On a Generalization of the Exchange Property to Modules with Semilocal Endomorphism Rings

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Introduction

The well-known Krull-Schmidt-Azumaya theorem gives sufficient conditions for a module to have an essentially unique decomposition as a direct sum of indecomposable submodules. A lot of work has been done over the years to extend as far as possible this theorem and to see whether particular classes of modules have essentially unique decomposition.

Recently, though, the attention has been pointed in another direction. Instead of looking for other "very good" classes of modules, a great deal of attention has been posed on "good" classes of modules and on ways to measure how different is "good" from "very good". Namely, for every full subcategory $\mathcal{C}$ of Mod-R, a reduced commutative monoid $V(\mathcal{C})$ carrying all the information about direct sum decompositions in $\mathcal{C}$ has been considered. The elements of $V(\mathcal{C})$ are the isomorphism classes $\langle A\rangle$ of the modules A in $\mathcal{C}$ and the sum is given by $\langle A\rangle + \langle B\rangle = \langle A \oplus B\rangle$ for every $A, B \in \mathcal{C}$.

It is clear that the Krull-Schmidt theorem holds in $\mathcal{C}$ if and only if the monoid $V(\mathcal{C})$ is free, the point being we can consider weaker, though controllable, conditions, such as the monoid $V(\mathcal{C})$ being a Krull monoid.

In 1964, P. Crawley and B. Jónsson introduced the exchange property of a module and, in 1969, R. B. Warfield Jr. proved that the exchange property is equivalent to the endomorphism ring of the module being local for indecomposable module. These two equivalent properties are a natural property to ask to the indecomposable modules belonging to a class $\mathcal{C}$ for $V(\mathcal{C})$ to be a free monoid.

In 2002, A. Facchini proved that a sufficient condition for $V(\mathcal{C})$ to be a Krull monoid is that every module in $\mathcal{C}$ has semilocal endomorphism ring. What about the exchange property? Is there any analogue property which is equivalent for M to the fact that $\mathrm{End}(M)$ is semilocal?

The semiexchange property was born as an attempt to give a positive answer to this question. In Section 1 we define the semiexchange property. Given a ring R, a right R-module M and a positive integer m, we say M has the semiexchange property

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with respect to m if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i$$

where $M' \cong M$, there are a partition $I = \dot{\bigcup}_{j \in J} I_j$ with $|I_j| \leq m$ for any $j \in J$ and R-submodules B_j of $\oplus_{i \in I_j} A_i$, $j \in J$, such that $G = M' \oplus (\oplus_{j \in J} B_j)$. In Section 1 we also give the definition and prove the basic properties of the semiexchange property for elements of a cancellative monoid. The theory of cancellative monoids has been extensively developed in recent years, with the study of non-unique factorization in domains as main motivation. It turns out it is very useful to study non-unique decompositions of modules as well. In this respect we think it is wise to compare tools and results in the two fields. We gave elementary proofs for our results for seek of semplicity, but we gave also references to results in the literature which have our claims as simple corollaries.

In Section 2 we prove that the dual Goldie dimension of a module M is the smallest integer n (if any) such that M has the semiexchange property with respect to n. Thus an indecomposable module M whose endomorphism ring is not semilocal does not have the semiexchange property with respect to n for any integer n. In Section 3, finally, as an application of the semiexchange property, we will prove a stronger version of the Weak Krull-Schmidt Theorem for biuniform modules.

Throughout the paper rings will be associative rings with identity $1 \neq 0$ and modules will be right modules. Mod-R will denote the category of right modules over a ring R, mod-R will denote the category of finitely presented right modules, proj-R will denote the category of finitely generated projective right modules and add-M will denote the full subcategory of Mod-R which elements are isomorphic to a direct summand of a finite direct sum of copies of the module M.

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1 Definitions and first properties

A few preliminars are in order to make the paper as self-contained as possible.

We begin with a well known immediate consequence of the modular identity that will be used repeatedly in the sequel.

Lemma 1.1 *If $A \subseteq B \subseteq A \oplus C$ are modules, then $B = A \oplus D$, where $D = B \cap C$.*

Other lemmas we use extensively are the following ones. We recall them here for the readers' convenience.

Lemma 1.2 ([2, Lemma 3.8]) *If G, M', N, P, A_i $(i \in I)$, B_i $(i \in I)$ are modules, $B_i \subseteq A_i$ for every $i \in I$,*

$$G = M' \oplus N \oplus P = (\oplus_{i \in I} A_i) \oplus P$$

and

$$G/P = \left(\left(M' + P\right)/P\right) \oplus \left(\oplus_{i\in I}\left(\left(B_i + P\right)/P\right)\right),$$

then

$$G = M' \oplus (\oplus_{i\in I} B_i) \oplus P.$$

Lemma 1.3 ([3, Lemma 2.6]) *Let A be a module and let M_1, M_2, M be submodules of A such that $A = M_1 \oplus M_2$. Let $\pi_2 \colon A = M_1 \oplus M_2 \to M_2$ be the canonical projection. Then $A = M_1 \oplus M$ if and only if $\pi_2|_M \colon M \to M_2$ is an isomorphism. If these two equivalent conditions hold, then the canonical projection $\pi_M \colon A \to M$ with respect to the decomposition $A = M_1 \oplus M$ is $(\pi_2|_M)^{-1} \circ \pi_2$.*

We are now ready to start. We begin defining the semiexchange property, which is the object of study of the whole paper. Then we will prove some properties of the semiexchange property, trying to generalize as closely as possible the well-known properties of the exchange property.

Definition. Let R be a ring, M be a right R-module, $\aleph$ be a cardinal and m be a positive integer. We say that M has the $\aleph$-semiexchange property with respect to m if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i\in I} A_i$$

where $M' \cong M$ and $|I| \leq \aleph$, there is a partition $I = \dot\bigcup_{j\in J} I_j$ with $|I_j| \leq m$ for any $j \in J$ and R-submodules B_j of $\oplus_{i\in I_j} A_i$, $j \in J$, such that $G = M' \oplus (\oplus_{j\in J} B_j)$. Note that, by Lemma 1.1, the submodules B_j are direct summands of the $\oplus_{i\in I_j} A_i$'s.

Let X be a monoid, x be an element of X, $\aleph$ be a finite cardinal and m be a positive integer. Recall that X is naturally equipped with a pre-order given by $s \leq t$ if and only if there is an element $r \in X$ such that $s + r = t$. We say that x has the $\aleph$-semiexchange property with respect to m if whenever

$$x + y = \sum_{i\in I} a_i$$

where $|I| \leq \aleph$, there is a partition $I = \dot\bigcup_{j\in J} I_j$ with $|I_j| \leq m$ for any $j \in J$ and elements $b_j \leq \sum_{i\in I_j} a_i$, $j \in J$, such that $x + y = x + \left(\sum_{j\in J} b_j\right)$.

We say that an R-module (an element of X) has the finite semiexchange property with respect to m if it has the $\aleph$-semiexchange property with respect to m for any finite cardinal $\aleph$.

