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Introduction

The idea of decomposing a module as a direct sum of submodules is as old as Module Theory. Indeed, most of the information about a module can be retrieved by knowing one of its decompositions into direct summands and knowing these direct summands: the submodule lattice, several invariants and dimensions associated to the module (among which the Krull dimension, the Goldie and the dual Goldie dimension, the homological dimensions as the projective and the injective dimension), various functors associated to the module (among which Hom, the tensor product, Ext and Tor) can be easily calculated provided we have the corresponding information about its direct summands.

A milestone in this technique is the Krull-Schmidt-Azumaya theorem, which gives sufficient conditions for a module to have an essentially unique decomposition as a direct sum of indecomposable submodules. A lot of work has been done over the years to extend as far as possible this theorem and to see wether particular classes of modules have essentially unique decomposition.

Recently, though, the attention has been pointed in another direction. Instead of looking for other "very good" classes of modules, a great deal of attention has been posed on "good" classes of modules and on ways to measure how different is "good" from "very good". Namely, for every full subcategory $\mathcal{C}$ of Mod-R, a reduced commutative monoid $V(\mathcal{C})$ carrying all the information about direct sum decompositions in $\mathcal{C}$ has been considered. The elements of $V(\mathcal{C})$ are the isomorphism classes $\langle A\rangle$ of the modules A in $\mathcal{C}$ and the sum is given by $\langle A\rangle + \langle B\rangle = \langle A \oplus B\rangle$ for every $A, B \in \mathcal{C}$.

It is clear that the Krull-Schmidt theorem holds in $\mathcal{C}$ if and only if the monoid $V(\mathcal{C})$ is free, the point being we can consider weaker, though controllable, conditions, such as the monoid $V(\mathcal{C})$ being a Krull monoid.

This thesis aims to present a (definitely not comprehensive) collection of recent results in the field obtained by the author and others. There are some well-known theorems and techniques, some results among the very recent (hence not well-known) results and some aside results which show how the literature is, as it is natural, full of related material that could help spreading our comprehension and finding new approaches and ideas, with the additional aim, obviously, of pointing out the author's own contribution to the research in the field.

Organization of the thesis

The thesis is organized as follows.

In Chapter 1 and 2 we present some basic and well-known material, presenting some recent developments and strengthening some of the results. Namely, in Chapter 1 we prove the Krull-Schmidt theorem with particular attention to the classical, finite case and in Chapter 2 we prove the weak Krull-Schmidt theorem for biuniform modules [Fac98], strengthening its finite version (the author's proof) and presenting an infinite version due to P. Příhoda [Pří05].

In Chapter 3 we analyze some concrete examples. On the one hand, we present an interesting theorem by Lady ([Lad74]) which takes into account the behaviour of the direct sum decompositions of torsion-free abelian groups of finite rank. On the other hand, we compute the Krull-Schmidt monoid of a ring of polynomials in two non-commuting indeterminates.

The succeeding chapters are devoted to presenting some of the author's research in the field.

Chapter 4 deals with direct sum decompositions of uniform (respectively, couniform) objects of an abelian category, rather than dealing with direct sum decompositions of biuniform objects as Chapter 2. In particular, the Krull-Schmidt theorem for monogeny classes of uniform objects (Theorem 4.1.4) states that, given A_1, A_2, ..., A_n, B_1, B_2, ..., B_t uniform objects of an abelian category C, one has $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_m = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_m$ if and only if $n = t$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i = 1, 2, \ldots, n$ (recall that two modules A and B are in the same monogeny class, in notation $[A]_m = [B]_m$, if there is a monomorphism $A \to B$ and a monomorphism $B \to A$). Thus, for an abelian category $\mathcal{C}$, if $\mathcal{U}$ is the category of finite direct sums of uniform objects, the monoid $V(\mathcal{U})/\sim_m$ (here $A \sim_m B$ if and only if A and B are in the same monogeny class) is free. As a corollary, we retrieve the "only if" implication of the Weak Krull-Schmidt Theorem for biuniform objects. The Chapter contains applications of the theorem to a number of significant abelian categories and examples that show that the theorem cannot be strengthened to direct sums of indecomposable modules of finite Goldie dimension, nor can it be strengthened to infinite direct sums of uniform objects.

In Chapter 5 we investigate the analogies and the differences between the Krull-Schmidt Theorem and the Jordan-Hölder Theorem, trying to give a general theory which takes into account both of them and a number of interesting in-between situations. Our input data are a class $\mathcal{C}$ of right modules over a fixed ring R, a class $\mathcal{R}$ of short exact sequences in $\mathcal{C}$, and a congruence $\equiv$ on the monoid $V(\mathcal{C})$. More precisely, suppose that we have an arbitrary class $\mathcal{C}$ of right R-modules closed under isomorphism and finite direct sums and with only a set of isomorphism classes. If we fix a class $\mathcal{R}$ of exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, we can construct the quotient monoid $V(\mathcal{C})/{\sim_{\mathcal{R}}}$, where $\sim_{\mathcal{R}}$ is the congruence relation on $V(\mathcal{C})$ generated by all pairs $(\langle B \rangle, \langle A \rangle + \langle C \rangle)$ with $0 \to A \to B \to C \to 0$ in $\mathcal{R}$. If $A, B \in \mathcal{C}$ and $A \leq B$, we write $A \leq_{\mathcal{R}} B$ if the canonical exact sequence $0 \to A \to B \to B/A \to 0$ belongs to $\mathcal{R}$. Now let $\equiv$ be an arbitrary congruence on $V(\mathcal{C})$. Our main objects of study are the descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$, with $A_i \leq_{\mathcal{R}} A_{i-1}$ for every i, up to the congruence $\equiv$, that is, we identify two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ if $n = m$ and there exists a permutation σ such that $\langle A_{i-1}/A_i \rangle \equiv \langle A'_{\sigma(i)-1}/A'_{\sigma(i)} \rangle$ for every $i = 1, 2, \ldots, n$. In this case, we say that the two descending series are *equivalent modulo* $\equiv$. Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\equiv$ and $\sim_R$. If $A, B \in \mathcal{C}$ and there exist a descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A with $A_i \leq_{\mathcal{R}} A_{i-1}$ for every i, a descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ of submodules of B with $B_i \leq_{\mathcal{R}} B_{i-1}$ for every i and a permutation σ of $\{1, 2, \ldots, n\}$ such that $\langle A_{i-1}/A_i \rangle \equiv \langle B_{\sigma(i)-1}/B_{\sigma(i)} \rangle$ for every $i = 1, 2, \ldots, n$, then $\langle A \rangle \equiv_{\mathcal{R}} \langle B \rangle$. We study the correspondence between the existence of such descending series (*descending series in* $\mathcal{R}$) and the quotient monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$. We give sufficient conditions on $\mathcal{R}$ and $\equiv$ to retrieve the Schreier and the Jordan-Hölder Theorems.

In Chapter 6, we investigate in further detail the monoid $V(\text{mod-}R)$, especially dealing with the relations between $V(\text{mod-}R)$ and $V(R\text{-mod})$, proving that the two are isomorphic in the special case when R is a semiperfect ring.

The final Chapter 7 is a bit far from the others. It was born to give relations between two rings A and B, weaker then "B is a direct summand of A", which would preserve the cohomology of the module categories. The result is the idea of *split exact pair* of functors between the two module categories which is investigated in the Chapter. We give a characterization of exact split pairs as the "composition" of three basic (and natural) classes of examples, prove that an exact split pair between two abelian categories induces a split pair between the respective derived categories and we use this split pair to compare cohomology in the original abelian categories. As an application of this machinery, we prove the Strong No Loops Conjecture for some classes of finite dimensional algebras and we prove non-trivial results on Brauer Algebras.

Results' fatherhood

We tried to keep note of the fatherhood af the results presented in the thesis in two different ways.

In the introduction to each chapter we recorded where the material of the chapter comes from. Moreover, in the title of each major result we recorded its fatherhood.

Notations

For the reader's convenience, we record here the assumptions we are taking for granted throughout the thesis and the usage of the simbols that could be multivocally interpreted.

All rings we consider are associative rings R with identity $1_R \neq 0_R$. Modules are right unital modules. All monoids are commutative additive monoids, that is, commutative additive semigroups with an identity 0_R.

Proper subsets will be denoted by $\subset$.

Similarly, proper substructures will be denoted by $<$.

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ will denote, respectively, the set of the non-negative integers, of the integers, of the rationals, of the real numbers and of the complex numbers.

When writing $M^{(I)}$ or M^I we will mean the direct sum and the direct product of $\mid I \mid$ copies of M, respectively.

When writing Mod-R, R-Mod, mod-R or R-mod we will mean the category of right R-modules, of left R-modules, of finitely presented right R-modules and of finitely presented left R-modules, respectively.

The injective and projective dimensions of a module M will be denoted by $id(M)$ and $pd(M)$, respectively. The global dimension of a ring R will be denoted by $gl.dim(R)$.

We will use calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots)$ to denote categories, capital letters $(A, B, C, \ldots)$ to denote rings, modules and objects of a category and small letters $(a, b, c, \ldots)$ to denote elements of a ring or a module.

Introduzione

L'idea di scomporre un modulo come somma diretta di sottomoduli è nata assieme alla teoria dei moduli. Questo perché buona parte delle informazioni su di un modulo possono essere ricavate dalla conoscenza di una scomposizione del modulo in somma diretta di sottomoduli e dalla conoscenza di questi addendi diretti: il reticolo dei sottomoduli, molti invarianti e dimensioni associate al modulo (tra cui la dimensione di Krull, la dimensione di Goldie e la dimensione duale di Goldie, le dimensioni omologiche quali la dimensione proiettiva e la dimensione iniettiva), vari funtori associati al modulo (tra cui Hom, il prodotto tensoriale, Ext e Tor) possono essere calcolati in modo molto semplice una volta che si conoscano le informazioni corrispondenti sugli addendi diretti.

Fondamentale a questo proposito è il teorema di Krull-Schmidt-Azumaya che dà condizioni sufficienti perchÈ un modulo si scomponga in modo unico come somma diretta di moduli indecomponibili. L'estensione di questo teorema e la ricerca di nuove classi di moduli che si scompongano in modo unico come somma diretta di moduli indecomponibili hanno rappresentato per decenni un importante filone di ricerca.

Recentemente l'attenzione si è spostata in un'altra direzione. Invece di cercare altre classi di moduli "molto buone", molti hanno puntato l'attenzione su delle "buone" classi di moduli e sui possibili modi per misurare in che modo queste siano diverse dalle classi "molto buone". In particolare si può associare ad ogni sottocategoria piena $\mathcal{C}$ di Mod-R un monoide commutativo e ridotto $V(\mathcal{C})$ che racchiude tutte le informazioni sulla scomposizione in somma diretta dentro a $\mathcal{C}$. Gli elementi di $V(\mathcal{C})$ sono le classi di isomorfismo $\langle A\rangle$ dei moduli A in $\mathcal{C}$ e la somma è definita da $\langle A\rangle + \langle B\rangle = \langle A \oplus B\rangle$ per ogni $A, B \in \mathcal{C}$.
È chiaro che il teorema di Krull-Schmidt-Azumaya vale in $\mathcal{C}$ se e solo se il monoide $V(\mathcal{C})$ è un monoide libero. Formulando la questione in questi termini, però, possiamo considerare delle condizioni più deboli ma ancora controllabili. Il monoide $V(\mathcal{C})$ potrebbe essere, per esempio, un monoide di Krull.

Lo scopo di questa tesi è quello di presentare una raccolta (decisamente non esaustiva) di risultati recenti in questo campo, ottenuti dall'autore o da altri. Ci sono alcuni risultati e alcune tecniche ormai classiche, alcuni risultati molto recenti e perciò non ancora molto conosciuti e alcuni risultati provenienti da aree di ricerca contigue che mostrano come la letteratura matematica sia, com'è naturale, piena di risultati collegati che possono aiutarci ad ampliare la nostra comprensione del problema e a trovare nuovi approcci e nuove idee. Un ulteriore scopo della tesi è, ovviamente, quello di presentare il contributo dell'autore alla ricerca in questo campo.

Organizzazione della tesi

La tesi è organizzata come segue.

Nei capitoli 1 e 2 è contenuto il materiale basilare e ben conosciuto, alcuni sviluppi recenti a esso direttamente connessi e alcune versioni più forti di alcuni risultati. In particolare nel capitolo 1 dimostriamo il teorema di Krull-Schmidt-Azumaya con particolare attenzione al caso classico di somme dirette finite di moduli, mentre nel capitolo 2 dimostriamo il teorema debole di Krull-Schmidt per moduli biuniformi [Fac98], presentando una versione più forte del caso finito (risultato originale dell'autore) e presentandone il caso infinito come dimostrato da P. Příhoda in [Pří05].

Nel capitolo 3 analizziamo alcuni esempi concreti. Da un lato riportiamo un interessante teorema di Lady ([Lad74]) che studia le scomposizioni in somme dirette dei gruppi abeliani senza torsione di rango finito. Dall'altro lato calcoliamo il monoide di Krull-Schmidt di un anello di polinomi con due indeterminate che non commutano tra loro.

I capitoli successivi presentano la ricerca dell'autore nell'area.

Il capitolo 4 esamina le scomposizioni in somme dirette di oggetti uniformi (rispettivamente couniformi) in una categoria abeliana, dividendo in due parti lo studio delle scomposizioni in somme dirette di moduli biuniformi fatto nel capitolo 2. In particolare dimostriamo il teorema di Krull-Schmidt per classi di monogenia di oggetti uniformi (teorema 4.1.4) che afferma che, dati A_1, A_2, $\dots$, A_n, B_1, B_2, $\dots$, B_t oggetti uniformi di una categoria abeliana $\mathcal{C}$, si ha che $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_m = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_m$ se e solo se $n = t$ e esiste una permutazione σ di $\{1, 2, \dots, n\}$ tale che $[A_i]_m = [B_{\sigma(i)}]_m$ per ogni $i = 1, 2, \dots, n$ (due moduli A e B sono nella stessa classe di monogenia, in simboli $[A]_m = [B]_m$, se esistono un monomorfismo $A \to B$ e un monomorfismo $B \to A$). Quindi per una categoria abeliana $\mathcal{C}$, se $\mathcal{U}$ è la categoria delle somme dirette finite di oggetti uniformi, il monoide $V(\mathcal{U})/\sim_m$ (dove $A \sim_m B$ se e solo se A e B sono nella stessa classe di monogenia) è libero. Come corollario si ottiene l'implicazione "solo se" del teorema debole di Krull-Schmidt per oggetti biuniformi. Il capitolo contiene applicazioni del teorema ad alcune categorie abeliane significative ed esempi che mostrano come il teorema non possa essere generalizzato né a somme dirette di moduli indecomponibili di dimensione di Goldie finita né a somme dirette infinite di oggetti uniformi.

