a surjective homomorphism of groups.*

- *Let* $\mathrm{def}_k^{\phi} := H_k(\mathfrak{def}^{\phi}(\_[\![0]\!]))$*. Then one has*

$$\mathrm{def}_k^{\phi}(\mathbb{Z}) = H_k(\ker(\phi), \mathbb{Z}).$$

- *In particular, if* $\pi\colon F(X) \to G$ *is a presentation, then*
  - $\mathrm{def}_0^{\pi}(\mathbb{Z}) = \mathbb{Z}$,
  - $\mathrm{def}_1^{\pi}(\mathbb{Z}) = \mathfrak{R}_{\pi}$,
  - $\mathrm{def}_k^{\pi}(\mathbb{Z}) = 0$ *for* $k \neq 0, 1$.

Deflation as unitary functor

Abstract madness . . .

Proposition

Let $\phi\colon A \to B$ be a surjective homomorphism of groups. Then

$$\mathfrak{def}\colon \mathfrak{D}(\mathbb{Z}[A], \sigma)[d] \longrightarrow \mathfrak{D}(\mathbb{Z}[B], \sigma)[d]$$

is a **unitary functor**[1] *for all $d \in \mathbb{Z}$. In particular, the functor $\mathfrak{def}^{\phi}(\_)$ maps an isomorphism of distinguished triangles of type $(\ddagger)$ to an isomorphism of distinguished triangles of type $(\ddagger)$.*

[1] Unitary $\Leftrightarrow$ $\exists$ a natural transformation between $\mathfrak{def}^{\phi}(\_^{\circledast})$ and $\mathfrak{def}^{\phi}(\_)^{\circledast}$ satisfying certain properties.

Dualizing modules of finite groups, I

Bringing the nonsense to life . . .

Proposition (Eckmann-Shapiro lemma)

Let G be a finite group and $M \in ob({}_G\mathrm{Mod})$. Then one has a natural isomorphism

$$M^{\circledast} = {}^{\times}\mathrm{Hom}_G(M, \mathbb{Z}[G]) \simeq \mathrm{Hom}(M, \mathbb{Z}) = M^*.$$

Dualizing modules of finite groups, II

Bringing the nonsense to life . . .

Corollary

Let G be a finite group. Then

$$\_^{\circledast} \colon {}_G\mathrm{Mod}^{\mathrm{op}} \longrightarrow {}_G\mathrm{Mod}$$

is an additive, contravariant, relatively exact functor, i.e. if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is a short exact sequence of left* $\mathbb{Z}[G]$*-modules, which splits as sequence of abelian groups, then*

$$0 \longrightarrow C^{\circledast} \xrightarrow{\beta^{\circledast}} B^{\circledast} \xrightarrow{\alpha^{\circledast}} A^{\circledast} \longrightarrow 0$$

is short exact.

A 5-term exact sequence

Applying homology to abstract nonsense . . .

Theorem (T.W. [4])

Let $\pi\colon F(X) \to G$ *be a finite presentation of the finite group G. The homology functor applied to the distinguished triangle* $\mathfrak{Def}^{\pi}(\Delta)$ *yields a long exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \eta\ } \mathfrak{R}_\pi \xrightarrow{\ \bar\iota\ } Q_\pi \xrightarrow{\ \bar\iota_\circ\ } \mathfrak{R}^*_\pi \xrightarrow{\ \eta^*\ } \mathbb{Z} \longrightarrow 0\ .$$

where $\bar\iota_\circ := \bar\iota^* \circ (\mu_\pi)_\circ$*.*

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