We say that an R-module has the semiexchange property with respect to m if it has the $\aleph$-semiexchange property with respect to m for any cardinal $\aleph$.

For every cardinal $\aleph$, an R-module has the $\aleph$-exchange property [2] if and only if it has the $\aleph$-semiexchange property with respect to 1.

Similarly we will say that, for a finite cardinal $\aleph$, an element of a monoid X has the $\aleph$-exchange property if it has the $\aleph$-semiexchange property with respect to 1 and that it has the exchange property if it has the $\aleph$-semiexchange property with respect to 1 for every finite cardinal $\aleph$.

Lemma 1.4 *An indecomposable R-module M has the $\aleph$-semiexchange property with respect to m if and only if for any R-module G and any two direct sum decompositions $G = M' \oplus N = \bigoplus_{i\in I} A_i$ where $|I| \leq \aleph$ and $M' \cong M$, there are indices $i_1, \dots, i_t \in I$ for some $t \leq m$ and a submodule B of $\bigoplus_{k=1}^{t} A_{i_k}$ such that $G = M' \oplus B \oplus \left(\bigoplus_{j\neq i_1,\dots,i_t} A_j\right)$.*

Proof. Let M be an indecomposable R-module. If M has the $\aleph$-semiexchange property with respect to m and

$$M \oplus N = \bigoplus_{i\in I} A_i$$

where $|I| \leq \aleph$, there is a partition $I = \dot{\bigcup}_{j\in J} I_j$ with $|I_j| \leq m$ for any $j \in J$ and decompositions $\bigoplus_{i\in I_j} A_i = B_j \oplus C_j$, $j \in J$, such that $\bigoplus_{i\in I} A_i = M \oplus \left(\bigoplus_{j\in J} B_j\right)$. Therefore $M \cong \bigoplus_{j\in J} C_j$ and, since M is indecomposable, $C_j = 0$ for any j but for one index j_0. We conclude that $M \oplus N = \bigoplus_{j\in J}\left(\bigoplus_{i\in I_j} A_i\right) = M \oplus B_{j_0} \oplus \left(\bigoplus_{i\notin I_{j_0}} A_i\right)$ with $|I_{j_0}| \leq m$. ■

Lemma 1.5 *An indecomposable element x of a cancellative monoid X has the $\aleph$-semiexchange property with respect to m if and only if whenever there are $y, a_i \in X$ ($i \in I$, $|I| \leq \aleph$) such that $a = x + y = \sum_{i\in I} a_i$, there are indices $i_1, \dots, i_t \in I$ for some $t \leq m$ and an element b of X such that $a = x + b + \sum_{j\neq i_1,\dots,i_m} a_j$.*

Proof. The proof is a straight translation of the previous proof in the monoid language. ■

Remark 1.6 Using the notations of [7, Definition 2.8.14], if $\omega(y) < \infty$ for some element y of a cancellative monoid X, then y has the semiexchange property with respect to $\omega(y)$. On the other hand, the previous Lemma essentially says that an indecomposable element x has the finite semiexchange property with respect to m if and only if $\omega(x) \leq m$.

Proposition 1.7 *Let M be a module and let $M = M_1 \oplus M_2$ be a decomposition of M. If M has the $\aleph$-semiexchange property with respect to m, then M_1 has the $\aleph$-semiexchange property with respect to m.*

Proof. Suppose M has the $\aleph$-semiexchange property with respect to m and suppose $G = M_1' \oplus N = \bigoplus_{i\in I} A_i$ with $M_1' \cong M_1$ and $|I| \leq \aleph$. Then $G' = M_2 \oplus G = M' \oplus N = M_2 \oplus \bigoplus_{i\in I} A_i$ with $M' \cong M$. Let $k \in I$ be any index and define $A_i' = A_i$ for every $i \neq k$ and $A_k' = M_2 \oplus A_k$. One has $G' = M' \oplus N = \bigoplus_{i\in I} A_k'$. Thus there is a partition $I = \dot{\bigcup}_{j\in J} I_j$ with $|I_j| \leq m$ and decompositions $\bigoplus_{i\in I_j} A_i = B_j \oplus C_j$, $j \in J$, such that $\bigoplus_{i\in I} A_i' = M' \oplus \left(\bigoplus_{j\in J} B_j\right)$. We will denote by j_0 the index $j \in J$ such that $k \in I_{j_0}$. Since $M_2 \subseteq M_2 \oplus B_{j_0} \subseteq M_2 \oplus G$, we have by Lemma 1.1 that $M_2 \oplus B_{j_0} = M_2 \oplus B_{j_0}'$ where $B_{j_0}' = (M_2 \oplus B_{j_0}) \cap G \subseteq G$. Thus $M' \oplus B_{j_0} = M_1' \oplus M_2 \oplus B_{j_0} = M_1' \oplus M_2 \oplus B_{j_0}'$ and, denoting the B_j's by B_j' for every $j \neq j_0$ one has $G' = M' \oplus \left(\bigoplus_{j\in J} B_j'\right)$. Note that $B_j' \subseteq G$ for every $j \in J$ and that $M_1' \subseteq G$. Thus using the modular identity we get $G = G \cap \left(M_2 \oplus \left(M_1' \oplus \left(\bigoplus_{j\in J} B_j'\right)\right)\right) = (G \cap M_2) \oplus \left(M_1' \oplus \left(\bigoplus_{j\in J} B_j'\right)\right) = M_1' \oplus \left(\bigoplus_{j\in J} B_j'\right)$. This shows that M_1 has the $\aleph$-semiexchange property with respect to m. ■

Proposition 1.8 *Let M be a module and let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ be a decomposition of M into indecomposable modules. If M_x has the $\aleph$-semiexchange property with respect to m_x for every x, then M has the $\aleph$-semiexchange property with respect to $\sum_{x=1}^{k}(m_x - 1) + 1$.*

PROOF. Suppose

$$G = \bigoplus_{j=1}^{k} M_j' \oplus N = \bigoplus_{i\in I} A_i$$

with $M_j' \cong M_j$ for every $j = 1, \ldots, k$ and with $|I| \leq \aleph$.

We will recursively define for every $x = 1, 2, \ldots, k$ sets $I_x, J_x, R_x, K_x, S_x, T_x, T_x'$ and modules B_x, C_x, D_x and $A_{x,i}$ for every $i \in I_x$ such that $G = M_1' \oplus \ldots \oplus M_x' \oplus B_x \oplus \left(\bigoplus_{i\notin J_x} A_{x-1,i}\right)$.

As a start consider $I_0 = I$, $K_0 = \{A_i\}_{i\in I}$ and $A_{0,i} = A_i$ for every $i \in I$.