Nel capitolo 5 vengono studiate analogie e differenze tra il teorema di Krull-Schmidt e il teorema di Jordan-Hölder e si cerca di ottenere una teoria generale che comprenda entrambe queste situazioni (che possono essere considerate situazioni limite in un senso che specificheremo) e molte interessanti situazioni intermedie. Si parte da una classe $\mathcal{C}$ di moduli destri su di un fissato anello R, una classe $\mathcal{R}$ di sequenze esatte brevi in $\mathcal{C}$, e da una congruenza $\equiv$ sul monoide $V(\mathcal{C})$. Più precisamente, sia $\mathcal{C}$ una classe arbitraria di R-moduli destri chiusa per isomorfismi e somme dirette finite che abbia un insieme (e non una classe propria) di classi di isomorfismo. Se fissiamo una classe $\mathcal{R}$ di sequenze esatte brevi $0 \to A \to B \to C \to 0$ con $A, B, C \in \mathcal{C}$, possiamo considerare il monoide quoziente $V(\mathcal{C})/{\sim_{\mathcal{R}}}$, dove $\sim_{\mathcal{R}}$ è la congruenza su $V(\mathcal{C})$ generata dalle coppie $(\langle B\rangle, \langle A\rangle + \langle C\rangle)$ con $0 \to A \to B \to C \to 0$ in $\mathcal{R}$. Se $A, B \in \mathcal{C}$ e $A \leq B$, allora scriviamo $A \leq_{\mathcal{R}} B$ se la sequenza canonica $0 \to A \to B \to B/A \to 0$ appartiene a $\mathcal{R}$. Sia ora $\equiv$ una congruenza arbitraria su $V(\mathcal{C})$. Il nostro oggetto di studio principale sono le serie discendenti $A_0 \geq A_1 \geq \cdots \geq A_n = 0$, con $A_i \leq_{\mathcal{R}} A_{i-1}$ per ogni i, a meno della congruenza $\equiv$, identificando cioè $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ e $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ se $n = m$ ed esiste una permutazione σ tale che $\langle A_{i-1}/A_i\rangle \equiv \langle A'_{\sigma(i)-1}/A'_{\sigma(i)}\rangle$ per ogni $i = 1, 2, \ldots, n$. In questo caso diciamo che le due serie discendenti sono *equivalenti modulo* $\equiv$. Sia $\equiv_{\mathcal{R}}$ la congruenza su $V(\mathcal{C})$ generata dalle due congruenze $\equiv$ e $\sim_R$. Se $A, B \in \mathcal{C}$ ed esistono una serie discendente $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ di sottomoduli di A con $A_i \leq_{\mathcal{R}} A_{i-1}$ per ogni i, una serie discendente $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ di sottomoduli di B con $B_i \leq_{\mathcal{R}} B_{i-1}$ per ogni i e una permutazione σ di $\{1, 2, \ldots, n\}$ tale che $\langle A_{i-1}/A_i\rangle \equiv \langle B_{\sigma(i)-1}/B_{\sigma(i)}\rangle$ per ogni $i = 1, 2, \ldots, n$, allora $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$. Il nostro obiettivo è quello di studiare la relazione tra l'esistenza di una siffatta serie discendente (*serie discendente in* $\mathcal{R}$) e il monoide quoziente $V(\mathcal{C})/\equiv_{\mathcal{R}}$. Diamo condizioni sufficienti su $\mathcal{R}$ e $\equiv$ per ottenere i teoremi di Schreier e di Jordan-Hölder.

Nel capitolo 6 analizziamo più in dettaglio il monoide $V(\text{mod-}R)$, considerando in particolar modo la relazione tra $V(\text{mod-}R)$ e $V(R\text{-mod})$, e dimostriamo che i due monoidi sono isomorfi se l'anello R è semiperfetto.

Il capitolo 7 definisce a analizza una proprietà più debole della proprietà di scambio che viene usata per provare il teorema di Krull-Schmidt. In particolare si cerca di mettere in luce la relazione eventualmente esistente tra la dimensione duale di Goldie dell'anello degli endomorfismi di un modulo e questa proprietà. Le due sono strettamente correlate nei casi in cui l'anello degli endomorfismi del modulo abbia dimensione duale di Goldie 1 o 2, mentre la correlazione si indebolisce per dimensione duale di Goldie più alta.

Il conclusivo capitolo 8 è un po' discosto dagli altri. È nato per ottenere, tra due anelli A e B, una relazione più debole di "B è un addendo diretto di A" che conservasse la coomologia delle rispettive categorie di moduli. Il risultato è l'idea di *coppia esatta spezzante* di funtori tra le due categorie di moduli, idea che viene esaminata nel capitolo. Diamo una caratterizzazione delle coppie esatte spezzanti come composizione di tre classi di esempi estremamente naturali, dimostriamo che una coppia esatta spezzante tra due categorie abeliane induce una coppia spezzante tra le rispettive categorie derivate e usiamo queste coppie per confrontare la coomologia nelle categorie abeliane da cui siamo partiti. Applicazioni di questa tecnica sono la dimostrazione della Strong No Loops Conjecture per alcune classi di algebre di dimensione finita e alcuni risultati sulle algebre di Brauer.

Paternità dei risultati

Abbiamo messo in risalto la paternità dei risultati contenuti nella tesi in due modi diversi.

Nell'introduzione ad ogni capitolo abbiamo segnalato la fonte da cui è stato tratto il materiale inserito nel capitolo. Inoltre abbiamo annotato la paternità di ogni teorema nella sua intestazione.

Notazioni

Raccogliamo qui, per comodità del lettore, le ipotesi che daremo per scontate nella tesi e il significato dei simboli che potrebbero esere interpretati in modi differenti.

Tutti gli anelli che considereremo saranno anelli associativi R con unità $1_R \neq 0_R$.

I sottoinsiemi propri saranno denotati dal simbolo $\subset$.

Analogamente le sottostrutture proprie saranno denotate dal simbolo $<$.

I simboli $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ rappresenteranno, rispettivamente, gli insiemi dei numeri interi non negativi, degli interi, dei razionali, dei reali, dei numeri complessi.

Le scritture $M^{(I)}$ e M^I indicheranno rispettivamente la somma diretta e il prodotto diretto di $\mid I \mid$ copie di M.

Le scritture Mod-R, R-Mod, mod-R e R-mod indicheranno rispettivamente le categorie degli R-moduli destri, degli R-moduli sinistri, degli R-moduli destri finitamente presentati e degli R-moduli sinistri finitamente presentati.

La dimensione iniettiva e la dimensione proiettiva di M saranno rispettivamente indicate con $id(M)$ e $pd(M)$. La dimensione globale di un anello R sarà indicata con $gl.dim(R)$.

Useremo le lettere corsive $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots)$ per indicare le categorie, le lettere maiuscole $(A, B, C, \ldots)$ per indicare gli anelli, i moduli e gli oggetti di una categoria e le lettere minuscole $(a, b, c, \ldots)$ per indicare gli elementi di un anello o di un modulo.

Chapter 1

The Krull-Schmidt-Azumaya Theorem

As a start, we are going to prove the Krull-Schmidt-Azumaya theorem following [Fac98]. This approach uses a well-known property of the modules which have a local endomorphism ring, namely, the exchange property.

We chose this approach both because of its elegance and because biuniform modules have a "sloppy version" of the exchange property which will allow us to prove a weak version of the Krull-Schmidt theorem for biuniform modules.

Since we are following this approach closely, we are not giving full proofs of the statements unless they are particularly significant in view of the generalization to biuniform modules, referring the reader to [Fac98] for the proofs we omitted.

1.1 The exchange property

Let M be an R-module. Recall that the lattice $\mathcal{L}(M)$ of the submodules of M is a *modular lattice*, i.e., if A, B, C are submodules of M and $C \leq A$, then $A \cap (B + C) = (A \cap B) + C$.

A useful consequence of the modular identity is the following Lemma.

Lemma 1.1.1. *If $A \subseteq B \subseteq A \oplus C$ are modules, then $B = A \oplus D$, where $D = B \cap C$.*

Proof. See [Fac98, Lemma 2.1]. ■

Definition. Let R be a ring, M be a right R-module and $\aleph$ be a cardinal. We say M *has the* $\aleph$*-exchange property* if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i$$

where $M' \cong M$ and $\mid I \mid \leq \aleph$, there are R-submodules B_i of A_i, $i \in I$, such that $G = M' \oplus (\oplus_{i \in I} B_i)$.

We say an R-module *has the finite exchange property* if it has the $\aleph$-exchange property for every finite cardinal $\aleph$.

We say an R-module *has the exchange property* if it has the $\aleph$-exchange property for every cardinal $\aleph$.

Note that, by Lemma 1.1.1, since

$$B_i \subseteq A_i \subseteq B_i \oplus \big(M' \oplus (\oplus_{j \neq i} B_j)\big)$$

one has $A_i = B_i \oplus D_i$, where $D_i = A_i \cap (M' \oplus (\oplus_{j \neq i} B_j))$. Therefore the B_i's in Definition 1.1 are necessarily direct summands of A_i.

Also note that a finitely generated module has the exchange property if and only if it has the finite exchange property.

In the rest of the section we will prove some properties of the exchange property. In this respect the next Lemma and its Corollary will be very helpful.

Lemma 1.1.2. *If* G, M', N, X, A_i $(i \in I)$, B_i $(i \in I)$ *are modules,* $B_i \subseteq A_i$ *for every* $i \in I$*,*

$$G = M' \oplus N \oplus X = (\oplus_{i \in I} A_i) \oplus X \tag{1.1.1}$$

and

$$G/X = ((M' + X)/X) \oplus (\oplus_{i \in I}((B_i + X)/X)), \tag{1.1.2}$$

then

$$G = M' \oplus (\oplus_{i \in I} B_i) \oplus X.$$

Proof. See [Fac98, Lemma 2.2]. ∎

Corollary 1.1.3. *Let* G, M', N, X, A_i $(i \in I)$ *be* R*-modules such that* $|I| \leq \aleph$*,*

$$G = M' \oplus N \oplus X = (\oplus_{i \in I} A_i) \oplus X$$

and M' *has the* $\aleph$*-exchange property. Then for every* $i \in I$ *there is a direct summand* B_i *of* A_i *such that*

$$G = M' \oplus (\oplus_{i \in I} B_i) \oplus X. \quad ∎$$

Lemma 1.1.4 (The Exchange Property and Direct Sums). *Suppose $M = M_1 \oplus M_2$. Then the module M has the $\aleph$-exchange property for some cardinal $\aleph$ if and only if both M_1 and M_2 have the $\aleph$-exchange property.*

PROOF. Suppose $M = M_1 \oplus M_2$ has the $\aleph$-exchange property, $G = M_1' \oplus N = \oplus_{i\in I} A_i$, $M_1' \cong M_1$ and $|I| \leq \aleph$.

Then $G' = M_2 \oplus G = M' \oplus N = M_2 \oplus (\oplus_{i\in I} A_i)$, where $M' = M_1' \oplus M_2 \cong M$. Let k be an element of I, and set $I' = I \setminus \{k\}$. Then $G' = M' \oplus N = (M_2 \oplus A_k) \oplus (\oplus_{i\in I'} A_i)$. Since M has the $\aleph$-exchange property, there exist submodules $B \subseteq M_2 \oplus A_k$ and $B_i \subseteq A_i$ for every $i \in I'$ such that

$$G' = M' \oplus B \oplus (\oplus_{i\in I'} B_i)\,. \tag{1.1.3}$$

Now $M_2 \subseteq M_2 \oplus B \subseteq M_2 \oplus A_k$ so that, by Lemma 1.1.1, $M_2 \oplus B = M_2 \oplus B_k$, where $B_k = (M_2 \oplus B) \cap A_k$. Thus $M' \oplus B = (M_1' \oplus M_2) \oplus B = M_1' \oplus M_2 \oplus B_k$. Substituting this into (1.1.3) one has

$$G' = M_1' \oplus M_2 \oplus (\oplus_{i\in I} B_i)\,. \tag{1.1.4}$$

Let us now apply the modular identity to the modules $M_1' \oplus (\oplus_{i\in I} B_i) \subseteq G$ and M_2 to get $G \cap (M_2 + (M_1' \oplus (\oplus_{i\in I} B_i))) = (G \cap M_2) + (M_1' \oplus (\oplus_{i\in I} B_i))$, that is, $G = M_1' \oplus (\oplus_{i\in I} B_i)$. Therefore M_1 has the $\aleph$-exchange property.

Conversely, if M_1 and M_2 have the $\aleph$-exchange property and

$$G = M_1' \oplus M_2' \oplus N = \oplus_{i\in I} A_i,$$

where $M_1' \cong M_1$, $M_2' \cong M_2$ and $|I| \leq \aleph$, then, using the $\aleph$-exchange property of M_1, we get submodules $A_i' \subseteq A_i$ such that $G = M_1' \oplus M_2' \oplus N = M_1' \oplus (\oplus_{i\in I} A_i')$. Since M_2 has the $\aleph$-exchange property, too, from Corollary 1.1.3 it follows that for every $i \in I$ there exists a submodule $B_i \subseteq A_i'$ such that

$$G = M_2' \oplus (\oplus_{i\in I} B_i) \oplus M_1'.$$

Thus $M = M_1 \oplus M_2$ has the $\aleph$-exchange property. ∎

Obviously, every module has the 1-exchange property. The next Lemma shows that modules with the 2-exchange property have the finite exchange property.

Lemma 1.1.5. *If a module M has the 2-exchange property, then M has the finite exchange property.*

Proof. We will prove, for an integer $n \geq 2$, that if M has the n-exchange property, then it has the $(n+1)$-exchange property.

Let M be a module with the n-exchange property for some $n \geq 2$ and suppose

$$G = M' \oplus N = A_1 \oplus A_2 \oplus \cdots \oplus A_{n+1},$$

where $M' \cong M$. Set $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, so that $G = M' \oplus N = A \oplus A_{n+1}$. Since M has the 2-exchange property, there exist submodules $A' \subseteq A$ and $B_{n+1} \subseteq A_{n+1}$ such that $G = M' \oplus A' \oplus B_{n+1}$. Apply Lemma 1.1.1 to the modules $A' \subseteq A \subseteq A' \oplus (M' \oplus B_{n+1})$ and $B_{n+1} \subseteq A_{n+1} \subseteq B_{n+1} \oplus (M' \oplus A')$ to get $A = A' \oplus A''$ and $A_{n+1} = B_{n+1} \oplus A'_{n+1}$, where $A'' = A \cap (M' \oplus B_{n+1})$ and $A'_{n+1} = A_{n+1} \cap (M' \oplus A')$. Since

$$G = M' \oplus A' \oplus B_{n+1} = (A'' \oplus A'_{n+1}) \oplus (A' \oplus B_{n+1}),$$

one has A'' is isomorphic to a direct summand of M'. Thus A'' has the n-exchange property by Lemma 1.1.4. Now

$$A = A' \oplus A'' = A_1 \oplus A_2 \oplus \cdots \oplus A_n,$$

so that there exist submodules $B_i \subseteq A_i$ $(i = 1, 2, \ldots, n)$ such that

$$A = A'' \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n.$$

By Lemma 1.1.1 applied to the modules

$$A'' \subseteq M' \oplus B_{n+1} \subseteq G = A'' \oplus (A' \oplus A_{n+1}),$$

one has $M' \oplus B_{n+1} = A'' \oplus A'''$, where $A''' = (M' \oplus B_{n+1}) \cap (P' \oplus A_{n+1})$. Thus

$$\begin{aligned} G &= M' \oplus A' \oplus B_{n+1} = A' \oplus A'' \oplus A''' = A \oplus A''' \\ &= B_1 \oplus \cdots \oplus B_n \oplus A'' \oplus A''' = B_1 \oplus \cdots \oplus B_n \oplus B_{n+1} \oplus M', \end{aligned}$$

Therefore M has the $(n+1)$-exchange property. ■

1.2 Indecomposable modules with the exchange property

Indecomposable modules with the (finite) exchange property are very special. They are exactly those with a local endomorphism ring, which gives the link between the exchange property and the Krull-Schmidt theorem.

We start proving two easy lemmas of independent interest.

Lemma 1.2.1. *Let A be a module and let M_1, M_2, M' be submodules of A such that $A = M_1 \oplus M_2$. Let $\pi_2 \colon A = M_1 \oplus M_2 \to M_2$ be the canonical projection. Then $A = M_1 \oplus M'$ if and only if $\pi_2|_{M'} \colon M' \to M_2$ is an isomorphism. If these equivalent conditions hold, then the canonical projection $\pi_{M'} \colon A \to M'$ with respect to the decomposition $A = M_1 \oplus M'$ is $(\pi_2|_{M'})^{-1} \circ \pi_2$.*

Proof. See [Fac98, Lemma 2.6]. ∎

Lemma 1.2.2. *Let $M, N, A_1, \dots, A_n$ be modules with $M \oplus N = A_1 \oplus \dots \oplus A_n$. If M is an indecomposable module with the finite exchange property, then there is an index $j = 1, 2, \dots, n$ and a direct sum decomposition $A_j = B \oplus C$ of A_j such that $M \oplus N = M \oplus B \oplus (\oplus_{i \neq j} A_i)$, $M \cong C$ and $N \cong B \oplus (\oplus_{i \neq j} A_i)$.*

Proof. See [Fac98, Lemma 2.7]. ∎

We are now ready to prove the main theorem of this section.

Theorem 1.2.3 (Warfield, Crawley and Jónsson).
Let M_R be an indecomposable module. Then the following conditions are equivalent.