Suppose that we defined all the mentioned sets and modules for some $x - 1 = 1, 2, \ldots k-1$. Since M_x' is an indecomposable module with the $\aleph$-semiexchange property with respect to m_x, by Lemma 1.2, there is a subset $J_x \subseteq I_{x-1}$ with $|J_x| = m_x$ and a decomposition $\bigoplus_{i\in J_x} A_{x,i} = B_x + C_x$ such that

$$G = M_1' \oplus \ldots \oplus M_{x-1}' \oplus \left(\bigoplus_{i\in I_{x-1}} A_{x-1,i}\right) = M_1' \oplus \ldots \oplus M_x' \oplus B_x \oplus \left(\bigoplus_{i\notin J_x} A_{x-1,i}\right).$$

Define

$$R_x = \{A_{x,i}\}_{i\in J_x} \cup \left(\bigcup_{y\in\{1,\ldots,x-1\} \text{ such that } B_y\in\{A_{x,i}\}_{i\in J_x}} R_y\right),$$

$$S_x = R_x \cap \{A_i\}_{i\in I}, \quad T_x = R_x \setminus S_x \quad \text{and} \quad T_x' = T_x \cup \{B_x\}$$

and set $K_x = \{A_{x-1,i}\}_{i\in I_{x-1}\setminus J_x} \cup \{B_x\}$. We do not want to tell B_x and the $A_{x-1,i}$'s apart, so we are defining the $A_{x,i}$'s just by renaming the elements of K_x. Choose a set I_x with the same cardinality of K_x and use it to rename the elements of K_x as $\{A_{x,i}\}_{i\in I_x}$.

Consider now the partial order $\preceq$ given by $x \preceq y$ if $B_x \in T_y'$.

We will prove that:

(a) if $x \preceq y$, then $x \leq y$;

(b) if $x \preceq y$ and $x \preceq z$, then $z \preceq y$ or $y \preceq z$;

(c) one has $|S_x| \leq \sum_{y\preceq x}(m_y - 1) + 1$;

(d) if $x = 1, 2, \ldots, k$ and $x_1, x_2, \ldots, x_t$ are the maximal elements of $\{1, 2, \ldots, x\}$ with respect to $\preceq$, then $S_{x_1}, S_{x_2}, \ldots, S_{x_t}$ form a partition of $I \setminus K_x$.

If for every $x = 1, 2, \ldots, k$ one notes that $\bigoplus_{i \in S_x} A_i = B_x \oplus D_x$ where $D_x = \bigoplus_{y \preceq x} C_y$ and that $G = \bigoplus_{j=1}^{k} M'_j \oplus N = M'_1 \oplus \ldots \oplus M'_x \oplus \bigoplus_{y \text{ maximal wrt } \preceq} B_y \oplus \bigoplus_{i \notin \bigcup_{z=1}^{x} J_z} A_i$, then the conclusion of the proof follows.

Let us now show claims (a) – (d).

(a) Straightforward.

(b) If $x \preceq y$, then $B_x \in T'_y$. This means that $B_x \notin K_y$ and the only possibility for B_x to be in some T_z for $z \geq y$, is that $B_y \in T'_z$, i.e. $y \preceq z$.

(c) Since $S_x = (\{A_{x-1,i}\}_{i \in J_x} \cap K_0) \cup \left(\bigcup_{y \text{ such that } B_y \in \{A_{x-1,i}\}_{i \in J_x}} S_y\right)$, one has

$$|S_x| \leq m_x - \left| \left\{ B_y \mid B_y \in \{A_{x-1,i}\}_{i \in J_x} \right\} \right| + \sum_{B_y \in \{A_{x-1,i}\}_{i \in J_x}} |S_y| \leq$$

$$\leq m_x + \sum_{B_y \in \{A_{x-1,i}\}_{i \in J_x}} \sum_{z \preceq y} (m_z - 1) = \sum_{y \preceq x} (m_y - 1) + 1.$$

(d) All the A_i's eventually substituted (i.e. the A_i's which are not in K_x) are in some S_y. Since $S_x \supseteq S_y$ for every $y \preceq x$, they all are in some S_z with z maximal with respect to $\preceq$. The same idea of (a) shows these S_z's are disjoint. ■

Proposition 1.9 *Let x be an element of a cancellative monoid X and let $x = x_1 + x_2$ be a decomposition of x. If x_1, x_2 have respectively the $\aleph$-semiexchange property with respect to m_1 and the $\aleph$-semiexchange property with respect to m_2, then x has the $\aleph$-semiexchange property with respect to $m_1 m_2$.*

Proof. Suppose

$$a = x_1 + x_2 + y = \sum_{i \in I} a_i.$$

There is a partition $I = \dot{\bigcup}_{j \in J} I_j$ with $|I_j| \leq m_1$ for any $j \in J$ and decompositions $\sum_{i \in I_j} a_i = b_j + c_j$, $j \in J$, such that $\sum_{i \in I} a_i = x_1 + \left(\sum_{j \in J} b_j\right)$. By the cancellativity of X we have $x_2 + y = \sum_{j \in J} b_j$ and, by the $\aleph$-semiexchange property with respect to m_2 of x_2, there is a partition $J = \dot{\bigcup}_{k \in K} J_k$ with $|J_k| \leq m_2$ for any $k \in K$ and decompositions $\sum_{j \in J_k} b_j = d_k + e_k$, $k \in K$, such that

$$a = \sum_{i \in I} a_i = x_1 + x_2 + \left(\sum_{k \in K} d_k\right).$$

Therefore x has the $\aleph$-semiexchange property with respect to $m_1 m_2$. ■

Let us now turn our attention to free monoids and Krull monoids. The reason why the exchange property is "a natural property to ask to the modules belonging to a class $\mathcal{C}$ for $V(\mathcal{C})$ to be a free monoid" is that a monoid is free if and only if it is atomic and all its elements have the finite exchange property.

Remark 1.10 *A atomic monoid is free if and only if all its elements have the finite exchange property.*

In fact, let F be a free monoid and let $x, y, a_1, a_2, \dots, a_n$ be elements of F such that $x + y = \sum_{i=1}^{n} a_i$. By Proposition 1.9 it is sufficient to assume x indecomposable. Being F free, there exist $a_{1,1}, a_{1,2}, \dots, a_{1,t_1}, a_{2,1}, a_{2,2}, \dots, a_{2,t_2}, \dots, a_{n,1}, a_{n,2}, \dots, a_{n,t_n}$ indecomposable elements of F such that $a_i = a_{i,1} + a_{i,2} + \dots + a_{i,t_i}$ $(i = 1, 2, \dots, t_i)$. Moreover there are k, h such that $x = a_{k,h}$, so that $x \leq a_k$ and x has the finite exchange property.

Conversely, if every element $x \in F$ has the exchange property, it is easy to see that, if $a = a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_m$ where the a_i's and the b_j's are indecomposable, one has $m = n$ and $a_i = b_i$ after a suitable rearrangement of the indices. This is equivalent to the fact that F is free (this is very well known, see for example [8, p. 7]).

This naturality, however, seems to disappear in the Krull case, at least for monoids, as the next example shows. It is recovered, however, for classes of modules (see Corollary 2.7).

Proposition 1.11 *If x is an element of a Krull monoid X, then x has the finite semiexchange property with respect to m for some m.*

Proof. Let X be a Krull monoid, let I be a set, let $\varphi \colon X \to \mathbb{N}^{(I)}$ be a divisor monoid homomorphism and let x be an element of X. Again by Proposition 1.9 it is sufficient to think x indecomposable. Let n be a positive integer and let $y, a_1, a_2, \dots, a_n \in X$ such that $x + y = a_1 + a_2 + \dots + a_n$. Let $x_1, x_2, \dots, x_m$ be indecomposable elements of $\mathbb{N}^{(I)}$ such that $\varphi(x) = x_1 + x_2 + \dots + x_m$. Since x_i has the finite exchange property for every i, one has $x_i \leq \varphi(a_{j[i]})$ for some $j[i]$, so that $\varphi(x) \leq \sum_{i=1}^{m} \varphi(a_{j[i]})$. Since φ is a divisor homomorphism one has $x \leq \sum_{i=1}^{m} a_{j[i]}$. ∎

Remark 1.12 Remark 1.10 and Proposition 1.11 are, by remark 1.6, easy corollaries of [7, Proposition 7.1.9].

Example 1.13 *There exists a non-Krull atomic monoid which is not a Krull monoid and whose elements have the finite semiexchange property with respect to m for some integer m depending on the element.*

Consider the indecomposable elements of the monoid $M = \mathbb{N}_{\geq 2} = \{2, 3, 4, \dots\}$. It is clear that every element $m \in M$ has the semiexchange property with respect to 3. Nevertheless the monoid M is not a Krull monoid since it is not even integrally closed ([8, Theorem 22.8]).

Proposition 1.14 *Every module has the m-semiexchange property with respect to m. If a module has the $(m+1)$-semiexchange property with respect to m, then it has the finite semiexchange property with respect to m.*

Proof. Obviously every module has the m-semiexchange property with respect to m. We will show that, for every $n > m$, if M has the n-semiexchange property with respect to m then it has the $(n+1)$-semiexchange property with respect to m. In fact if

$$M \oplus N = \bigoplus_{i=1}^{n+1} A_i,$$

then $M \oplus N = \bigoplus_{i=1}^{n} B_i$ where $B_i = A_i$ for $i = 1, 2, \dots, n-1$ and $B_n = A_n \oplus A_{n+1}$. Thus, there exists a partition $\{1, 2, \dots, n\} = \dot{\bigcup}_{j \in J} I_j$ with $|I_j| \leq m$ for any $j \in J$ and decompositions $\bigoplus_{i \in I_j} B_i = C_j \oplus C'_j$, $j \in J$, such that

$$M \oplus N = M \oplus \left(\bigoplus_{j \in J} C_j \right).$$

One has $B_n \in I_{j_0}$ for some index j_0. Set $I'_j = I_j$ for every $j \neq j_0$ and $I'_{j_0} = I_{j_0} \cup \{n+1\}$. If $|I_{j_0}| < m$ we are done. If $|I_{j_0}| = m$, then $\left| I'_{j_0} \right| = m+1$. Since C'_{j_0} is a direct summand of M, it has the n-semiexchange property with respect to m and, since $n > m$, it has the $(m+1)$-semiexchange property with respect to m. Now $\bigoplus_{i \in I'_{j_0}} A_i = C_{j_0} + C'_{j_0}$, so that there is a partition $I'_{j_0} = \dot{\bigcup}_{j \in J'} I'_j$ with $\left| I'_j \right| \leq m$ for any $j \in J'$ and decompositions $\bigoplus_{i \in I'_j} A_i = D_j + D' j$, $j \in J'$, such that

$$\bigoplus_{i \in I'_{j_0}} A_i = C_{j_0} + C'_{j_0} = C_{j_0} + \bigoplus_{j \in J'} D_j,$$

so that

$$M \oplus N = M \oplus \left(\bigoplus_{j \in J} C_j \right) = M \oplus \left(\bigoplus_{j \in J \setminus \{j_0\}} C_j \right) \oplus \left(\bigoplus_{j \in J'} D_j \right),$$

and we are done. ■

2 Modules with semilocal endomorphism rings

In this section we investigate the link between the semiexchange property of a module and the dual Goldie dimension of its endomorphism ring. For the definition and the basic properties of the dual Goldie dimension of a module we refere the reader to [3, chapter 2]. For our pourposes the main thing we should keep in mind is that a ring R is semilocal if and only if the regular module R_R has finite dual Goldie dimension and this dimension turns out to be the length of the right semisimple module $R/J(R)$. The corresponding left-hand condition holds as well.

We start with the following Lemma which is a restatement and rearrangement of Lemma 1.3 and Proposition 1 of [10].

Lemma 2.1 *Let A be a module and let M_1, M_2, M be submodules of A such that $A = M \oplus M_1$. Let $\pi_1 \colon A \to M_1$ be the canonical projection with respect to this decomposition and $\varepsilon_i \colon M_i \to A$ be the embeddings for $i = 1, 2$. Then:*

(1) *one has* $A = M \oplus M_2$ *if and only if there is a homomorphism* $\pi_2 \colon A \to M_2$ *such that* $\pi_1\varepsilon_2\pi_2\varepsilon_1 = \mathrm{id}_{M_1}$ *and* $\pi_2\varepsilon_1\pi_1\varepsilon_2 = \mathrm{id}_{M_2} = \pi_2\varepsilon_2$*;*

(2) *there exists a direct summand* M' *of* M_1 *such that* $A = M_2 \oplus M' \oplus M$ *if and only if there is an epimorphism* $\pi_2 \colon A \to M_2$ *such that* $\mathrm{id}_{M_2} = \pi_2\varepsilon_2$ *and* $\pi_2\varepsilon_1\pi_1\varepsilon_2$ *is an isomorphism.*

Proof. (1) Follows from Lemma 1.3. If $A = M \oplus M_2$, then, by Lemma 1.3, $\pi_1\varepsilon_2$ is an isomorphism and the projection onto M_2 associated to this decomposition is $\pi_2 = (\pi_1\varepsilon_2)^{-1}\pi_1$. Hence $\pi_1\varepsilon_2\pi_2 = \pi_1$, so that $\pi_1\varepsilon_2\pi_2\varepsilon_1 = \pi_1\varepsilon_1 = \mathrm{id}_{M_1}$. Similarly from $\pi_2 = (\pi_1\varepsilon_2)^{-1}\pi_1$ we get $\pi_2\varepsilon_1\pi_1\varepsilon_2 = (\pi_1\varepsilon_2)^{-1}\pi_1\varepsilon_1\pi_1\varepsilon_2 = \mathrm{id}_{M_2} = \pi_2\varepsilon_2$.

Conversely, if $\pi_1\varepsilon_2\pi_2\varepsilon_1 = \mathrm{id}_{M_1}$ and $\pi_2\varepsilon_1\pi_1\varepsilon_2 = \mathrm{id}_{M_2}$, then it is clear that $\pi_1\varepsilon_2$ is an isomorphism. Hence, again by Lemma 1.3, we get $A = M \oplus M_2$.

(2) Suppose there is an epimorphism π_2 such that $\mathrm{id}_{M_2} = \pi_2\varepsilon_2$ and $\pi_2\varepsilon_1\pi_1\varepsilon_2$ is an isomorphism. One has $A = M_2 \oplus \ker(\pi_2)$ and π_2 is the canonical projection onto M_2 associated with this decomposition.