(a) *The endomorphism ring of M_R is local.*

(b) *The module M_R has the finite exchange property.*

(c) *The module M_R has the exchange property.*

Proof. (a) $\Rightarrow$ (b). Let M_R be a module with local endomorphism ring $\mathrm{End}(M_R)$. By Lemma 1.1.5 it is sufficient to show that M has the 2-exchange property. Let G, N, A_1, A_2 be modules such that $G = M \oplus N = A_1 \oplus A_2$. Let $\varepsilon_M, \varepsilon_{A_1}, \varepsilon_{A_2}, \pi_M, \pi_{A_1}, \pi_{A_2}$ be the embeddings of M, A_1, A_2 into G and the canonical projections of G onto M, A_1, A_2 with respect to these two decompositions. We need to show that there are submodules $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ such that $G = M \oplus B_1 \oplus B_2$. Now

$$1_M = \pi_M \varepsilon_M = \pi_M(\varepsilon_{A_1}\pi_{A_1} + \varepsilon_{A_2}\pi_{A_2})\varepsilon_M = \pi_M\varepsilon_{A_1}\pi_{A_1}\varepsilon_M + \pi_M\varepsilon_{A_2}\pi_{A_2}\varepsilon_M.$$

Since $\mathrm{End}(M)$ is local, one of these two summands has to be an automorphism of M. Say $\pi_M\varepsilon_{A_1}\pi_{A_1}\varepsilon_M$ is invertible. Let H be the image of the monomorphism

$$\varepsilon_{A_1}\pi_{A_1}\varepsilon_M \colon M \to G,$$

so that $\varepsilon_{A_1}\pi_{A_1}\varepsilon_M$ induces an isomorphism $M \to H$ and $\pi_M|_H\colon H \to M$ is an isomorphism. From Lemma 1.2.1 it follows that $G = N \oplus H$ and that the projection $G \to H$ with respect to this decomposition is $(\pi_M|_H)^{-1}\pi_M$. Now

$$H = \varepsilon_{A_1}\pi_{A_1}\varepsilon_M(M) \subseteq A_1 \subseteq N \oplus H,$$

so that, by Lemma 1.1.1, one has $A_1 = H \oplus B_1$, where $B_1 = A_1 \cap N$, and the projection $A_1 \to H$ with respect to this decomposition is $(\pi_M|_H)^{-1}\pi_M|_{A_1}$. Therefore $G = A_1 \oplus A_2 = H \oplus (B_1 \oplus A_2)$. With respect to this last decomposition the projection $G \to H$ is $(\pi_M|_H)^{-1}\pi_M|_{A_1}\pi_{A_1} = (\pi_M|_H)^{-1}\pi_M\varepsilon_{A_1}\pi_{A_1}$, which, when restricted to M, is $(\pi_M|_H)^{-1}\pi_M\varepsilon_{A_1}\pi_{A_1}\varepsilon_M$. This is an isomorphism. Again by Lemma 1.2.1 we get that $G = M \oplus B_1 \oplus A_2$.

(b) $\Rightarrow$ (c). Let M_R be an indecomposable module with the finite exchange property and suppose $G = M \oplus N = \oplus_{i\in I} A_i$. Fix a non-zero element $x \in M$. There is a finite subset F of I such that $x \in \oplus_{i\in F} A_i$. Set $A' = \oplus_{i\in I\setminus F} A_i$, so that $G = M \oplus N = (\oplus_{i\in F} A_i) \oplus A'$. By Lemma 1.2.2 either there is an index $j \in F$ and a direct sum decomposition $A_j = B \oplus C$ of A_j such that

$$G = M \oplus B \oplus (\oplus_{i\in F,\ i\neq j} A_i) \oplus A',$$

or there is a direct sum decomposition $A' = B' \oplus C'$ of A' such that $G = M \oplus B' \oplus (\oplus_{i\in F} A_i)$. Since $M \cap (\oplus_{i\in F} A_i) \neq 0$, the second possibility cannot occur. Therefore there is an index $j \in F$ and a submodule B of A_j such that

$$G = M \oplus B \oplus (\oplus_{i\in F,\ i\neq j} A_i) \oplus A' = M \oplus B \oplus (\oplus_{i\in I,\ i\neq j} A_i).$$

(c) $\Rightarrow$ (a). Let M be an indecomposable R-module such that $\mathrm{End}(M)$ is not a local ring. There exist two non-invertible elements $\varphi, \psi \in \mathrm{End}(M)$ such that $\varphi - \psi = 1_M$. Set $A = M_1 \oplus M_2$, where M_1, M_2 are both equal to M, and let $\pi_i\colon A \to M_i$, $i = 1, 2$ be the canonical projections. Consider the maps

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} : M \to M_1 \oplus M_2 \text{ and } (1_M \ \ -1_M) : M_1 \oplus M_2 \to M.$$

The composite is the identity mapping of M, so that $A = M' \oplus K$, where M' denotes the image of $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and K denotes the kernel of $(1_M \ \ -1_M)$. If M had the exchange property, there would be direct summands B_1 of M_1 and B_2 of M_2 such that $A = M' \oplus K = M' \oplus B_1 \oplus B_2$. Since M_1 and M_2 are indecomposable, we would have either $A = M' \oplus M_1$ or $A = M' \oplus M_2$. If $A = M' \oplus M_1$, then $\pi_2|_{M'}\colon M' \to M_2$ would be an isomorphism (Lemma 1.2.1). Then the composite morphism $\pi_2 \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} : M \to M_2$ would be an isomorphism. But

$\pi_2 \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \psi$, contradiction. Similarly if $A = M' \oplus M_2$. This shows that M does not have the exchange property. ■

1.3 The Krull-Schmidt-Azumaya Theorem

Summing up what we did so far, it is not difficult to get the finite version of the Krull-Schmidt theorem.

Theorem 1.3.1 (Krull-Schmidt Theorem – finite case; Krull, Schmidt).
Let $M_1, \dots, M_n, N_1, \dots, N_m$ be modules with local endomorphism rings. If

$$G = M_1 \oplus \dots \oplus M_n = N_1 \oplus \dots \oplus N_m,$$

then $m = n$ and there is a permutation $\sigma \in S_n$ such that for every $i = 1, 2, \dots, n$ we have

$$G = M_1 \oplus \ldots \oplus M_i \oplus N_{\sigma(i+1)} \oplus \ldots \oplus N_{\sigma(n)}.$$

Therefore $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \dots, n$.

Proof. Let us construct step by step an injective map $\sigma \colon \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$ such that for every $i = 1, 2, \dots, n$ we have

$$G = M_1 \oplus \ldots \oplus M_i \oplus N_{\sigma(i+1)} \oplus \ldots \oplus N_{\sigma(n)}.$$

Suppose we have an injective map $\sigma_{i-1} \colon \{1, 2, \dots, i-1\} \to \{1, 2, \dots, m\}$ such that

$$G = M_1 \oplus \dots \oplus M_n = M_1 \oplus \ldots \oplus M_{i-1} \oplus \bigoplus_{j \notin \sigma_{i-1}\{1,2,\dots,i-1\}} N_j.$$

Note that $\sigma_{i-1}\{1, 2, \dots, i-1\} \neq \{1, 2, \dots, m\}$ since $\bigoplus_{j \notin \sigma_{i-1}\{1,2,\dots,i-1\}} N_j \cong \bigoplus_{h=i,i+1,\dots,n} M_h$. Thus by Corollary 1.1.3 there is an index $k \notin \sigma_{i-1}\{1, 2, \dots, i-1\}$ such that

$$G = M_1 \oplus \dots \oplus M_n = M_1 \oplus \ldots \oplus M_i \oplus \bigoplus_{j \notin \{k\} \cup \sigma_{i-1}\{1,2,\dots,i-1\}} N_j.$$

Set $\sigma_i \colon \{1, 2, \dots, i\} \to \{1, 2, \dots, m\}$ to be the same as σ_{i-1} on $1, 2, \dots, i-1$ and set $\sigma_i(i) = k$.

Obviously $\sigma = \sigma_n \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ is an injective map. Moreover

$$G = M_1 \oplus \cdots \oplus M_n = M_1 \oplus \ldots \oplus M_n \oplus \bigoplus_{j \notin \sigma\{1,2,\ldots,n\}} N_j,$$

so that $\sigma\{1, 2, \ldots, n\} = \{1, 2, \ldots, m\}$ and σ is a permutation. ∎

We now state the infinite version of the Krull-Schmidt theorem although it would take us some more work to give its proof.

Theorem 1.3.2 (Krull-Schmidt-Azumaya Theorem – infinite case; Krull, Schmidt, Azumaya).
Let M be a module that is a direct sum of modules with local endomorphism rings. Then any two direct sum decompositions of M into indecomposable direct summands are isomorphic.

Chapter 2

Biuniform modules

In [War75], R. B. Warfield asked whether the Krull-Schmidt Theorem holds for direct sums of uniserial modules. Warfield's problem was solved completely in [Fac96] by giving a counterexample.

Nevertheless, there is a weak form of the Krull-Schmidt Theorem which holds for these modules. This chapter is devoted to proving this result.

The results in Sections 2.1-2.6 come from [Fac98], except for Proposition 2.2.1 and Theorem 2.3.1. These results, in fact, are stronger versions of the corresponding results in the book. Again we give full proofs only for these two original results and for the results which seem to be particularly significant to us.

The results in Section 2.7 come from [Příí05]. In that section we are giving all the proofs both because the source is not as well-known as [Fac98] and because we rearranged the material in order to make it more clear and more consistent with the notation we have been using so far.

2.1 First properties of biuniform modules

Definition. A ring E is said to have *stable range* 1 if, whenever $a, b \in E$ and $Ea + Eb = E$, there exists $t \in E$ with $a + tb \in U(E)$.

This definition is part of a rich theory which is interesting in itself but, since we will not need it, we are not giving any more details about it.

Recall a ring is said to be *semilocal* if the quotient modulo the Jacobson radical is a semisimple artinian ring.

Theorem 2.1.1 (Bass).
A semilocal ring has stable range 1*.*

Proof. See [Fac98, Theorem 4.4]. ■

Let M be a module whose endomorphism ring has stable range 1. Then M cancels from direct sums.

Theorem 2.1.2 (Evans).
Let R be a ring and let M_R be an R-module. Suppose that $E = \mathrm{End}(M_R)$ has stable range 1*. If A_R and B_R are R-modules such that $M \oplus A \cong M \oplus B$, then $A \cong B$.*

Proof. Throughout the proof, $f_{M,N}$ will denote an R-homomorphism from N to M.

Let

$$\varphi = \begin{pmatrix} f_{M,M} & f_{M,A} \\ f_{B,M} & f_{B,A} \end{pmatrix} : M \oplus A \to M \oplus B$$

and

$$\psi = \begin{pmatrix} g_{M,M} & g_{M,B} \\ g_{A,M} & g_{A,B} \end{pmatrix} : M \oplus B \to M \oplus A$$

be two isomorphisms such that $\psi\varphi$ is the identity on $M \oplus A$. We have

$$\begin{pmatrix} g_{M,M}f_{M,M} + g_{M,B}f_{B,M} & g_{M,M}f_{M,A} + g_{M,B}f_{B,A} \\ g_{A,M}f_{M,M} + g_{A,B}f_{B,M} & g_{A,M}f_{M,A} + g_{A,B}f_{B,A} \end{pmatrix} = \mathrm{id}_{\mathrm{A\oplus B}}.$$

Since $g_{M,M}f_{M,M} + g_{M,B}f_{B,M} = 1_M$, we get

$$Ef_{M,M} + Eg_{M,B}f_{B,M} = E.$$

Thus there exists some $t \in E$ such that $u = f_{M,M} + tg_{M,B}f_{B,M}$ is an automorphism of M. Consider the morphism

$$\psi' = \begin{pmatrix} 1_M & tg_{M,B} \\ g_{A,M} & g_{A,B} \end{pmatrix} : M \oplus B \to M \oplus A.$$

One has

$$\psi'\varphi = \begin{pmatrix} u & v_{M,A} \\ 0 & 1_A \end{pmatrix}$$

is an automorphism of $M \oplus A$.

Now $\varphi\colon M\oplus A\to M\oplus B$ is an isomorphism, so that $\psi'\colon M\oplus B\to M\oplus A$ is an isomorphism, too. But then

$$\psi''=\begin{pmatrix} 1_M & 0\\ -g_{A,M} & 1_A\end{pmatrix}\psi'\begin{pmatrix} 1_M & -tg_{M,B}\\ 0 & 1_B\end{pmatrix}=\begin{pmatrix} 1_M & 0\\ 0 & g_{A,B}-g_{A,M}tg_{M,B}\end{pmatrix}$$

is an isomorphism. Thus the morphism $g_{A,B}-g_{A,M}tg_{M,B}$ is an isomorphism of B onto A. ■

Corollary 2.1.3. *Let M_R be a module over a ring R such that* $\mathrm{End}(M_R)$ *is a semilocal ring. If A_R and B_R are R-modules with $M\oplus A\cong M\oplus B$, then $A\cong B$.* ■

Recall that a module is said to be *uniform* if every two non-zero submodules have a non-zero intersection, it is said to be *couniform* if every two proper submodules have a proper sum and it is said to be *biuniform* if it is both uniform and couniform, that is to say if every submodule is both essential and superfluous. As an example we can consider *uniserial modules*, i.e. modules whose submodules are totally ordered by inclusion.

Recall also that for every module it is well defined its Goldie dimension. A module *has Goldie dimension n* if and only if it has an essential submodule which is the direct sum of n uniform modules. Dually a module *M has dual Goldie dimension n* if and only if there exists a coindependent set $\{N_1,N_2,\dots,N_n\}$ of submodules of M such that $N=N_1\cap N_2\cap\cdots\cap N_n$ is superfluous in M and $M/N\cong\oplus_{i=1}^n M/N_i$ is a direct sum of n couniform modules. Here by coindependent we mean that $N_i+(\bigcap_{j\neq i}N_j)=M$ for every $i\in I$.

Lemma 2.1.4. *Let R be a ring, let A,B,C be non-zero R-modules and let $\alpha\colon A\to B$, $\beta\colon B\to C$ be homomorphisms. Then*

(a) *If B is uniform, the composite $\beta\alpha$ is a monomorphism if and only if β and α are both monomorphisms;*

(b) *If B is couniform, the composite $\beta\alpha$ is an epimorphism if and only if β and α are both epimorphisms.*

Proof. See [Fac98, Lemma 6.26]. ■

We are now turning our attention towards the endomorphism ring of a biuniform module. The next theorem shows that the endomorphism ring of a biuniform module is semilocal. More precisely it has at most two maximal ideals, namely the set of non-surjective endomorphisms and the set of non-injective endomorphisms (in particular, biuniform modules cancel from direct sums by Corollary 2.1.3). With this in mind our middle-term goal is to prove a weaker version of the Krull-Schmidt theorem for biuniform modules.

Theorem 2.1.5 (Facchini).
Let A_R be a biuniform module over an arbitrary ring R and let $E = \mathrm{End}(A_R)$ be its endomorphism ring. Let I be the subset of E whose elements are all the endomorphisms of A_R that are not injective, and K be the subset of E whose elements are all the endomorphisms of A_R that are not surjective. Then I and K are two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover exactly one of the following two conditions hold:

(a) *Either the ideals I and K are comparable, so that E is a local ring and $I \cup K$ is its maximal ideal, or*

(b) *I and K are not comparable, $J(E) = I \cap K$, and $E/J(E)$ is canonically isomorphic to the direct product of the two division rings E/I and E/K.*

Proof. The subset I of E is additively closed since A_R is uniform. Similarly K is additively closed since A_R is couniform. By Lemma 2.1.4 the subsets I and K of E are two-sided completely prime ideals.

Let J be any proper right or left ideal of E. The set $I \cup K$ is exactly the set of non-invertible elements of E, so that $J \subseteq I \cup K$. If there exist $x \in J \setminus I$ and $y \in J \setminus K$, then $x + y \in J$, $x \in K$, and $y \in I$. Thus $x + y \notin I$ and $x + y \notin K$. Thus $x + y \notin I \cup K$. But $x + y \in J$, a contradiction. This shows J is contained either in I or in K. In particular, the unique maximal right ideals of E are (at most) I and K. Similarly, the unique maximal left ideals of E are (at most) I and K.

If I and K are comparable, then $I \cup K$ is the unique maximal right (and left) ideal of E and case (a) holds. If I and K are not comparable, then E has exactly two maximal right ideals I and K, so that $J(E) = I \cap K$, and there is a canonical injective ring homomorphism $E/J(E) \to E/I \times E/K$. But $I + K = E$, hence this ring homomorphism is surjective by the Chinese Remainder Theorem. ∎

Definition. A biuniform module is said to be *of type* 1 if its endomorphism ring is local, and *of type* 2 otherwise.