Let H be the image of the homomorphism $\varepsilon_1\pi_1\varepsilon_2 \colon M_2 \to A$. Since $\pi_2\varepsilon_1\pi_1\varepsilon_2$ is an isomorphism, $\pi_2|_H \colon H \to M_2$ is an isomorphism as well and $A = H \oplus \ker(\pi_2)$ by Lemma 1.3. The projection onto H relative to this decomposition is $\pi_H = (\pi_2|_H)^{-1}\pi_2$. Now $H = \varepsilon_1\pi_1\varepsilon_2(M_2) \subseteq M_1 \subseteq H \oplus \ker(\pi_2)$, so that, by Proposition 1.1, $M_1 = H \oplus M'$ where $M' = M_1 \cap \ker(\pi_2)$. The projection $\pi'_H \colon M_1 \to H$ relative to this decomposition is $\pi'_H = (\pi_2|_H)^{-1}\pi_2|_{M_1}$. Thus

$$A = M_1 \oplus M = H \oplus M' \oplus M$$

with projection $\pi''_2 \colon A \to H$, where $\pi''_2 = (\pi_2|_H)^{-1}\pi_2|_{M_1}\pi_1 = (\pi_2|_H)^{-1}\pi_2\varepsilon_1\pi_1$, which is, when restricted to M_2, the map $(\pi_2|_H)^{-1}\pi_2\varepsilon_1\pi_1\varepsilon_2$, hence it is an isomorphism. Therefore, again by Lemma 1.3,

$$A = M_2 \oplus M' \oplus M.$$

Conversely, if there exists a direct summand M' of M_1 such that $A = M_2 \oplus M' \oplus M$, then, by (1), there is an epimorphism $\pi : A \to M_2 \oplus M'$ such that $\pi\varepsilon_1\pi_1\varepsilon_{M_2\oplus M'} = \mathrm{id}_{M_2\oplus M'}$ and $\pi\varepsilon_{M_2\oplus M'} = \mathrm{id}_{M_2\oplus M'}$. Therefore, if we denote by π'_2 the canonical projection $M_2 \oplus M' \to M_2$ with kernel M' and we define $\pi_2 = \pi'_2\pi$, we get $\pi_2\varepsilon_1\pi_1\varepsilon_2 = \pi'_2\pi\varepsilon_1\pi_1\varepsilon_{M_2\oplus M'}\varepsilon_2|^{M_2\oplus M'} = \pi'_2\mathrm{id}_{M_2\oplus M'}\varepsilon_2|^{M_2\oplus M'} = \mathrm{id}_{M_2}$ and $\pi_2\varepsilon_2 = \pi'_2\pi\varepsilon_{M_2\oplus M'}\varepsilon_2|^{M_2\oplus M'} = \pi'_2\mathrm{id}_{M_2\oplus M'}\varepsilon_2|^{M_2\oplus M'} = \mathrm{id}_{M_2}$, and this completes the proof. ∎

Lemma 2.2 *Let* R *be a ring, let* J *be its Jacobson radical and let* P, Q *be two projective finitely generated right modules. Let* $\pi_P \colon P \to P/PJ$ *and* $\pi_Q \colon Q \to Q/QJ$ *be the canonical projections. For each* $f \colon P \to Q$ *there is a unique morphism* $\bar{f} \colon P/PJ \to Q/QJ$ *such that* $\pi_Q f = \bar{f}\pi_P$ *and for every* $g \colon P/PJ \to Q/QJ$ *there is a morphism* $\underline{g} \colon P \to Q$ *such that* $\pi_Q\underline{g} = g\pi_P$*. Moreover:*

- *for each* $f \colon P \to Q$*,* f *is an epimorphism if and only if* $\bar{f}$ *is an epimorphism;*
- *for each* $g \colon P/PJ \to Q/QJ$*,* $\overline{(\underline{g})} = g$*.*

- *for each $f\colon P \to Q$, if f is an isomorphism then $\bar{f}$ is an isomorphism;*

Finally, if J is superfluous in R or P is an indecomposable projective module, then $f\colon P \to Q$, f is an isomorphism if and only if $\bar{f}$ is an isomorphism.

PROOF. Let $f\colon P \to Q$ be an homomorphism. Since $QJ \supseteq f(P)J = f(PJ)$, one has $\ker \pi_Q f \supseteq \ker \pi_P$ so that, by the factor theorem [1, Theorem 3.6], there exists a unique homomorphism $\bar{f}\colon P/PJ \to Q/QJ$ such that $\pi_Q f = \bar{f}\pi_P$ and f is an epimorphism if and only if $\bar{f}$ is an epimorphism. Moreover, if f is an isomorphism, then $\ker \pi_P = \ker \pi_Q f$ and $\bar{f}$ is also injective.

Let now $g\colon P/PJ \to Q/QJ$ be a homomorphism. Since P is projective and π_Q is an epimorphism, there is an homomorphism $\underline{g}\colon P \to Q$ such that $\pi_Q \underline{g} = g\pi_P$.

Obviously (denoting by $[x]$ the equivalence class of x in X/XJ) one has $g[x] = [\underline{g}(x)] = \overline{(\underline{g})}[x]$ for every $[x] \in P/PJ$.

If $\bar{f}$ is an isomorphism and J is superfluous in R, then f is an epimorphism and $(\ker f + PJ)/PJ = 0$, so that $\ker f \subseteq PJ$ which is a superfluous submodule of P. Now $\ker f$ is a direct summand of P, so that $\ker f = 0$ and f is an isomorphism.

Finally, if $\bar{f}$ is an isomorphism and P is an indecomposable projective module, then, being $\ker f$ a direct summand of P, the kernel of f is either equal to 0 or equal to P. But f is surjective, so that $\ker f$ cannot be equal to P. Hence $\ker f = 0$ and f is an isomorphism. ∎

The next Lemma is a collection of bits and pieces from [4, proof of Lemma 3.1] and [6, Lemma 2.1]. We decided to state and prove it since we did not find the whole, natural statement anywhere in the literature.

Lemma 2.3 *Let R be a ring, e be an idempotent in R and $J(R)$ be the Jacobson radical. Then eRe is semilocal of dual Goldie dimension n if and only if $eR/eJ(R)$ is a semisimple $R/J(R)$-module of composition length n.*

PROOF. Suppose that eRe is semilocal of dual Goldie dimension n, that is $eRe/J(eRe) \cong (e + J(R))R/J(R)(e + J(R))$ is a semisimple Artinian ring of Goldie dimension n. Then, by [6, Lemma 2.1], $eR/eJ(R) \cong (e+J(R))R/J(R)$ is a semisimple $R/J(R)$-module of composition length n.

Conversely, if $eR/eJ(R)$ is a semisimple $R/J(R)$-module of composition length n, say $eR/eJ(R) \cong \bigoplus_{i=1}^m S_i^{j_i}$ for some pair-wise non-isomorphic simples S_i, then $\mathrm{End}_R(eR/eJ(R)) \cong eRe/eJ(R)e$ is isomorphic to the direct product $\prod_{i=1}^m \mathbb{M}_{j_i}(\mathrm{End}_R(S_i))$, where $\mathbb{M}_\alpha(S)$ denotes the ring of $\alpha \times \alpha$ matrices with coefficients in the ring S. As each $\mathbb{M}_{j_i}(\mathrm{End}_R(S_i))$ is a direct sum $T_i^{j_i}$ of j_i isomorphic simple modules T_i, it follows that $eRe/eJ(R)e = eRe/J(eRe) \cong \bigoplus_{i=1}^m T_i^{j_i}$ for some pair-wise non-isomorphic simples T_i. ∎

We are now ready to prove the main results about the semiexchange property, that is to say to link the semiexchange property of an indecomposable module to the dual Goldie dimension of its endomorphism ring. The link is as strict as one may wish, in the sense that the dual Goldie dimension of the endomorphism ring of an indecomposable module M is m if and only if M has the (finite) semiexchange property with respect to m and it does not have the (finite) semiexchange property with respect to $m-1$.

Theorem 2.4 *Let M be an indecomposable module whose endomorphism ring has finite dual Goldie dimension m. Then M has the semiexchange property with respect to m.*

Proof. Let $G, M', N, A_i (i \in I)$ be modules such that M' is isomorphic to M and $G = M' \oplus N = \bigoplus_{i \in I} A_i$ and let $\varepsilon : M' \to G$, $\pi : G \to M'$, $\varepsilon_i : A_i \to G$, $\pi_i : G \to A_i$ be the inclusions and the projections relative to these decompositions. Let R, R_i be the endomorphism rings of M', A_i respectively. Let $J(S)$ denote the jacobson radical of a ring S and let $R/J(R)$ be the direct sum $S_1 \oplus S_2 \oplus \ldots \oplus S_m$ where the S_j's are simple modules. We denote by F the natural category equivalence $\mathrm{Hom}_R(G, -)$: add-$G \to$ proj-End(G) given by, for every idempotent $e \in \mathrm{End}(G)$, the corrispondence of the object $e\mathrm{End}(G)$ of proj-End(G) to the direct summand eG of G_R, which is an object of add(G_R) (see [3, Theorem 4.7]). Let J denote $J(\mathrm{End}(G))$, let $e = F(\varepsilon), p = F(\pi), e_i = F(\varepsilon_i), p_i = F(\pi_i)$, for every $P, Q \in$proj-End(G) and every $f : P \to Q$ let $\overline{f} : P/PJ \to Q/QJ$ be the morphism induced in the category proj-End$(G)/J(\mathrm{End}(G))$ and for every $f : P/PJ \to Q/QJ$ in the category proj-End$(G)/J(\mathrm{End}(G))$ let $\underline{f} : P \to Q$ be a lifting of f. Finally, let $\overline{p}_{S_i} : R/J(R) \to S_i$ and $\overline{e}_{S_i} : S_i \to R/J(R)$ be the inclusions and the natural projections associated to the given decomposition of R/RJ.

By Lemma 2.3, one has $F(G)/F(G)J = T_1 \oplus T_2 \oplus \ldots \oplus T_m \oplus F(N)/F(N)J = \bigoplus_{i \in I} F(A_i)/F(A_i)J$. Since the T_j's have the exchange property, there are indices $i_1, i_2, \ldots, i_m \in I$ and a direct summand B of $\bigoplus_{j=1}^m F(A_{i_j})/F(A_{i_j})J$ such that $F(G)/F(G)J = T_1 \oplus T_2 \oplus \ldots \oplus T_m \oplus B \oplus \bigoplus_{i \in I \setminus \{i_1, \ldots, i_m\}} F(A_i)/F(A_i)J$ or, equivalently by Lemma 2.1, there is an epimorphism $t\colon F(G)/F(G)J \to T_1 \oplus T_2 \oplus \ldots \oplus T_m$ such that $t\overline{e}_{i_1 \oplus \ldots \oplus i_m} \overline{p}_{i_1 \oplus \ldots \oplus i_m} \overline{e}$ and $t\overline{e}$ are isomorphisms.

Therefore the morphism $\tau = F^{-1}(\underline{t})\colon G \to M$ is surjective and $\tau \varepsilon_{i_1 \oplus \ldots \oplus i_m} \pi_{i_1 \oplus \ldots \oplus i_m} \varepsilon$ and $\beta = \tau\varepsilon$ are isomorphisms by Lemma 2.2. Setting $\pi_{M'} = \beta^{-1}\tau$, one has that $\pi_{M'} \varepsilon_{i_1 \oplus \ldots \oplus i_m} \pi_{i_1 \oplus \ldots \oplus i_m} \varepsilon$ is an isomorphism and $\pi_{M'}\varepsilon = \mathrm{id}_{M'}$, so that by Lemma 2.1 $G = M' \oplus X \oplus \bigoplus_{i \in I \setminus \{i_1, \ldots, i_m\}} A_i$ for some direct summand X of $\bigoplus_{j=1}^m A_{i_j}$. Hence the conclusion. ∎

Theorem 2.5 *Let m be a positive integer and M be an indecomposable module whose endomorphism ring has dual Goldie dimension greater or equal to m (possibly infinite). Then M does not have the finite semiexchange property with respect to $m - 1$.*

Proof. Let M be a module whose endomorphism ring R has dual Goldie dimension greater or equal to m. This means that in the regular module R_R there is a set of m coindependent modules $\{A_1, A_2, \ldots, A_m\}$. We can consider, without loss of generality, that these modules are maximal right ideals.

If J is the intersection $A_1 \cap A_2 \cap \ldots \cap A_m$, then R/J is the direct sum of m simple modules $R/J_R = S_1 \oplus S_2 \oplus \ldots \oplus S_m$. Now let $M_1, M_2, \ldots, M_m$ be m modules isomorphic to M and set $G = M_1 \oplus M_2 \oplus \ldots \oplus M_m$. For every $i = 1, 2, \ldots, m$ denote by ε_i the inclusion $M_i \to G$ and by π_i the projection $G \to M_i$ relative to this decomposition. Denote by $R^{(i)}, J^{(i)}, S_1^{(i)}, S_2^{(i)}, \ldots, S_m^{(i)}$ the endomorphism ring of M_i, the intersection of the coindependent modules in $R^{(i)}$ and the simples summing up to $R^{(i)}/J^{(i)}$ respectively.

Let $F\colon$ add-M $\to$proj-R be the category equivalence described in Theorem 2.4. Set $p_i = F(\pi_i)$ and $e_i = F(\varepsilon_i)$ for every index $i = 1, 2, \ldots, m$ and finally, for every homomorphism $f\colon P \to Q$, let $\overline{f}\colon P/PJ \to Q/QJ$ be the map induced by f.