Lemma 2.1.6. *Let A be a uniform module and let B be a couniform module over a ring R.*

(a) *If $f, g\colon A \to B$ are two homomorphisms, f is injective and not surjective, and g is surjective and not injective, then $f + g$ is an isomorphism.*

(b) *If $f_1, \dots, f_n\colon A \to B$ are n homomorphisms and $f_1 + \dots + f_n$ is an isomorphism, then either one of the f_i is an isomorphism or there exist two distinct indices $i, j = 1, 2, \dots, n$ such that f_i is injective and not surjective, and f_j is surjective and not injective.*

Proof. See [Fac98, Lemma 9.2]. ∎

Definition. Let A and B be two modules.

We say that A and B *belong to the same monogeny class* (and in this case we will use the notation $[A]_m = [B]_m$), if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$.

We say that A and B *belong to the same epigeny class* (we will use the notation $[A]_e = [B]_e$), if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$.

Note that this defines two equivalence relations in the class of all right modules over a ring.

We will prove now an easy property of biuniform modules with respect to these definitions. Later we will prove the same property for finite direct sums of biuniform modules (see Corollary 4.1.7).

Proposition 2.1.7. *Let A be a uniform module and let B be a couniform module over a ring R. Then $A \cong B$ if and only if $[A]_m = [B]_m$ and $[A]_e = [B]_e$.*

Proof. See [Fac98, Proposition 9.3]. ∎

Lemma 2.1.8. *Let A be a module over a ring R and let B, C be biuniform R-modules such that $[A]_m = [B]_m$ and $[A]_e = [C]_e$. Then:*

(a) *A is biuniform;*

(b) *$A \oplus D \cong B \oplus C$ for some R-module D;*

(c) *the module D in* (b) *is unique up to isomorphism and is biuniform;*

(d) *if B and C are uniserial, then A and D are uniserial.*

Proof. (a) By hypothesis there exist two monomorphisms $\alpha\colon A \to B$ and $\beta\colon B \to A$ and two epimorphisms $\gamma\colon A \to C$ and $\delta\colon C \to A$. Since $B \neq 0$, it follows that $A \neq 0$, and $\dim(A) \leq \dim(B) = 1$, where $\dim(M)$ is the Goldie dimension of the module M. Recall if there is a monomorphism $f\colon M \to N$, then $\dim(M) \leq \dim(N)$. Hence $\dim(A) = 1$, i.e. A is uniform. Similarly, A is couniform, and so biuniform.

(b) By Lemma 2.1.6(a) one of the three morphisms $\beta\alpha$, $\delta\gamma$ or $\varphi = \beta\alpha + \delta\gamma$ is an isomorphism. If $\beta\alpha\colon A \to A$ is an isomorphism, then $\alpha\colon A \to B$ is an isomorphism by Lemma 2.1.4(b). Thus the module $D = C$ has the required property. Similarly, if $\delta\gamma$ is an isomorphism, then $D = B$ has the required property.

Suppose finally φ is an isomorphism. Since $\varphi = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \begin{pmatrix} \beta & \delta \end{pmatrix}$, the composite

$$\left(\gamma^{-1} \begin{pmatrix} \alpha_2 & \beta_2 \end{pmatrix}\right) \circ \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} : A \to B \oplus C \to A$$

is the identity mapping of A, so that A is isomorphic to a direct summand of $B \oplus C$ which has a complement D with the required property.

(c) Uniqueness is clear because biuniform modules cancel from direct sums.

As $A \oplus D \cong B \oplus C$, it follows that $1 + \dim(D) = \dim(A) + \dim(D) = \dim(B) + \dim(C) = 2$ and $1 + \operatorname{codim}(D) = \operatorname{codim}(A) + \operatorname{codim}(D) = \operatorname{codim}(B) + \operatorname{codim}(C) = 2$, so that D is biuniform as well.

(d) Suppose that B and C are uniserial. In order to prove that A and D are uniserial, it is sufficient to prove that every uniform submodule U of $B \oplus C$ is uniserial. Let $\pi_1\colon B \oplus C \to B$ and $\pi_2\colon B \oplus C \to C$ be the canonical projections. If U is a uniform submodule of $B \oplus C$, then from $U \cap \ker(\pi_1) \cap \ker(\pi_2) = 0$ it follows that either $U \cap \ker(\pi_1) = 0$ or $U \cap \ker(\pi_2) = 0$. Thus either the restriction of π_1 to U or the restriction of π_2 to U is a monomorphism. Hence U is isomorphic to a submodule of B or C. In both cases U is uniserial. ∎

2.2 Some technical lemmas

In this section, we prove a series of technical results that will be used in the subsequent sections. The first proposition is the "two-dimensional analogue" of Lemma 1.2.2 and it is a stronger version of [Fac98, Proposition 9.5]. Thanks to it we will be able to prove in Theorem 2.3.1 a stronger version of [Fac98, Theorem 9.13] which is one of the main results of [Fac98].

Proposition 2.2.1. *Let $A, B, C_1, \dots, C_n$ ($n \geq 2$) be modules. Suppose that A is biuniform and $G = A \oplus B = C_1 \oplus \cdots \oplus C_n$. Then there are two distinct indices $i, j = 1, \dots, n$ and a direct summand B' of $C_i \oplus C_j$ such that $G = A \oplus B' \oplus (\oplus_{k \neq i,j} C_k)$.*

Proof. Let $\varepsilon_A, \pi_A, \varepsilon_B, \pi_B$ and ε_i, π_i ($i = 1, 2, \dots, n$) be the embeddings and the canonical projections with respect to the two direct sum decompositions $A \oplus B$ and $C_1 \oplus \cdots \oplus C_n$. In the ring $E = \mathrm{End}(A)$ we have that

$$1_E = \pi_A \varepsilon_A = \pi_A \left(\sum_{i=1}^{n} \varepsilon_i \pi_i \right) \varepsilon_A = \sum_{i=1}^{n} \pi_A \varepsilon_i \pi_i \varepsilon_A.$$

By Lemma 2.1.6(b) either one of summands $\pi_A \varepsilon_i \pi_i \varepsilon_A$ is an isomorphism or there exist two distinct indices $i, j = 1, 2, \dots, n$ such that $\pi_A \varepsilon_i \pi_i \varepsilon_A$ is injective and not surjective, and $\pi_A \varepsilon_j \pi_j \varepsilon_A$ is surjective and not injective.

Suppose there exists an i such that $\pi_A \varepsilon_i \pi_i \varepsilon_A$ is an isomorphism. Let H be the image of the homomorphism $\varepsilon_i \pi_i \varepsilon_A \colon A \to C_i$. Since $\pi_A \varepsilon_i \pi_i \varepsilon_A$ is an isomorphism, $\pi_A \mid_H$ is an isomorphism as well and $G = H \oplus B$ by Proposition 1.2.1. The projection relative to this decomposition is $\pi_H = (\pi_A \mid_H)^{-1} \pi_A$. Now $H = \varepsilon_i \pi_i \varepsilon_A(A) \subseteq C_i \subseteq H \oplus B$, so that, by Proposition 1.1.1, $C_i = H \oplus B'$ where $B' = C_i \cap B$. The projection $\pi'_H \colon C_i \to H$ relative to this decomposition is $\pi'_H = (\pi_A \mid_H)^{-1} \pi_A \mid_{C_i}$. Thus

$$G = C_1 \oplus \ldots \oplus C_n = H \oplus B' \oplus (\bigoplus_{j \neq i} C_j)$$

with projection $(\pi_A \mid_H)^{-1} \pi_A \mid_{C_i} \pi_{C_i} = (\pi_A \mid_H)^{-1} \pi_A \varepsilon_{C_i} \pi_{C_i}$ which is, when restricted to A, the map $(\pi_A \mid_H)^{-1} \pi_A \varepsilon_{C_i} \pi_{C_i} \varepsilon_A$, hence it is an isomorphism. Therefore, again by Proposition 1.2.1,

$$G = A \oplus B' \oplus (\bigoplus_{j \neq i} C_j).$$

On the other hand if there exist two distinct indices $i, j = 1, 2, \dots, n$ such that $\pi_A \varepsilon_i \pi_i \varepsilon_A$ is injective and not surjective and $\pi_A \varepsilon_j \pi_j \varepsilon_A$ is surjective and not injective, then define H to be the image of $(\varepsilon_i \pi_i \varepsilon_A + \varepsilon_j \pi_j \varepsilon_A) \colon A \to C_i \oplus C_j$. Since $\pi_A \varepsilon_i \pi_i \varepsilon_A + \pi_A \varepsilon_j \pi_j \varepsilon_A$ is an isomorphism, $\pi_A \mid_H$ is an isomorphism as well and $G = H \oplus B$ by 1.2.1. The projection relative to this decomposition is $\pi_H = (\pi_A \mid_H)^{-1} \pi_A$. Now $H = (\varepsilon_i \pi_i \varepsilon_A + \varepsilon_j \pi_j \varepsilon_A)(A) \subseteq C_i \oplus C_j \subseteq H \oplus B$, so that, by Proposition 1.1.1, $C_i \oplus C_j = H \oplus B'$ where $B' = (C_i \oplus C_j) \cap B$. The projection

$\pi'_H \colon C_i \oplus C_j \to H$ relative to this decomposition is $\pi'_H = (\pi_A \mid_H)^{-1} \pi_A \mid_{C_i \oplus C_j}$. Thus

$$G = C_1 \oplus \ldots \oplus C_n = H \oplus B' \oplus (\bigoplus_{k \neq i,j} C_k)$$

with projection $(\pi_A \mid_H)^{-1} \pi_A \mid_{C_i \oplus C_j} \pi_{C_i \oplus C_j} = (\pi_A \mid_H)^{-1} (\pi_A \varepsilon_i \pi_i + \pi_A \varepsilon_j \pi_j)$ which is, when restricted to A, the map $(\pi_A \mid_H)^{-1} (\pi_A \varepsilon_i \pi_i \varepsilon_A + \pi_A \varepsilon_j \pi_j \varepsilon_A)$, hence it is an isomorphism. Therefore, once again by Proposition 1.2.1, we get

$$G = A \oplus B' \oplus (\bigoplus_{k \neq i,j} C_k)$$

and we are done. ■

We now consider the direct sum of a (possibly infinite) set $\{ A_i \mid i \in I \}$ of uniform modules.

Proposition 2.2.2. *Suppose that $M = \oplus_{i \in I} A_i = B \oplus C$, where B and A_i are uniform modules for every $i \in I$. Let $\varepsilon_i \colon A_i \to M$, $\varepsilon_B \colon B \to M$, $\pi_i \colon M \to A_i$ and $\pi_B \colon M \to B$ be the embeddings and the canonical projections relative to these direct sum decompositions of M. Then there exists $k \in I$ such that $\pi_B \varepsilon_k \pi_k \varepsilon_B$ is a monomorphism. In particular, $[B]_m = [A_k]_m$.*

Proof. See [Fac98, Proposition 9.6]. ■

The dual Proposition holds for *finite* sets of couniform modules.

Proposition 2.2.3. *Suppose that $M = A_1 \oplus A_2 \ldots \oplus A_n = B \oplus C$, where B and A_i are couniform modules for every $i = 1, 2 \ldots, n$. Let $\varepsilon_i \colon A_i \to M$, $\varepsilon_B \colon B \to M$, $\pi_i \colon M \to A_i$ and $\pi_B \colon M \to B$ be the embeddings and the canonical projections relative to these direct sum decompositions of M. Then there exists $k = 1, 2, \ldots, n$ such that $\pi_B \varepsilon_k \pi_k \varepsilon_B$ is an epimorphism. In particular, $[B]_e = [A_k]_e$.*

Proof. See [Fac98, Proposition 9.7]. ■

Our aim is to show that if two direct sums of biuniform modules are isomorphic, then the monogeny and the epigeny classes of the two decompositions are the same, although the isomorphism classes can be different as the Krull-Schmidt theorem does not hold in general. The next lemma shows this happens for two direct sums of two biuniform modules each.

Lemma 2.2.4. *Let A, B, C, D be biuniform modules such that $A \oplus B \cong C \oplus D$. Then $\{[A]_m, [B]_m\} = \{[C]_m, [D]_m\}$ and $\{[A]_e, [B]_e\} = \{[C]_e, [D]_e\}$.*

Proof. See [Fac98, Lemma 9.8]. ∎

2.3 Weak Krull-Schmidt Theorem for biuniform modules

Now we are ready to prove the Weak Krull-Schmidt theorem for finite direct sums of biuniform modules. This version is a bit stronger then the usual one proved by Facchini in [Fac98]. In particular it is a closer generalization of the Krull-Schmidt-Azumaya thoerem as stated in Theorem 1.3.1.

Theorem 2.3.1 (Weak Krull-Schmidt Theorem for biuniform modules; Facchini, Diracca).
Let $M_1, \dots, M_n, N_1, \dots, N_m$ be biuniform modules. If

$$G = M_1 \oplus \cdots \oplus M_n = N_1 \oplus \cdots \oplus N_m,$$

then $m = n$, there are two permutations $\sigma, \tau \in S_n$ and there are modules $B_2, B_3, \dots, B_n$ such that for every $i = 1, 2, \dots, n-1$ we have

$$G = M_\sigma(1) \oplus \ldots \oplus M_\sigma(i) \oplus B_{i+1} \oplus N_{\tau(i+2)} \oplus \ldots \oplus N_{\tau(n)}.$$

Moreover, if we set $\varphi = \sigma^{-1}\tau$ and $\psi(i) = \sigma^{-1}\tau(i+1)$ for every $i = 1, 2, \dots, n-1$ and $\psi(n) = \sigma^{-1}\tau(1)$, then for every $i = 1, 2, \dots, n$ we get

$$[M_i]_m = [N_{\varphi(i)}]_m \quad \textit{and} \quad [M_i]_e = [N_{\psi(i)}]_e.$$

Proof. First of all note that $n = m$ is obvious since $n = \dim G = m$.

For every direct sum decomposition $X = Y \oplus Z$ define $\varepsilon_Y \colon Y \to X$ to be the embedding and $\pi_Y \colon Y \to X$ to be the canonical projection. Throughout the proof we will use the composite morphisms $\pi_{M_i}\varepsilon_{N_j}\pi_{N_j}\varepsilon_{M_i}$. Note that, being M_i and N_j biuniform, the morphism $\pi_{M_i}\varepsilon_{N_j}\pi_{N_j}\varepsilon_{M_i}$ is surjective (injective) if and only if both $\pi_{M_i}\varepsilon_{N_j}$ and $\pi_{N_j}\varepsilon_{M_i}$ are surjective (injective) if and only if $\pi_{N_j}\varepsilon_{M_i}\pi_{M_i}\varepsilon_{N_j}$ is surjective (injective). From now on we will not make any difference between the two maps when saying whether they are surjective (injective) or not.

Let I be the set $\{j = 1, 2, \dots, n \mid \exists i(\pi_{N_j}\varepsilon_{M_i}\pi_{M_i}\varepsilon_{N_j})$ is an isomorphism$\}$. If $i \in I$, then by Proposition 2.2.1 one has $G = M_j \oplus (\bigoplus_{\ell \neq i} N_\ell)$, so that $\pi_{M_j}\varepsilon_{N_\ell} = 0$ for every $\ell \neq i$.

Finally define $B_1 = N_1$, $\alpha = 1$ and $\sigma_1 = \mathrm{id}_{S_n} = \tau_1$.

With all this in mind we can procede step by step along the index i.

While $0 \leq i < n$, procede as follows: thanks to the previous step we already got

$$G = M_{\sigma_i(1)} \oplus \ldots \oplus M_{\sigma_i(i-1)} \oplus B_i \oplus N_{\tau_i(i+1)} \oplus \ldots \oplus N_{\tau_i(n)}.$$

There are two possibilities: either there is no ℓ such that $\pi_{B_i}\varepsilon_{M_\ell}\pi_{M_\ell}\varepsilon_{B_i}$ is an isomorphism or there is such an ℓ.