Consider the morphisms

$$\alpha\colon \bigoplus_{i=1}^{m} \left(S_1^{(i)} \oplus S_2^{(i)} \oplus \ldots \oplus S_m^{(i)}\right) \to S_1 \oplus S_2 \oplus \ldots \oplus S_m$$

given by

$$\alpha(s_1^{(1)}, s_2^{(1)}, \ldots, s_m^{(1)}, s_1^{(2)}, s_2^{(2)}, \ldots, s_m^{(2)}, \ldots, s_1^{(m)}, s_2^{(m)}, \ldots, s_m^{(m)}) = (s_1^{(1)}, s_2^{(2)}, \ldots, s_m^{(m)})$$

and

$$\beta\colon S_1 \oplus S_2 \oplus \ldots \oplus S_m \to \bigoplus_{i=1}^{m} \left(S_1^{(i)} \oplus S_2^{(i)} \oplus \ldots \oplus S_m^{(i)}\right)$$

given by

$$\beta(s_1, s_2, \ldots, s_m) = (s_1, 0, \ldots, 0, 0, s_2 \ldots, 0, \ldots, 0, 0, \ldots, s_m).$$

Obviously $\alpha\beta$ is the identity of R/J, and it can be lifted to a morphism $\underline{\beta}\underline{\alpha}\colon R \to \oplus_{i=1}^{m} R^{(i)} \to R$ which is an isomorphism since R is an indecomposable projective module. Therefore the morphism $F^{-1}(\alpha)F^{-1}(\beta)\colon M \to G \to M$ is an isomorphism as well. Hence $M' = F^{-1}(\beta)(M)$ is a direct summand of G isomorphic to M. Assume, by way of contraddiction, that M has the finite semiexchange property with respect to $m-1$. Since M is indecomposable, according to Lemma 1.4 there is a subset $I \subset \{1, \ldots, m\}$ such that M' can be substituted to $\oplus_{i \in I} M_i$ i.e., there exists an epimorphism $\pi\colon G \to M'$ such that the morphism $\sum_{i \in I} \pi\varepsilon_i\pi_i\varepsilon_{M'} = \pi_{M'}\varepsilon_{\oplus_{i\in I}M_i}\pi_{\oplus_{i\in I}M_i}\varepsilon_{M'}$ is an isomorphism. Thus the morphism $\sum_{i\in I} \overline{F(\pi)}\bar{e}_i\bar{p}_i\bar{e}_{M'}$ is an isomorphism.

But it is clear that, for every and every epimorphism $p\colon F(G)/F(G)J(\mathrm{End}G) \to F(M')/F(M')J(\mathrm{End}G)$, the morphism $\sum_{i\in I} p\bar{e}_i\bar{p}_i\bar{e}_{M'}$ is not an isomorphism and this yelds a contraddiction.

Thus M does not have the finite semiexchange property with respect to $m-1$. ■

We can sum up the previous results as this theorem.

Theorem 2.6 *For an indecomposable module M and for a positive integer m the following are equivalent:*

(a) *the endomorphism ring of M has dual Goldie dimension m;*

(b) *the module M has the finite semiexchange property with respect to m but it does not have the finite semiexchange property with respect to $m-1$;*

(c) *the module M has the semiexchange property with respect to m but it does not have the semiexchange property with respect to $m-1$.*

This naturally implies that, if the endomorphism ring of M has infinite dual Goldie dimension, then M does not have the finite semiexchange property with respect to m for any positive integer m.

By [4, Theorem 3.4] we get for free the already mentioned "come back of naturality".

Corollary 2.7 *Let $\mathcal{C}$ be an* add*-close class of modules such that every $C \in \mathcal{C}$ is a finite direct sum of indecomposable modules and has the semiexchange property with respect to n for some n depending on C. Then $V(\mathcal{C})$ is a Krull monoid.*

3 Weak Krull-Schmidt Theorem for biuniform modules

In this section we show an application of the semiexchange property which has been one of the motivations behind its definition.

In 1996 A. Facchini proved a weak version of the Krull-Schmidt theorem for biuniform modules. Direct sums of biuniform modules do not decompose in a unique way as direct sum of indecomposables up to a permutation and up to isomorphism. However, they decompose in a unique way up to two permutations and up to monogeny and epigeny (recall that two modules A and B are said to be in the same monogeny class, in notation $[A]_m = [B]_m$, if there exist monomorphisms from A to B and viceversa, and, dually, they are said to be in the same epigeny class, in notation $[A]_e = [B]_e$, if there exist an epimorphism form A to B and an epimorphism from B to A; both are equivalence relations).

We will prove a version of the Weak Krull-Schmidt theorem for finite direct sums of biuniform modules which is stronger then the usual one proved by Facchini in [3]. In particular it is a closer generalization of the Krull-Schmidt thoerem as stated for example in [1, Theorem 12.9].

Before stating the main result it could be useful to recall some facts about biuniform modules.

(1) [3, Corollary 4.16] The endomorphism ring of a biuniform module has dual Goldie dimension ≤ 2, so that any biuniform module has the semiexchange property with respect to 2.

(2) [3, Lemma 9.8] Let A, B, C, D be biuniform modules such that $A \oplus B \cong C \oplus D$. Then $\{[A]_m, [B]_m\} = \{[C]_m, [D]_m\}$ and $\{[A]_e, [B]_e\} = \{[C]_e, [D]_e\}$.

(3) [3, Lemma 9.2(b)] If $f_1, \dots, f_n \colon A \to B$ are n homomorphisms and $f_1 + \dots + f_n$ is an isomorphism, then either one of the f_i is an isomorphism or there exist two distinct indices $i, j = 1, 2, \dots, n$ such that f_i is injective and not surjective, and f_j is surjective and not injective.

Theorem 3.1 (Weak Krull-Schmidt Theorem for biuniform modules) *Let $M_1, \dots, M_n, N_1, \dots, N_m$ be biuniform modules. If*

$$G = M_1 \oplus \dots \oplus M_n = N_1 \oplus \dots \oplus N_m,$$

then $m = n$, there are two permutations σ, τ of $\{1, 2, \dots, n\}$ and there are modules $B_1, B_2, \dots, B_n$ such that

(1) *for every $i = 1, 2, \dots, n$ we have*

$$G = M_{\sigma(1)} \oplus \ldots \oplus M_{\sigma(i-1)} \oplus B_i \oplus N_{\tau(i+1)} \oplus \ldots \oplus N_{\tau(n)};$$

(2) *if we set* $\varphi = \sigma^{-1}\tau$ *and* $\psi(i) = \sigma^{-1}\tau(i+1)$ *for every* $i = 1, 2, \ldots, n-1$ *and* $\psi(n) = \sigma^{-1}\tau(1)$*, then for every* $i = 1, 2, \ldots, n$ *we get*

$$[M_i]_m = [N_{\varphi(i)}]_m \quad \text{and} \quad [M_i]_e = [N_{\psi(i)}]_e.$$

PROOF. First of all note that $n = m$ is obvious since $n = \dim G = m$.

For every direct sum decomposition $X = Y \oplus Z$ define $\varepsilon_Y : Y \to X$ to be the embedding and $\pi(Z)_Y : X \to Y$ to be the canonical projection. We will often write π_Y instead of $\pi(Z)_Y$ if there is no possibility of confusion. Throughout the proof we will use the composite morphisms $\pi_{M_i}\varepsilon_{N_j}\pi_{N_j}\varepsilon_{M_i}$. Note that, as M_i and N_j are biuniform, the morphism $\pi_{M_i}\varepsilon_{N_j}\pi_{N_j}\varepsilon_{M_i}$ is surjective (injective) if and only if both $\pi_{M_i}\varepsilon_{N_j}$ and $\pi_{N_j}\varepsilon_{M_i}$ are surjective (injective) if and only if $\pi_{N_j}\varepsilon_{M_i}\pi_{M_i}\varepsilon_{N_j}$ is surjective (injective) (see [3, Lemma 6.26]).

Let I be the set $\{j | j = 1, 2, \ldots, n, \exists i(\pi_{N_j}\varepsilon_{M_i}\pi_{M_i}\varepsilon_{N_j})$ is an isomorphism$\}$. If $i \in I$, then by Lemma 2.1(2) one has $G = M_j \oplus \left(\bigoplus_{\ell \neq i} N_\ell\right)$, so $\pi_{M_j}\varepsilon_{N_\ell} = 0$ for every $\ell \neq i$.

Finally define $B_1 = N_1$, $\alpha = 1$ and $\sigma_1 = \mathrm{id}_{S_n} = \tau_1$.

With all this in mind we can proceed step by step along the index i.

While $0 \leq i < n$, procede as follows: thanks to the previous step we already got

$$G = M_{\sigma_i(1)} \oplus \ldots \oplus M_{\sigma_i(i-1)} \oplus B_i \oplus N_{\tau_i(i+1)} \oplus \ldots \oplus N_{\tau_i(n)}.$$

There are two possibilities: either there is no ℓ such that

$$\pi(M_{\sigma_i(1)} \oplus \ldots \oplus M_{\sigma_i(i-1)} \oplus N_{\tau_i(i+1)} \oplus \ldots \oplus N_{\tau_i(n)})_{B_i}\varepsilon_{M_\ell}\pi(\oplus_{h \neq \ell} M_h)_{M_\ell}\varepsilon_{B_i}$$

is an isomorphism or there is such an ℓ.

In the first case, since $\sum_{j=1}^n \pi_{B_i}\varepsilon_{M_j}\pi_{M_j}\varepsilon_{B_i}$, by Fact (3) there is an index h such that $\pi_{B_i}\varepsilon_{M_h}\pi_{M_h}\varepsilon_{B_i}$ is injective and not surjective. Note that (a) $h \notin \{\sigma_i(1), \ldots, \sigma_i(i-1)\}$ (since $\pi_{M_{\sigma_i(\ell)}}\varepsilon_{B_i} = 0$ for every $\ell = 1, \ldots, i-1$) and (b) $h \notin I$ (for the same reason). By (a) there is a permutation $\sigma_{i+1} \in S_n$ such that $\sigma_{i+1}(\ell) = \sigma_i(\ell)$ for every $\ell = 1, \ldots, i-1$ and $\sigma_{i+1}(i) = h$. By (b) there is a module $X \in \{M_{\sigma_i(1)}, \ldots, M_{\sigma_i(i-1)}, B_i, N_{\tau_i(i+1)}, \ldots, N_{\tau_i(n)}\}$ such that $\pi_{M_h}\varepsilon_X\pi_X\varepsilon_{M_h}$ is surjective and non-injective. Now we have $X \neq M_{\sigma_i(1)}, \ldots, M_{\sigma_i(i-1)}$, because $\pi_{M_{\sigma_i(\ell)}}\varepsilon_{M_h} = 0$ for every $\ell = 1, \ldots, i-1$. Moreover $X \neq B_i$ since $\pi_{B_i}\varepsilon_{M_h}\pi_{M_h}\varepsilon_{B_i}$ is injective and not surjective. Therefore $X \in \{N_{\tau_i(i+1)}, \ldots, N_{\tau_i(n)}\}$, say $X = N_k$. Hence there is a permutation $\tau_{i+1} \in S_n$ such that $\tau_{i+1}(\ell) = \tau_i(\ell)$ for every $\ell = 1, 2, \ldots, i$ and that $\tau_{i+1}(i+1) = k$. Finally, by Fact (1) and Lemma 1.2, there is a module $B_{i+1} \subseteq B_i \oplus N_k$ such that

$$G = M_{\sigma_{i+1}(1)} \oplus \ldots \oplus M_{\sigma_{i+1}(i)} \oplus B_{i+1} \oplus N_{\tau_{i+1}(i+2)} \oplus \ldots \oplus N_{\tau_{i+1}(n)}.$$

Note that, by Fact (2), one has $[B_{i+1}]_m = [N_{\tau_{i+1}(i+1)}]_m$, $[B_{i+1}]_e = [B_i]_e$, $[M_{\sigma_{i+1}(i)}]_e = [N_{\tau_{i+1}(i+1)}]_e$ and $[B_i]_m = [M_{\sigma_{i+1}(i)}]_m$.

In the latter case note $\ell \notin \{\sigma_i(1), \ldots, \sigma_i(i-1)\}$ since $\pi_{M_{\sigma_i(k)}}\varepsilon_{B_i} = 0$ for every $k = 1, \ldots, i-1$. Thus there is a permutation $\sigma_{i+1} \in S_n$ such that $\sigma_{i+1}(k) = \sigma_i(k)$ for every $k = 1, \ldots, i-1$ and $\sigma_{i+1}(i) = \ell$. For the usual reason there is a permutation $\tau_{i+1} \in S_n$ such that $\tau_{i+1}(\ell) = \tau_i(\ell)$ for every $\ell = 2, \ldots, i$ and that $\tau_{i+1}(i+1) = \alpha$. Note that one has $[B_i]_m = [M_{\sigma_{i+1}(i)}]_m$, $[B_i]_e = [M_{\sigma_{i+1}(i)}]_e$ and $N_\alpha = N_{\tau_{i+1}(i+1)}$. Reset $\alpha \colon = \tau_{i+1}(1)$ and set $B_{i+1} = N_{\tau_{i+1}(1)}$. By Fact (1) and Lemma 1.2 we get

$$G = M_{\sigma_{i+1}(1)} \oplus \ldots \oplus M_{\sigma_{i+1}(i)} \oplus B_{i+1} \oplus N_{\tau_{i+1}(i+2)} \oplus \ldots \oplus N_{\tau_{i+1}(n)}.$$

Note that one has $[B_{i+1}]_m = [N_\alpha]_m$ and $[B_{i+1}]_e = [N_\alpha]_e$.

Finally compute the n-th step to check the epigeny and monogeny classes of $[N_{\tau_n(n)}]$ without defining neither B_{n+1}, σ_{n+1} nor τ_{n+1}.

To conclude it is enough to run through the n steps, set $\sigma = \sigma_n$ and $\tau = \tau_n$ and check the monogeny classes and the epigeny classes of the modules M_i, N_i and B_i. ■

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