In the first case there is an index h such that $\pi_{B_i}\varepsilon_{M_h}\pi_{M_h}\varepsilon_{B_i}$ is injective and not surjective. Note that (1) $h \notin \{\sigma_i(1), \ldots, \sigma_i(i-1)\}$ (since $\pi_{M_{\sigma_i(\ell)}}\varepsilon_{B_i} = 0$ for every $\ell = 1, \ldots, i-1$) and (2) $h \notin I$ (for the same reason). By (1) there is a permutation $\sigma_{i+1} \in S_n$ such that $\sigma_{i+1}(\ell) = \sigma_i(\ell)$ for every $\ell = 1, \ldots, i-1$ and $\sigma_{i+1}(i) = h$. By (2) there is a module $X \in \{M_{\sigma_i(1)}, \ldots, M_{\sigma_i(i-1)}, B_i, N_{\tau_i(i+1)}, \ldots, N_{\tau_i(n)}\}$ such that $\pi_{M_h}\varepsilon_X\pi_X\varepsilon_{M_h}$ is surjective and non-injective. Now we have $X \neq M_{\sigma_i(1)}, \ldots, M_{\sigma_i(i-1)}$, because $\pi_{M_{\sigma_i(\ell)}}\varepsilon_{M_h} = 0$ for every $\ell = 1, \ldots, i-1$. Moreover $X \neq B_i$ since $\pi_{B_i}\varepsilon_{M_h}\pi_{M_h}\varepsilon_{B_i}$ is injective and not surjective. Therefore $X \in \{N_{\tau_i(i+1)}, \ldots, N_{\tau_i(n)}\}$, say $X = N_k$. Hence there is a permutation $\tau_{i+1} \in S_n$ such that $\tau_{i+1}(\ell) = \tau_i(\ell)$ for every $\ell = 1, 2, \ldots, i$ and that $\tau_{i+1}(i+1) = k$. Finally, by Proposition 2.2.1 and Lemma 1.1.2, there is a module $B_{i+1} \subseteq B_i \oplus N_k$ such that

$$G = M_{\sigma_{i+1}(1)} \oplus \ldots \oplus M_{\sigma_{i+1}(i)} \oplus B_{i+1} \oplus N_{\tau_{i+1}(i+2)} \oplus \ldots \oplus N_{\tau_{i+1}(n)}.$$

Note that, by Lemma 2.2.4, one has $[B_{i+1}]_m = [N_{\tau_{i+1}(i+1)}]_m$, $[B_{i+1}]_e = [B_i]_e$, $[M_{\sigma_{i+1}(i)}]_e = [N_{\tau_{i+1}(i+1)}]_e$ and $[B_i]_m = [M_{\sigma_{i+1}(i)}]_m$.

In the latter case note $\ell \notin \{\sigma_i(1), \ldots, \sigma_i(i-1)\}$ since $\pi_{M_{\sigma_i(k)}}\varepsilon_{B_i} = 0$ for every $k = 1, \ldots, i-1$. Thus there is a permutation $\sigma_{i+1} \in S_n$ such that $\sigma_{i+1}(k) = \sigma_i(k)$ for every $k = 1, \ldots, i-1$ and $\sigma_{i+1}(i) = \ell$. For the usual reason there is a permutation $\tau_{i+1} \in S_n$ such that $\tau_{i+1}(\ell) = \tau_i(\ell)$ for every $\ell = 2, \ldots, i$ and that $\tau_{i+1}(i+1) = \alpha$. Note that one has $[B_i]_m = [M_{\sigma_{i+1}(i)}]_m$, $[B_i]_e = [M_{\sigma_{i+1}(i)}]_e$ and $N_\alpha = N_{\tau_{i+1}(i+1)}$. Reset $\alpha\colon = \tau_{i+1}(1)$ and set $B_{i+1} = N_{\tau_{i+1}(1)}$. By Proposition 2.2.1 and Lemma 1.1.2 we get

$$G = M_{\sigma_{i+1}(1)} \oplus \ldots \oplus M_{\sigma_{i+1}(i)} \oplus B_{i+1} \oplus N_{\tau_{i+1}(i+2)} \oplus \ldots \oplus N_{\tau_{i+1}(n)}.$$

Note that one has $[B_{i+1}]_m = [N_\alpha]_m$ and $[B_{i+1}]_e = [N_\alpha]_e$.

Finally compute the n-th step to check the epigeny and monogeny classes of $[N_{\tau_n(n)}]$ without defining neither B_{n+1}, σ_{n+1} nor τ_{n+1}.

To conclude the proof it is sufficient to run thorough the n steps, to set $\sigma = \sigma_n$ and $\tau = \tau_n$ and to check the monogeny and epigeny classes of the modules M_i, N_i and B_i. ■

There are examples that show that for any two permutations σ, τ of $\{1, 2, \ldots, n\}$, there is a suitable serial ring R and $2n$ finitely presented uniserial R-modules $U_1, \ldots, U_n, V_1, \ldots, V_n$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \ldots, n$ (see [Fac98, Example 9.20]). Thus if a module M is a finite direct sum of n biuniform modules, then the isomorphism classes of the biuniform direct summands may depend on the decomposition. This proves that Theorem 2.3.1 cannot be improved even if the base ring R is serial and the modules in question are finitely presented and uniserial.

2.4 A sufficient condition

Now we turn our attention to the infinite case.

The aim of this section is to give a sufficient condition for two infinite sets of biuniform modules to sum up to two isomorphic modules.

We start the section extending some results of the previous one to the infinite case in a rather technical way. This will lead to the main theorem.

Proposition 2.4.1. *Let R be an arbitrary ring, let $\{ A_i \mid i \in I \}$ be a set of biuniform R-modules and let $B_1, B_2, \ldots, B_n$ be uniform R-modules. If $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is a direct summand of $\oplus_{i \in I} A_i$, then there exist n distinct indices $k_1, \ldots, k_n$ in I such that $[B_i]_m = [A_{k_i}]_m$ for every $i = 1, 2, \ldots, n$.*

Proof. See [Fac98, Proposition 9.9]. ■

Proposition 2.4.2. *Let $A_1, \ldots, A_n, C_1, \ldots, C_m$ be biuniform right modules over an arbitrary ring R. If $A_1 \oplus \cdots \oplus A_n$ is isomorphic to a direct summand of $C_1 \oplus \cdots \oplus C_m$, then there exist n distinct indices $k_1, \ldots, k_n \in \{1, 2, \ldots, m\}$ such that $[A_i]_e = [C_{k_i}]_e$ for every $i = 1, \ldots, n$.*

Proof. See [Fac98, Proposition 9.10]. ■

Theorem 2.4.3 (Dung and Facchini).
Let $\{ A_i \mid i \in I \}$ be an arbitrary family of modules over a ring R and let $\{ B_j \mid j \in J \}$ be a family of biuniform R-modules. Assume that there exist two bijections $\sigma, \tau \colon I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_e = [B_{\tau(i)}]_e$ for every $i \in I$. Then all the modules A_i are biuniform and

$$\oplus_{i \in I} A_i \cong \oplus_{j \in J} B_j.$$

Proof. From $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_e = [B_{\tau(i)}]_e$, it follows that the module A_i is non-zero, isomorphic to a submodule of $B_{\sigma(i)}$ and a homomorphic image of $B_{\tau(i)}$. Hence every A_i is biuniform. Let S_I be the symmetric group on the set I, that is, the group whose elements are all bijections $\alpha\colon I \to I$. The mapping $S_I \times I \to I$, $(\alpha, i) \mapsto \alpha(i)$, defines a natural action of S_I on the set I. Let $C = \{\, (\tau^{-1}\sigma)^z \mid z \in \mathbb{Z}\,\}$ be the cyclic subgroup of S_I generated by $\tau^{-1}\sigma$. Then the action of S_I on I restricts to an action of C on I. For every element $i \in I$ let $Ci = \{\, (\tau^{-1}\sigma)^z(i) \mid z \in \mathbb{Z}\,\}$ be the C-orbit of i and let $\sigma(Ci) \subseteq J$ be the image of Ci via the bijection $\sigma\colon I \to J$.

We claim that

$$\oplus_{k \in Ci} A_k \cong \oplus_{\ell \in \sigma(Ci)} B_\ell \tag{2.4.1}$$

for every $i \in I$. If we prove the claim, we are done, because the set

$$\mathcal{F} = \{\, Ci \mid i \in I\,\}$$

is a partition of I, so that its image $\mathcal{G} = \{\, \sigma(Ci) \mid i \in I\,\}$ via the bijection $\sigma\colon I \to J$ is a partition of J. Hence the conclusion follows immediately from (2.4.1).

In order to prove the claim, fix an index $i \in I$. For simplicity of notation, for every $z \in \mathbb{Z}$ define $i_z = (\tau^{-1}\sigma)^z(i)$, $j_z = \sigma(i_z)$, $A_z = A_{i_z}$ and $B_z = B_{j_z}$. Thus if the orbit $Ci = \{\, i_z \mid z \in \mathbb{Z}\,\}$ is infinite, then $\sigma(Ci) = \{\, j_z \mid z \in \mathbb{Z}\,\}$ is infinite, and $A_z = A_w$ if and only if $z = w$. Whereas if the orbit $Ci = \{\, i_z \mid z \in \mathbb{Z}\,\}$ is a finite set with q elements, then $A_z = A_w$ if and only if $z \equiv w \pmod q$. For every $z \in \mathbb{Z}$ we have

$$\tau(i_z) = \tau(\tau^{-1}\sigma)^z(i) = \sigma(\tau^{-1}\sigma)^{z-1}(i) = \sigma(i_{z-1}) = j_{z-1}.$$

Hence from the hypothesis $[A_k]_m = [B_{\sigma(k)}]_m$ and $[A_k]_e = [B_{\tau(k)}]_e$ for every $k \in I$ we have

$$[A_z]_m = [B_z]_m \quad \text{and} \quad [A_z]_e = [B_{z-1}]_e \tag{2.4.2}$$

for every $z \in \mathbb{Z}$.

We shall argue by induction on the integer $n \geq 0$ and show that for every $n \geq 0$ there exist biuniform modules C_n, D_n satisfying the following properties:

(a) $[C_n]_m = [A_{-n-1}]_m$ and $[C_n]_e = [A_{n+1}]_e$ for every $n \geq 0$;

(b) $C_n \oplus D_n \cong A_{n+1} \oplus A_{-n-1}$ for every $n \geq 0$;

(c) $B_0 \oplus B_{-1} \cong A_0 \oplus C_0$ and $B_n \oplus B_{-n-1} \cong C_n \oplus D_{n-1}$ for every $n \geq 1$.

Since $[A_0]_m = [B_0]_m$ and $[A_0]_e = [B_{-1}]_e$, by Lemma 2.1.8 there is a biuniform module C_0 such that $A_0 \oplus C_0 \cong B_0 \oplus B_{-1}$. Thus C_0 satisfies property (c), and from Lemma 2.2.4 we have that $[C_0]_m = [B_{-1}]_m$ and $[C_0]_e = [B_0]_e$. Hence $[C_0]_m = [A_{-1}]_m$ and $[C_0]_e = [A_1]_e$ because of (2.4.2), that is, property (a) is satisfied. By Lemma 2.1.8 there exists a biuniform module D_0 such that $C_0 \oplus D_0 \cong A_1 \oplus A_{-1}$, i.e., D_0 satisfies property (b) as well.

Fix an integer $n \geq 1$ and suppose that there exist C_{n-1}, D_{n-1} satisfying properties (a) and (b), i.e., such that $[C_{n-1}]_m = [A_{-n}]_m$, $[C_{n-1}]_e = [A_n]_e$ and $C_{n-1} \oplus D_{n-1} \cong A_n \oplus A_{-n}$. From Lemma 2.2.4 we obtain $[D_{n-1}]_m = [A_n]_m$ and $[D_{n-1}]_e = [A_{-n}]_e$. From (2.4.2) we get that

$$[D_{n-1}]_m = [B_n]_m \quad \text{and} \quad [D_{n-1}]_e = [B_{-n-1}]_e.$$

Hence by Lemma 2.1.8 there exists a biuniform module C_n such that

$$D_{n-1} \oplus C_n \cong B_n \oplus B_{-n-1},$$

that is, (c) holds. From Lemma 2.2.4 it follows that $[C_n]_m = [B_{-n-1}]_m$ and $[C_n]_e = [B_n]_e$. Thus $[C_n]_m = [A_{-n-1}]_m$ and $[C_n]_e = [A_{n+1}]_e$ by (2.4.2), i.e., property (a) holds. By Lemma 2.1.8 there exists a biuniform module D_n such that $C_n \oplus D_n \cong A_{n+1} \oplus A_{-n-1}$. This shows that (b) holds and completes the construction of the modules C_n and D_n.

Now we shall prove the claim (2.4.1) distinguishing the following cases: the orbit Ci is infinite, or finite with an even number of elements, or finite with one element, or finite with an odd number $q \geq 3$ of elements.

If the orbit Ci is infinite, then

$$\begin{aligned}
\oplus_{k\in Ci} A_k &= \oplus_{z\in\mathbb{Z}} A_z = A_0 \oplus (\oplus_{n\geq 0}(A_{n+1} \oplus A_{-n-1})) \\
&\cong A_0 \oplus (\oplus_{n\geq 0}(C_n \oplus D_n)) = A_0 \oplus C_0 \oplus (\oplus_{n\geq 1}(C_n \oplus D_{n-1})) \\
&\cong B_0 \oplus B_{-1} \oplus (\oplus_{n\geq 1}(B_n \oplus B_{-n-1})) = \oplus_{z\in\mathbb{Z}} B_z = \oplus_{\ell\in\sigma(Ci)} B_\ell.
\end{aligned}$$

If the orbit Ci is a finite set with an even number $q = 2r$ of elements, where $r \geq 1$, then $-r \equiv r \pmod{q}$, so that $A_{-r} = A_r$. From (a) we have that $[C_{r-1}]_m = [A_{-r}]_m = [A_r]_m$ and $[C_{r-1}]_e = [A_r]_e$. Hence $C_{r-1} \cong A_r$ by Proposition 2.1.7. Thus

$$\begin{aligned}
\oplus_{k\in Ci} A_k &= A_0 \oplus (\oplus_{n=1}^{r-1}(A_n \oplus A_{-n})) \oplus A_r \\
&\cong A_0 \oplus (\oplus_{n=1}^{r-1}(C_{n-1} \oplus D_{n-1})) \oplus C_{r-1} \\
&= A_0 \oplus C_0 \oplus (\oplus_{n=1}^{r-1}(C_n \oplus D_{n-1})) \\
&\cong B_0 \oplus B_{-1} \oplus (\oplus_{n=1}^{r-1}(B_n \oplus B_{-n-1})) = \oplus_{\ell\in\sigma(Ci)} B_\ell.
\end{aligned}$$

If the orbit Ci has exactly one element, then $0 \equiv -1 \pmod{1}$ forces $B_0 = B_{-1}$, so that $[A_0]_m = [B_0]_m$ and $[A_0]_e = [B_{-1}]_e = [B_0]_e$ by (2.4.2). Hence $A_0 \cong B_0$ by Proposition 2.1.7 as desired.

If the orbit Ci is a finite set with an odd number $q = 2r + 1$ of elements ($r \geq 1$), then $-r - 1 \equiv r \pmod q$, so that $B_{-r-1} = B_r$. Hence

$$C_r \oplus D_{r-1} \cong B_r \oplus B_{-r-1} = B_r \oplus B_r.$$

It follows that $[D_{r-1}]_m = [B_r]_m$ and $[D_{r-1}]_e = [B_r]_e$ (Lemma 2.2.4). Hence $D_{r-1} \cong B_r$ (Proposition 2.1.7). Therefore

$$\begin{aligned}\oplus_{k\in Ci} A_k &= A_0 \oplus (\oplus_{n=1}^{r}(A_n \oplus A_{-n})) \\ &\cong A_0 \oplus (\oplus_{n=1}^{r}(C_{n-1} \oplus D_{n-1})) \\ &= A_0 \oplus C_0 \oplus (\oplus_{n=1}^{r-1}(C_n \oplus D_{n-1})) \oplus D_{r-1} \\ &\cong B_0 \oplus B_{-1} \oplus (\oplus_{n=1}^{r-1}(B_n \oplus B_{-n-1})) \oplus B_r = \oplus_{\ell\in\sigma(Ci)} B_\ell.\end{aligned}$$

This concludes the proof. ∎

2.5 An attempt to find necessary conditions

If we try to reverse the implication proved in Theorem 2.4.3, half of that implication can be reversed in general, as the next theorem shows.

Theorem 2.5.1 (Dung and Facchini).
Let $\{ U_i \mid i \in I \}$, $\{ V_j \mid j \in J \}$ be two families of biuniform right modules over an arbitrary ring R such that $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$. Then there exists a bijection $\sigma\colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ for every $i \in I$.

Proof. See [Fac98, Theorem 9.12]. ∎

Unfortunately the other half of the implication cannot be reversed in general. The next section is devoted to prove that.

2.6 Uniserial modules that are not quasi-small

Definition. An R-module N_R is *small* if for every family

$$\{ M_i \mid i \in I \}$$

of R-modules and any homomorphism $\varphi\colon N_R \to \oplus_{i\in I} M_i$, there is a finite subset $F \subseteq I$ such that $\pi_j \varphi = 0$ for every $j \in I \setminus F$. Here the $\pi_j\colon \oplus_{i\in I} M_i \to M_j$ are the canonical projections.

DEFINITION. A uniserial module is said to be *quasi-small* if for every family $\{M_i \mid i \in I\}$ of (uniserial) modules such that U is isomorphic to a direct summand of $\oplus_{i\in I} M_i$ there is a finite set $I' \subseteq I$ such that U is isomorphic to a direct summand of $\oplus_{i\in I'} M_i$.

Proposition 2.6.1. *Every uniserial module that is not small can be generated by $\aleph_0$ elements.*

PROOF. See [Fac98, Proposition 2.45]. ■

The next Proposition shows that, if not all the modules involved are quasi-small, Theorem of 2.5.1 cannot be dualized.

Proposition 2.6.2. *Let N be a uniserial R-module that is not quasi-small. Then the following statements hold true:*

(a) *There exists a countable family $\{ A_n \mid n \geq 1 \}$ of uniserial R-modules such that $N \oplus (\oplus_{n\geq 1} A_n) \cong \oplus_{n\geq 1} A_n$ and $[A_n]_e \neq [N]_e$ for every $n \geq 1$.*

(b) *Every non-zero homomorphic image of N is not quasi-small.*

PROOF. See [Fac98, Proposition 9.30]. ■

For some time the existence of non-quasi-small modules is been in doubt, untill in [Pun01b] Puninski showed that such a module exists.

Thus in general there is no result perfectly corresponding to 2.5.1, which is to say if $\{ U_i \mid i \in I \}$, $\{ V_j \mid j \in J \}$ are two families of non-zero uniserial modules such that $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$, then there does not need to exist a bijection $\tau \colon I \to J$ such that $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$.

There is, though, a one-to-one correspondence that preserves the epigeny classes of quasi-small uniserial modules (Theorem 2.6.4). Moreover this two one-to-one correspondences are sufficient condition for the two direct sums $\bigoplus_{i\in I} U_i$ and $\bigoplus_{j\in J} V_j$ to be isomorphic.

We start with a lemma which is an extension of Proposition 2.4.2 to the case of an infinite family of C_j's. It holds when the modules A_i's are quasi-small.

Lemma 2.6.3. *Let R be a ring, let $A_1, \dots, A_n$ be biuniform quasi-small R-modules and let $\{ C_j \mid j \in J \}$ be a set of biuniform modules. If $A_1 \oplus \cdots \oplus A_n$ is isomorphic to a direct summand of $\oplus_{j\in J} C_j$, then there exist n distinct indices $j_1, j_2, \dots, j_n \in J$ such that $[A_i]_e = [C_{j_t}]_e$ for every $t = 1, 2, \dots, n$.*

Proof. See [Fac98, Lemma 9.31]. ■

Theorem 2.6.4 (Dung and Facchini).
Let R be a ring and let $\{ U_i \mid i \in I \}$, $\{ V_j \mid j \in J \}$ be two sets of non-zero uniserial R-modules such that

$$\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j.$$

If $I' = \{ i \in I \mid U_i \text{ is quasi-small} \}$ and $J' = \{ j \in J \mid V_j \text{ is quasi-small} \}$, then there exists a bijection $\tau' \colon I' \to J'$ such that $[U_i]_e = [V_{\tau'(i)}]_e$ for every $i \in I'$.

Proof. We may assume $M = \oplus_{i\in I} U_i = \oplus_{j\in J} V_j$.
Let N be a non-zero uniserial quasi-small direct summand of M. Set

$$I_N = \{ i \in I' \mid [U_i]_e = [N]_e \} \quad \text{and} \quad J_N = \{ j \in J' \mid [V_j]_e = [N]_e \}.$$

Since N is quasi-small, there is a finite subset $F \subseteq I$ such that N is isomorphic to a direct summand of $\oplus_{i\in F} U_i$. By Proposition 2.2.3 there exists $k \in F$ with $[N]_e = [U_k]_e$. In particular, U_k is quasi-small by Proposition 2.6.2(b). Thus $k \in I_N$, so that the set I_N is non-empty. Similarly, J_N is non-empty. It is obvious that if N ranges in the set of all non-zero uniserial quasi-small direct summands of M, then the I_N form a partition of I' and the J_N form a partition of J'. In order to prove the statement it suffices to show that $|I_N| = |J_N|$ for every non-zero uniserial quasi-small direct summand N of M.

If either I_N or J_N is a finite set and $|I_N| \neq |J_N|$ we may assume $|I_N| < |J_N|$ by symmetry. Set $n = |I_N|$. Then in J_N there are $n+1$ indices $j_1, \dots, j_{n+1}$ with $[V_{j_t}]_e = [N]_e$. By Lemma 2.6.3 $[U_i]_e = [N]_e$ for at least $n+1$ distinct indices $i \in I$. By Proposition 2.6.2(b) these $n+1$ modules U_i are quasi-small, so that $|I_N| \geq n+1$, a contradiction. Hence $|I_N| = |J_N|$ if either I_N or J_N is a finite set.

If I_N and J_N are both infinite, it is sufficient to prove that $|J_N| \leq |I_N|$. Let $\varepsilon_k \colon U_k \to \oplus_{i\in I} U_i$ and $e_\ell \colon V_\ell \to \oplus_{j\in J} V_j$ be the embeddings, and $\pi_k \colon \oplus_{i\in I} U_i \to U_k$ and $p_\ell \colon \oplus_{j\in J} V_j \to V_\ell$ be the canonical projections. For every $t \in I$ set $A(t) = \{ j \in J \mid \pi_t e_j p_j \varepsilon_t \colon U_t \to U_t \text{ is an epimorphism} \}$.

Each set $A(t)$, $t \in I$, is countable, because by Proposition 2.6.1 the uniserial module U_t is either small or countably generated, so that there is a countable subset C of J such that $U_t \subseteq \oplus_{j\in C} V_j$, and then $\pi_t e_\ell p_\ell \varepsilon_t(U_t) = 0$ for every $\ell \in J \setminus C$. Hence $A(t) \subseteq C$ is countable.

We claim that $J' \subseteq \bigcup_{t\in I} A(t)$. To prove this, suppose the contrary, so that there exists $j \in J'$ such that $j \notin A(t)$ for every $t \in I$. Then $\pi_t e_j p_j \varepsilon_t \colon U_t \to U_t$

is not an epimorphism for every $t \in I$, so that

$$p_j \varepsilon_t \pi_t e_j \colon V_j \to V_j$$

is not an epimorphism for every $t \in I$ by Lemma 2.1.4(b). Hence for every $t \in I$ there is a cyclic proper submodule $C_t \subset V_j$ such that $p_j \varepsilon_t \pi_t e_j(V_j) \subseteq C_t$. For every $x \in V_j$ there are only a finite number of $t \in I$ such that $\pi_t e_j(x) \neq 0$, so that it is possible to define a homomorphism $\psi \colon V_j \to \oplus_{t \in I} C_t$ by

$$\psi(x) = (p_j \varepsilon_t \pi_t e_j(x))_{t \in I}.$$

Let $\omega \colon \oplus_{t \in I} C_t \to V_j$ be defined by $\omega((c_t)_{t \in I}) = \sum_{t \in I} c_t$. Then $\omega\psi = 1_{V_j}$, so that V_j is isomorphic to a direct summand of $\oplus_{t \in I} C_t$. As $j \in J'$, the module V_j is quasi-small. Hence there is a finite subset $F \subseteq I$ such that V_j is isomorphic to a direct summand of $\oplus_{t \in F} C_t$. In particular, V_j is finitely generated, and so cyclic. Let v be a generator of V_j. There exists a finite subset $G \subseteq I$ such that $v \in \oplus_{t \in G} U_t$. Then $\sum_{t \in G} p_j \varepsilon_t \pi_t e_j(v) = v$ forces

$$v \in \sum_{t \in G} p_j \varepsilon_t \pi_t e_j(V_j) \subseteq \sum_{t \in G} C_t \subset V_j,$$

a contradiction. This proves the claim.

Now $J_N \subseteq \bigcup_{t \in I_N} A(t)$, because if $j \in J_N$, then by the claim there exists $t \in I$ such that $j \in A(t)$. The mapping $\pi_t e_j p_j \varepsilon_t$ is an epimorphism, so that $[U_t]_e = [V_j]_e = [N]_e$ by Lemma 2.1.4(b). Since N is quasi-small, U_t must be quasi-small by Proposition 2.6.2(b), that is, $t \in I_N$. Therefore

$$|J_N| \leq \aleph_0 |I_N| = |I_N|. \quad \blacksquare$$

2.7 The Weak Krull-Schmidt Theorem for Uniserial Modules

In [Pří05], Pavel Prihoda found the correct version of the Weak Krull-Schmidt Theorem for infinite families of uniserial modules. To present his result we shall need some more definitions.

Let U be a uniserial module, let $S \subseteq \mathrm{End}_R(U)$ be the set of all injective endomorphisms of U and let $T \subseteq \mathrm{End}_R(U)$ be the set of all surjective endomorphisms of U. Let us denote $U_m = \bigcap_{f \in S} \mathrm{Im} f$ and $U_e = \bigcap_{f \in T} \mathrm{Ker} f$. Then U_m, U_e are fully invariant submodules of U.

Two uniserial modules are said to be *in the same component* (written $U \sim V$) if there is a module W such that $[U]_m = [W]_m$ and $[V_e] = [W_e]$. Observe that uniserial modules of the same monogeny or epigeny class are in the same component. Obviously, if $U \sim V$, then $U = 0$ if and only if $V = 0$.

Let us show some properties of the concepts we have defined in this section.

Proposition 2.7.1. *Let U be a uniserial module and let V be a submodule of U.*

(i) Suppose there is an injective non-surjective endomorphism of U. Then $[V]_m = [U]_m$ if and only if $U_m \subset V$.

(ii) Suppose there is a surjective non-injective endomorphism of U. Then $[V]_e = [U]_e$ if and only if $V \subset U_e$.

(iii) If $[V]_m = [U]_m$, then $V_e \subseteq U_e$.

Proof. (i) If $U_m \subset V$, then there is a monomorphism from U to V. Clearly, the inclusion $V \subseteq U$ is also a monomorphism, thus we have $[U]_m = [V]_m$. Conversely if $f\colon U \to V$ and $g\colon U \to U$ are monomorphisms such that $g(U) \neq U$, then $\mathrm{Im}(f \circ g) \subset V$ and therefore $U_m \subset V$.

(ii) If $V \subset U_e$, then there is an epimorphism $f\colon U \to U$ such that $f(V) = 0$. Thus f induces an epimorphism $U/V \to U$. Clearly, the projection $U \subseteq U/V$ is also an epimorphism, thus we have $[U]_e = [U/V]_e$. Conversely if $f\colon U/V \to U$ is an epimorphism and $g\colon U \to U/V$ is the canonical projection, then $V \subset \mathrm{Ker}((f \circ g)^2)$ and therefore $U_e \supsetneq V$.

(iii) Let E be the injective envelope of U, $\varepsilon\colon U \to E$, $\nu\colon V \to U$ and $\mu = \varepsilon\nu\colon V \to E$ be the inclusions. Let $f\colon V \to V$ be an epimorphism. We want to show $\mathrm{Ker}(f) \subseteq U_e$. Since E is injective, we can extend $\varepsilon f\colon V \to E$ to $g\colon U \to E$. Now $g(U) \supseteq V$ so there is a monomorphism $U \to g(U)$. Since g is an epimorphism $U \to g(U)$, we have $U \cong g(U)$ and therefore $\mathrm{Ker}(f) \subseteq U_e$. ■

Proposition 2.7.2. *The relation $\sim$ is an equivalence relation on the class of uniserial modules.*

Proof. In order to show that $\sim$ is simmetric, suppose $U \sim V$. Thus there is a module W such that $[U]_m = [W]_m$ and $[V]_e = [W]_e$. By [Fac98, Lemma 9.4, Theorem 9.13] there is a module W' such that $[U]_e = [W']_e$ and $[V]_m = [W']_m$, thus $V \sim U$.

In order to prove transitivity, let U_1, U_2, U_3 be uniserial modules such that $U_1 \sim U_2$ and $U_2 \sim U_3$. There are V, W modules such that $[U_1]_m = [V]_m$,

$[U_2]_e = [V]_e$, $[U_2]_m = [W]_m$ and $[U_3]_e = [W]_e$. By [Fac98, Lemma 9.4] there is X such that $U_2 \oplus X \cong V \oplus W$ and $[X]_m = [V]_m = [U_1]_m$ and $[X]_e = [W]_e = [U_3]_e$. Therefore $U_1 \sim U_3$. ∎

Proposition 2.7.3. *Let U be a uniserial module. Then U is not quasi-small if and only if $U_m \subset U_e = U$ and U is countably generated.*

Proof. Let U be a uniserial module that is not quasi-small. Not being small, it has to be countably generated. any module with local endomorphism ring has to be countably generated by [Fac98, Theorem 9.29], so U has to be of type 2 and thus $U_m \subset U$. Finally, let $u \in U$. By [DF97, Lemma 4.5] there is a non-surjective monomorphism $f \colon U \to U$ such that $f(u) = u$. Then $1 - f$ is an epimorphism having u in its kernel, so $u \in U_e$.

Conversely, let U be a countably generated uniserial module of type 2 satisfying $U_e = U$. Let $g \colon U \to U$ be a non-surjective monomorphism and let $0 \neq u$ be an element of U. Then there is an epimorphism $f \colon U \to U$ such that $f(u) = 0$. Then $f + g$ is an automorphism and $(f + g)(u) = g(u)$. Now $(f + g)^{-1} \circ g \colon U \to U$ is a monomorphism that is not an automorphism and $(f + g)^{-1} g(u) = u$. Since this is true for every $0 \neq u \in U$, we conclude by [DF97, Lemma 4.5]. ∎

Proposition 2.7.4. *Let U be a uniserial module that is not quasi-small.*

(i) Any nonzero factor of U has the same epigeny class as U.

(ii) Let V be a uniserial module of the same monogeny class as U. Then $V_m \subset V_e$ and U is the union of its proper submodules isomorphic to V.

(iii) If $[V]_m = [U]_m$ and V is not quasi-small either, then $V \cong U$.

(iv) If $V \sim U$ and V is not quasi-small either, then $[V]_e = [U]_e$.

Proof. (i) This is a straightforward consequence of 2.7.1 (ii).

(ii) We ca suppose V is a sumodule of U such that $U_n \subset V \subset U_e$ since $[V]_e = [U_e]_m$. Now there is an epimorphism $f \colon U \to U$ such that $f(V) = 0$. The submodule $W = f^{-1}(V)$ is isomorphic to V because there is an epimorphism $W \to V$ (namely f) and a monomorphism $W \to V$ (since $V \subseteq W \subseteq U$ and therefore $[V]_m = [W]_m = [U]_m$). Let $g \colon V \to W$ be some isomorphism. Then $W_m = U_m = V_m$ and $gf\,|_W \colon W \to W$ is an epimorphism having V (and thus W_m) in its kernel. Hence $W_m \subset W_e$.

Let now X be the union of all proper submodules of U_e that are isomorphic to V. Suppose $U_e \neq X$. Then there is an epimorphism $f\colon U \to U$ such that $f(X) = 0$. Now $f^{-1}(V)$ is a proper submodule of U_e isomorphic to V. Since $X \subset f^{-1}(V)$, we have a contradiction and $X = U_e$.

(iii) Suppose $[U]_m = [V]_m$. All we have to do is to fine an epimorphism $f\colon U \to V$. By (ii) U is the union of all proper submodules isomorphic to V. As U is countably generated, there is a chain $X_1 \subset X_2 \subset \ldots \subset U$ such that $\bigcup_{i\geq 1} X_i = U$ and there are epimorphisms $f_i\colon V \to X_i$. The sum of these epimorphisms induces an epimorphism $\varphi\colon \oplus_{i\geq 1} V_i \to U$ where all the v_i's are equal to V and $\varphi(V_i) = X_i$. Since $V = V_e$ and V is countably generated, it is possible to constructby induction elements $v_1, v_2, \ldots \in V$ and homomorphisms $h_1, h_2, \ldots$ such that the following conditions are satisfied:

(a) $v_1, v_2, \ldots$ generate V;

(b) for any i the homomorphism $h_i\colon V \to V_i$ is an epimorphism and $h_{i+1}(v_i) = 0$;

(c) for any $i \geq 2$, $\varphi(h_i(v_i)) \notin X_{i-1}$.

The family $\{h_i\}_{i\in\mathbb{N}}$ is a summable family of homomorphisms $V \to \oplus_{i\geq 1} V_i$, since $h_j(v_i) = 0$ whenever $j > i$. Let $f = \varphi \circ h$, where $h = \sum_{i\in\mathbb{N}} h_i$. By properties (b) and (c) one has $f(v_i) \notin X_{i-1}$ for $i \geq 2$. Thus f is an epimorphism and we are done.

(iv) If $U \sim V$, then some nonzero factor U' of U has the same monogeny class as V. This factor can't be quasi-small and $U' \cong V$ follows by (iii). Moreover U and U' have the same epigeny class by (i). ∎

We are now ready, via a couple of technical lemmas, to prove the theorem.

Lemma 2.7.5. *Let U be a uniserial module that is not quasi-small and let $V \sim U$. Then for any $x, y \in U$ satisfying $U_m \subset yR \subseteq xR \subset U$ there are submodules $U_m \subset Y \subset yR$ and $xR \subset X \subset U$ such that $V \cong X/Y$.*

Proof. Since $V \sim U$, there is a submodule $U'' \subseteq U$ such that $[U/U'']m = [V]m$. Let $\pi\colon U \to U/U''$ be the canonical projection. Since U is not quasi-small, $U_m \subset U_e = U$, thus $[U/U_m]_e = [U]_e$ by Proposition 2.7.4(i), so that there exists an epimorphism $\alpha\colon U/U_m \to U$. Note this cannot be an isomorphism by Proposition 2.7.1(i). Defining $k = \alpha \circ \pi\colon U \to U$ we get an epimorphism such that $\ker k \supset U_m$.

Let $U' = \ker \pi \circ k$. Observe that $U_m \subset U'$ and $U/U' \cong U/U''$. There are a monomorphism $f\colon U \to U$ and an epimorphism $g\colon U \to U$ such that $\mathrm{Im} f \subset yR$ and $g(U') = 0$. Thus $h = f + g$ is an automorphism of U such that $Y = h(U') \subset yR$ and $W \cong U/U' \cong U/h(U') = U/Y$. Since U/Y is not quasi small and it is in the same monogeny class as V, U/Y is a union of its proper submodules isomorphic to V by Proposition 2.7.4(ii). Therefore, there is $xR \subset X \subset U$ such that $X/Y \cong V$. ■

Lemma 2.7.6. *Let $V_1, V_2, \dots$ and $W_1, W_2 \dots$ be uniserial modules such that $[V_i]_m = [W_i]_m$ and $[V_i]_e = [W_{i+1}]_e$ for every $i \geq 1$. Suppose W_1 is not quasi-small. Then $\oplus_{i\geq 1} W_i \cong \oplus_{i\geq 1} V_i$.*

Proof. First of all we show that there are $Z_1 \subset X_1 \subset Z_2 \subset X_2 \subset Z_3 \subset \dots \subset W_1$, $Z_1 \subset Y_1 \subset Y_2 \subset \dots \subset (W_1)_m$ such that $Z_1 \cong W_1$, $V_i \cong X_i/Y_i$ for any $i \in \mathbb{N}$, $[W_1]_e = [Z_i/Y_i]_e$ for any $2 \leq i \in \mathbb{N}$, $[V_i]_m = [Z_i/Y_i]_m$ for any $2 \leq i \in \mathbb{N}$ and $W_1 = \bigcup_{i\in\mathbb{N}} X_i$.

Let Z_1 be any submodule of W_1 isomorphic to W_1, i.e. any module such that $(W_1)_m \subset Z_1 \subset W_1$, and let $\{u_i \mid i \in \mathbb{N}\}$ be a countable set of generators of W_1. Note that $V_i \sim V_{i-1}$ for every $i > 1$ and $V_1 \sim W_1$ give us $V_i \sim W_1$ for every $i \in \mathbb{N}$.

From the previous Lemma we have $(W_1)_m \subset Y_1 \subset Z_1 \subset X_1 \subset W_1$ such that $V_1 \cong X_1/Y_1$.

Suppose we have found $Z_1 \subset X_1 \subset \dots \subset Z_n \subset X_n \subset W_1$, $Y_1 \subset \dots \subset Y_n \subset (W_1)_m$ and we want to define $X_{n+1}, Y_{n+1}, Z_{n+1}$. Since $W_1 \sim V_{n+1}$ there exists a non-quasismall U such that $[W_1]_e = [U]_e$ and $[U]_m = [V_{n+1}]_m$. By Lemma 2.7.5 there are $(W_1)_m \subset Y_{n+1} \subset Y_n$ and $Y_{n+1} \subseteq X' \subseteq W_1$ such taht $X'/Y_{n+1} \cong U$. Since $[U]_e = [W_1]_e = [W_1/Y_{n+1}]_e$ and X'/Y_{n+1} is a submodule of W_1/Y_{n+1}, there are a monomorphism $X'/Y_{n+1} \to W_1/Y_{n+1}$ and an epimorphism $X'/Y_{n+1} \to W_1/Y_{n+1}$. Therefore the two modules are isomorphic and we have $U \cong W_1/Y_{n+1}$. Now, by Lemma 2.7.4(ii), U is a union of its proper submodules isomorphic to U and it is also a union of its proper submodules isomorphic to V_{n+1}. Thus there is Z_{n+1} such that $u_{n+1} \in Z_{n+1}$, $X_n \subset Z_{n+1} \subset W_1$ and $Z_{n+1}/Y_{n+1} \cong U$. There exists also $Z_{n+1} \subset X_{n+1} \subset W_1$ such that $X_{n+1}/Y_{n+1} \cong V_{n+1}$.

By induction on $i \in \mathbb{N}$ we define homomorphisms $g_i \colon W_1 \to X_i/Y_i$ such that $\mathrm{Im} g_i = Z_i/Y_i$ for any $i \geq 2$ as follows: since $[V_1]_m = [W_1]_m$, there is a monomorphism $g_1 \colon W_1 \to X_1/Y_1$.

Suppose $g_1, \dots, g_k$ have already been defined. Let v_k be an element of W_1 such that $g_k(v_k) \notin Xk - 1/Y_k$ if $k \geq 2$ and let v_1 be any nonzero element of W_1. Since $(W_1)_e = W_1$ and $[Z_{k+1}/Y_{k+1}]_e = [W_1]_e$, there is an epimorphism $g'_{k+1} \colon W_1 \to Z_{k+1}/Y_{k+1}$ such that $u_k R + v_k R \subseteq \ker g'_{k+1}$. Now let g_{k+1} be the composition of g'_{k+1} and the inclusion $Z_{k+1}/Y_{k+1} \hookrightarrow X_{k+1}/Y_{k+1}$. Note that $0 = \ker g_1 \subset \ker g_2 \subset \dots$ and that $W_1 = \bigcup_{i\in\mathbb{N}} \ker g_i$.

For any $i \in \mathbb{N}$ let $h_i \colon X_i/Y_i \to X_i/Y_1$ be the natural projection. We will consider h_i as morphisms into W_1/Y_i. Let $g \colon W_1 \to \oplus_{i\in\mathbb{N}} X_i/Y_i$ be the sum

$\sum_{i\in\mathbb{N}} g_i$ and let $h\colon \oplus_{i\in\mathbb{N}} X_i/Y_i \to W_1/Y_1$ be the sum $\oplus_{i\in\mathbb{N}}h_i$. Since $h_1 \circ g_1$ is a monomorphism and $g_k(v_1) = 0$ for any $k \geq 2$, one has $\ker h \circ g \cap v_1R = 0$ and $h \circ g$ is a monomorphism. On the other hand $h \circ g(v_k) \notin X_{k-1}/Y_1$ for any $k > 2$. Therefore $h \circ g$ is an isomorphism and then g is a section for the short exact sequence $0 \to \ker h \hookrightarrow \oplus_{i\in\mathbb{N}}X_i/Y_i \xrightarrow{[} h \circ (h \circ g)^{-1}]W_1 \to 0$ and $\oplus_{i\in\mathbb{N}}V_i \cong \oplus_{i\in\mathbb{N}}X_i/Y_i \cong W_1 \oplus \ker h$.

We want now to show that $\ker h \cong \oplus_{i\geq 2}W_i$.

Let us denote $V_i' = X_i/Y_i$ for every $i \in \mathbb{N}$, $W_i' = X_{i-1}/Y_i$ for every $i \geq 2$ and $f_i\colon W_i' \to V_{i-1}'$ the natural projection for every $i \geq 2$. We have $h_i\,|_{W_i'}= h_{i-1} \circ f_i$ and $\ker h_i \subseteq W'i$ for any $i \geq 2$. Let $W_i'' \subseteq \oplus_{j\in\mathbb{N}}V_j'$ be $W_i'' = \{(0, \dots, 0, f_i(w), -w, 0, \dots) \mid w \in W_i'\}$. One can easily check that $W_i'' \cong W_i'$ and that $\ker h \supseteq \oplus_{i\geq 2}W_i''$. Let $M = \oplus_{i\geq 2}W_i''$ and let $x = v_1 + \dots + v_k \in \ker h$ with $v_i \in V_i'$. Since the f_i's are epimorphisms, there are $w_2, \dots, w_k$ ($w_i \in W_i''$) such that $x + w_2 + \dots + w_k \in \ker h \cap V_k' = \ker h_k$. Using $\ker h_j \subseteq W'j$, we find $w_k' \in W_k'', \dots, w_2' \in W_2''$ such that $x + w_2 + \dots + w_k + w_2' + \dots + w_k' \in V_1' \ker h = \ker h_1 = 0$. Therefore $x \in M$ and $\ker h \cong \oplus_{i\geq 2}W_i''$.

It remains to inspect the monogeny and the epigeny classes of the modules X_{i-1}/Y_i for $i \geq 2$. Since Z_1 is not quasismall, $[Z_1/Y_i]_e = [W_1]_e = [Z_i/Y_i]_e$. Of course Z_1/Y_i is a submodule of Z_i/Y_i, thus there exist a monomorphism and an epimorphism $Z_1/Y_i \to Z_i/Y_i$ and therefore $Z_1/Y_i \cong Z_i/Y_i$. Thus $[Z_1/Y_i]_m = [X_i/Y_i]_m = [X_{i-1}/Y_i]_m$. Since $[X_{i-1}/Y_i]_m = [Z_1/Y_i]_m$ and $(Z_1/Y_i)_e = Z_1/Y_i$, one has $Z_1/Y_i \subseteq (X_{i-1}/Y_i)_e$ by Proposition 2.7.1(iii). By 2.7.1(ii) $[X_{i-1}/Y_{i-1}]_e = [(X_{i-1}/Y_i)/(Y_{i-1}/Y_i)]_e = [X_{i-1}/Y_i]_e$, so $[X_{i-1}/Y_{i-1}]_e = [W_i']_e$. Since for any $i \geq 2$ is $W_i'' \cong W_i' \cong W_i$, we are done. ■

Theorem 2.7.7 (Příhoda).
Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be sets of nonzero uniserial modules. Let $I' = \{i \in I \mid U_i$ is quasi-small$\}$ and $J' = \{j \in J \mid U_j$ is quasi-small$\}$. Then $\oplus_{i\in I}U_i \cong \oplus_{j\in J}V_j$ if and only if there exists a bijection $\sigma\colon I \to J$ and a bijection $\tau\colon I' \to J'$ such that for any $i \in I$ one has $[U_i]_m = [V_{\sigma(i)}]_m$ and for any $i \in I'$ one has $[U_i]_e = [V_{\tau(i)}]_e$.

Proof. We proved the direct implication in 2.6.4. We shall prove the converse. We procede by transfinite induction. We will construct sets I_α, J_α, α ordinal, such that σ (resp. τ) induces a bijection between $I \setminus I_\alpha$ and $J \setminus J_\alpha$ (resp. between $(I \setminus I_\alpha) \cap I'$ and $(J \setminus J_\alpha) \cap J'$) and such that $\oplus_{i\in I_{\alpha+1}\setminus I_\alpha}U_i \cong \oplus_{j\in J_{\alpha+1}\setminus J_\alpha}V_j$ whenever $I_{\alpha+1}$ and $J_{\alpha+1}$ are defined.

For $\alpha = 0$ we put $I_0 = J_0 = \emptyset$. Suppose we have defined I_α, J_α. If $I \setminus I_\alpha \subseteq I', J \setminus J_\alpha \subseteq J'$, we finish the construction. Suppose there is $i \in I \setminus I_\alpha$ such that

U_i is not quasi-small. Let us define (finite or infinite) sequences of pairwise different elements $i_0, i_1, \ldots \in I \setminus I_\alpha$ and $j_0, j_1, \ldots \in J \setminus J_\alpha$ as follows: $i_0 = i$ and $j_k = \sigma(i_k)$ whenever i_k is defined, $i_{k+1} = \tau^{-1}(j_k)$ if $j_k \in J'$ and we stop if $j_k \notin J'$. Two cases may occur.

Either V_k is quasi-small for any $k \in \mathbb{N}$ and we define two infinite sequences of pairwise different elements. By Lemma 2.7.6, $\oplus_{k\in\mathbb{N}} U_{i_k} \cong \oplus_{k\in\mathbb{N}} V_{j_k}$. Moreover, U_i is the only module among $U_{i_k}, V_{j_k}, k \in \mathbb{N}$ that is not quasi-small. Thus σ induces a bijection between $\{i_k \mid k \in \mathbb{N}\}$ and $\{j_k \mid k \in \mathbb{N}\}$ and τ induces a bijection between $I' \cap \{i_k \mid k \in \mathbb{N}\}$ and $J' \cap \{j_k \mid k \in \mathbb{N}\}$. Thus we can define $I_{\alpha+1} = I_\alpha \cup \{i_k \mid k \in \mathbb{N}\}, J_{\alpha+1} = J_\alpha \cup \{j_k \mid k \in \mathbb{N}\}$.

The other case is that V_{j_h} is not quasi-small for some $h \in \mathbb{N}$. Thus we have defined only finite sequences $i_0, i_1, \ldots i_h$ and $j_0, j_1, \ldots j_h$. Since $U_{i_0} \sim V_{j_0} \sim U_{i_1} \sim \ldots \sim V_{j_h}$ and U_{i_0}, V_{j_h} are not quasi-small, one has $[U_{i_0}]_e = [V_{j_h}]_e$ according to Proposition 2.7.4(iv). Of course σ (respectively τ) induces a bijection between $\{i_0, i_1, \ldots i_h\}$ and $\{j_0, j_1, \ldots j_h\}$ (respectively$\{i_1, i_1, \ldots i_h\}$ and $\{j_0, j_1, \ldots j_{h-1}\}$ these possibly being the empty set). By Theorem 2.3.1, $\oplus_{k=0}^h U_{i_k} \cong \oplus_{k=0}^h V_{j_k}$. Therefore we can define $I_{\alpha+1} = I_\alpha \cup \{i_0, i_1, \ldots, i_h\}, J_{\alpha+1} = J_\alpha \cup \{j_0, j_1, \ldots, j_h\}$.

If $I \setminus I' \subseteq I_\alpha$ but there exists $j \in J \setminus (J' \cup J_\alpha)$ we proceed similarly starting with V_j.

If α is a limit ordinal and we have defined I_β, J_β for every $\beta < \alpha$, we simply define $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$ and $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$.

Of course the construction has to stop. Let α be the greatest ordinal for which I_α, J_α were defined. One can easily see that $\oplus_{i\in I_\alpha} U_i \cong \oplus_{j\in J_\alpha} V_j$. Since $I \setminus I_\alpha \subseteq I'$ and $J \setminus J_\alpha \subseteq J'$, σ and τ induce bijections between $I \setminus I_\alpha$ and $J \setminus J_\alpha$. Therefore $\oplus_{i\in I\setminus I_\alpha} U_i \cong \oplus_{j\in J\setminus J_\alpha} V_j$ by 2.3.1. Finally, $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$ as we wanted to prove. ■

Chapter 3

Two examples

3.1 Torsion free abelian groups of finite rank

In [Lad74], E. L. Lady, answering a question in [Fuc73], proved that torsion free abelian groups of finite rank have, up to isomorphisms, only finitely many direct summands.

The natural question, which is, as far as we know, still without answer, is whether the Krull-Schmidt monoid of torsion free abelian groups of finite rank has some kind of regularity i.e., whether it is a Krull monoid, a directly finite monoid and so on.

Throughout this chapter we will freely use definitions and results from [Fai73]. In particular for the definitions and the characterizations of the nilradical, the prime radical and the strongly nilpotent elements of a ring, the reader should refer to Faith's book, since the use of these terms is not consistent throughout the literature.

Theorem 3.1.1 (Lady).
If G is a torsion free abelian group of finite rank, then G has, up to isomorphism, only finitely many direct summands.

PROOF. Let $R = \mathrm{End}(G)$ be the ring of endomorphisms of the group G. It is well known that there is a category equivalence proj-R $\cong$add-G. Therefore it is sufficient for us to show that R_R has, up to isomorphisms, finitely many direct summands.

Recall that $(R, +)$, the additive group of R, is likewise a finite rank torsion free abelian group.

As a first step we will show we can suppose, without loss of generality, $J = J(R) = 0$. Let P_1, P_2 be two projective right R-modules such that $P_1/P_1J \cong$

P_2/P_2J (say $\theta\colon P_1/P_1J \to P_2/P_2J$ is an isomorphism) and let $\pi_i\colon P_i \to P_i/P_iJ$ ($i = 1, 2$) be the canonical projections. Being P_2 projective we can factor the map $\theta\pi_1$ as $\pi_2\psi$ for some $\psi\colon P_1 \to P_2$. The morphism ψ is surjective by Nakayama's Lemma (one has $\psi(P_1)/P_2J = P_2/P_2J$, i.e. $\psi(P_1) + P_2J = P_2$, thus $\psi(P_1) = P_2$, P_2J being superfluous in P_2 by Nakayama's Lemma), thus it splits and $\mathrm{Ker}(\psi)$ is a direct summand of P_1. Being $\mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\pi_1) = P_1J \ll P_1$, one has $\mathrm{Ker}(\psi) = 0$ and ψ is an isomorphism. Infinitely many non-isomorphic direct summands of R_R would thus give raise to infinitely many non-isomorphic direct summands of $R/J_{R/J}$. We can therefore suppose, without loss of generality, that $J = J(R) = 0$.

Now, being the rank of $(R, +)$ finite, the $\mathbb{Q}$-algebra $\mathbb{Q}R = \mathbb{Q} \otimes R$ is finite dimensional, hence artinian. Its Jacobson radical is, therefore, nilpotent and thus it coincides with the nilradical and with the prime radical. We want to show that the prime radical is zero, showing that $\mathbb{Q}R$ is a semisimple artinian $\mathbb{Q}$-algebra.

The prime radical is the set of all strongly nilpotent elements of the ring. Suppose there exists a non-zero strongly nilpotent element a of $\mathbb{Q}R$. There exists a non-zero integer n such that $na \in R$. Note that, being $(R, +)$ torsion free, one has $na \neq 0$. Moreover na is a strongly nilpotent element of $\mathbb{Q}R$, hence it is a non-zero strongly nilpotent element of R. But the prime radical of R is zero (being contained in the Jacobson radical), this giving raise to a contraddiction. Therefore the prime (hence the Jacobson) radical of $\mathbb{Q}R$ is zero.

Let now $\{a_1, a_2, \ldots, a_n\}$ be a maximal independent set in the additive group of $\mathbb{Q}R$. We can write

$$a_ia_j = \sum_{\ell=1}^{k} q_{ij\ell}a_\ell \qquad \text{with } \ q_{ij\ell} \in \mathbb{Q}.$$

If m is a common denominator of the $q_{ij\ell}$, then the elements $b_i = ma_i$ ($i = 1, 2, \ldots, k$) and 1 generate a subgroup S of $\mathbb{Q}R$ which is clearly a subring. As an abelian group it is finitely generated, hence free. Clearly $\mathbb{Q}R = \mathbb{Q}S$.

Moreover, since the ring $R \cap S$ has the same properties (it is a subring of $\mathbb{Q}R$, free as an abelian group, generating all $\mathbb{Q}R$ as a $\mathbb{Q}$-algebra) we can suppose $S \subseteq R$.

The ring $\mathbb{Q}R$ has, up to isomorphism, finitely many right ideals. Thus, by the Jordan-Zassenhaus Theorem [CR62, Theorem 79.1], in $\mathbb{Q}R$ there are, up to isomorphisms, finitely many right S-modules, say $N_1, N_2, \ldots, N_t$.

Now let $M = eR$ be a direct summand of R_R. For some integer n one has $ne \in S$ (since $\mathbb{Q}R = \mathbb{Q}S$). There is an S-isomorphism ϕ from neS to some N_i. We can extend ϕ to an R-isomorphism $\bar{\phi}\colon nM = neR \to N_iR$ by

$\bar{\phi}(ner) = \phi(ne)r$. As $eR \cong neR$ as right R-modules (via the R-isomorphism $er \mapsto ner$), the theorem is proved. ■

Thus the monoid $V(G)$ is certainly a directly finite monoid, since every torsion free abelian group of finite rank can be written as the direct sum of finitely many indecomposable subgroups.

On the other hand there are examples of torsion free abelian groups G whose Krull-Schmidt monoid $V(G)$ is not a Krull monoid (see [Arn82]).

3.2 The ring of polynomials in two non-commuting indeterminates

In this section we will compute explicitly the Krull-Schmidt monoid $V(R)$ where R is the ring of polynomials in two non-commuting indeterminates over a field k. For results and terminology on Von Neumann Regular Rings we refer the reader to [Goo91].

Let k be a field, $R = k < x, y >$ be the ring of polynomials in two non-commuting indeterminates over k, let $E = E(R_R)$ be the injective envelope of the R-module R_R, let $S = \mathrm{End}_R(E)$ be the endomorphism ring of E and let $J(S)$ be its Jacobson radical.

The ring $S/J(S)$ is a von Neumann regular right self-injective ring. We will show it is of Type III.

First of all note that idempotents lift modulo $J(S)$, so that $V(S) \cong V(S/J(S))$ and an idempotent is directly finite in S if and only if its projection on $S/J(S)$ is so.

For every $n \in \mathbb{N}$ let M_n be the set of all monomials of degree n. Now consider an idempotent $0 \neq \varepsilon \in S$. There is a polynomial $r \in R \cap \varepsilon E$, thus for every degree n we have $\varepsilon E \supseteq rR \supseteq \oplus_{m_i \in M_n} rm_iR$ so that $\oplus_{m_i \in M_n} E(rm_iR) \leq_{\oplus} \varepsilon E$. Therefore εE is not directly finite, which means εS is not so.

Therefore no non-zero idempotent in S is directly finite, thus no non-zero idempotent in $S/J(S)$ is directly finite, i.e. $S/J(S)$ is of Type III, hence it is purely infinite ([Goo91, p. 116]).

The Grothendieck group $K_0(S/J(S))$ turns out to be the trivial group 0 ([Goo91, Proposition 15.6]). The monoid $V(S/J(S))$, however, is not trivial since $S/J(S) \ncong 0$. We will show the monoid is not much more complicated since it is the monoid $(\mathbb{Z}/2\mathbb{Z}, \cdot)$. We will show this in two steps.

Lemma 3.2.1. *Given a cardinal $\xi \leq \aleph_0$, one has $E(R^{(\xi)}) \cong E(R)$.*

Proof. Since $xR \oplus yR$ is an essential submodule of R and $xR \cong yR \cong R$, one has $E(R) = E(xR \oplus yR) \cong E(R^2)$ and by induction $E(R) \cong E(R^n)$ for every $n \leq \aleph_0$. If $\xi = \aleph_0$, consider the polynomials $p_n = y^n x$ with $n \geq 0$. Note that $\oplus p_i R$ is an essential submodule of R, thus $E(\oplus p_i R) = E(R)$ and $p_i R \cong R$ give us $E(R^{(\xi)}) \cong E(R)$. ∎

Proposition 3.2.2. *Every non-zero direct summand X of $E(R)$ is isomorphic to $E(R)$.*

Proof. Let X be a direct summand of $E(R)$. Since R is essential in $E(R)$, the intersection $R \cap X$ is essential in X. Let $\{U_i\}_{i \in I}$ be a maximal set of independent submodules of $R \cap X$ (note that every U_i is a k-subspace of R, thus, being independent, they are at most $\dim_k(R) = \aleph_0$). Since for every i one has $R \cap U_i \neq 0$, there is a polynomial $p_i \in R \cap U_i$. The cyclic module $p_i R$ is essential in $U_i \cap R$ so that $\{p_i R\}_{i \in I}$ is a maximal set of independent submodules of $R \cap X$ and we can think, without loss of generality, that the U_i's are cyclic. Now every non-zero cyclic R-module is isomorphic to R, thus $R^{(I)}$ is an essential submodule of $R \cap X$. Therefore $R^{(I)}$ is an essential submodule of X and $X = E(R^{(I)}) \cong E(R)$. ∎

Corollary 3.2.3. *If k is a field, $R = k < x, y >$, $E = E(R_R)$ is the injective envelope of the right regular module and S is the ring of endomorphisms of E, then $V(S) = (\mathbb{Z}/2\mathbb{Z}, \cdot)$.*

Proof. Clear by Proposition 3.2.2 in view of the previous discussion. ∎

Chapter 4

Uniqueness of monogeny classes for uniform objects in abelian categories

So far we've been talking about biuniform modules. Nothing has been said about uniform or couniform modules. There are, though, similar results about uniform (respectively couniform) modules. All the results in this chapter come from [DF02].

4.1 Main results

In [DF02], we analyzed direct sums of uniform modules to show that if $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ are uniform objects of an abelian category C, then $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $B_1 \oplus B_2 \oplus \cdots \oplus B_t$ are in the same monogeny class if and only if $n = t$ and there is a permutation σ of $\{1, 2, \dots, n\}$ such that A_i and $B_{\sigma(i)}$ are in the same monogeny class for every $i = 1, 2, \dots, n$. This is proved using bipartite digraphs. We will show that, if the digraph has enough edges, given a bipartition of the digraph into two disjoint sets the strong components of the digraphs intersect the two disjoint sets of vertices of a bipartition in sets of the same cardinality. This may be viewed as a Krull-Schmidt Theorem for bipartite digraphs.

To go on we will need some notation.

Let X and Y be finite disjoint sets. We shall denote by $D(X, Y; E)$ the bipartite digraph having X and Y as disjoint sets of non-adjacent vertices and E as set of edges. That is, $V = X \cup Y$ is the vertex set of $D(X, Y; E)$, $E \subseteq X \times Y \cup Y \times X$ is the set of its edges, and $X \cap Y = \emptyset$. For every subset $T \subseteq V$

let $N^+(T) = \{ w \in V \mid (v, w) \in E \text{ for some } v \in T \}$ be the out-neighborhood of T. Define an equivalence relation $\sim_s$ on V by $v \sim_s w$ if there are both a path from v to w and a path from w to v ($v, w \in V$). The equivalence classes modulo $\sim_s$ are the vertex sets of the *strong components* of the digraph, that is, the maximal strongly connected subgraphs of the digraph $D(X, Y; E)$.

Lemma 4.1.1. (Krull-Schmidt Theorem for bipartite digraphs) *Let X and Y be disjoint sets of cardinality n and m respectively, let $V = X \cup Y$, and let $D = D(X, Y; E)$ be a bipartite digraph having X and Y as disjoint sets of non-adjacent vertices. If $|T| \leq |N^+(T)|$ for every subset T of V, then $n = m$ and, after a suitable numbering of the elements $x_1, \dots, x_n$ of X and $y_1, \dots, y_n$ of Y, $x_i \sim_s y_i$ for every $i = 1, \dots, n$.*

Notice that $|T| \leq |N^+(T)|$ for every subset T of V if and only if $|T'| \leq |N^+(T')|$ for every subset T' of X and $|T''| \leq |N^+(T'')|$ for every subset T'' of Y.

Proof. Since $|X| \leq |N^+(X)| \leq |Y|$ and $|Y| \leq |N^+(Y)| \leq |X|$, we get that $n = |X| = |Y| = m$, and we must number the sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ in such a way that $x_i \sim_s y_i$ for every $i = 1, 2, \dots, n$.

Consider the bipartite digraph $D' = D(X, Y; E \cap X \times Y)$. By Hall's Theorem (see [Wes01, Theorem 3.1.11]), the digraph D' has a perfect matching, that is, there exists a subset E' of $E \cap X \times Y$ such that for every $x \in X$ there is exactly one edge in E' with tail x and for every $y \in Y$ there is exactly one edge in E' with head y. Similarly, there exists a subset E'' of $E \cap Y \times X$ such that for every $y \in Y$ there is exactly one edge in E'' with tail y and for every $x \in X$ there is exactly one edge in E'' with head x. Thus the bipartite digraph $D_0 = D(X, Y; E' \cup E'')$ is a digraph with $2n$ vertices and $2n$ edges in which every vertex has outdegree one and indegree one. This means that D_0 is the functional digraph of a bijection $f \colon V \to V$ with $f(v) \neq v$ for every $v \in V$, that is, a permutation of V that leaves no point fixed. Therefore D_0 is a disjoint union of directed cycles $C_1, \dots, C_t$. Each of these directed cycles C_j passes through an even number of vertices of V, and it passes through the same number of vertices of X and vertices of Y. Therefore we may number the elements $x_1, \dots, x_n$ of X and $y_1, \dots, y_n$ of Y in such a way that for every $i = 1, \dots, n$ there exists one of these directed cycles C_j that passes through both x_i and y_i. Since the edges of the cycles C_j are edges of D, it follows that $x_i \sim_s y_i$ as desired. ∎

A partial converse of Lemma 4.1.1 holds as well:

Proposition 4.1.2. *Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be finite disjoint sets of the same cardinality, let $V = X \cup Y$, and let $D = D(X, Y; E)$ be a bipartite digraph having X and Y as disjoint sets of non-adjacent vertices. Suppose that:*

(a) *$x_i \sim_s y_i$ for every $i = 1, 2, \dots, n$;*

(b) *if $v_1, v_2, v_3, v_4 \in V$ and $(v_1, v_2), (v_2, v_3), (v_3, v_4) \in E$, then $(v_1, v_4) \in E$.*

Then $|T| \leq |N^+(T)|$ for every subset T of V.

Proof. If $T \subseteq X$ and $x_i \in T$, then $x_i \sim_s y_i$ by (a), so that there is a path of odd length from x_i to y_i. Thus $y_i \in N^+(T)$ by (b). This shows that $|T| \leq |N^+(T)|$. ■

We are ready to apply bipartite graphs to abelian categories, but first we need a further result about biproducts of uniform objects.

Lemma 4.1.3. *Let $A_1, A_2, \dots, A_n$ be non-zero objects of an abelian category C and let $B_1, B_2, \dots, B_t$ be uniform objects of C. If there is a monomorphism $\alpha\colon A_1 \oplus A_2 \oplus \dots \oplus A_n \to B_1 \oplus B_2 \oplus \dots \oplus B_t$, then $n \leq t$.*

Proof. Let $\mathcal{L} = \mathcal{L}(B_1 \oplus B_2 \oplus \dots \oplus B_t)$ be the class of all subobjects of $B_1 \oplus B_2 \oplus \dots \oplus B_t$. The class $\mathcal{L}$ satisfies the axioms of modular lattices [Pop73, Exercise 2.6.5] (apart from the fact that it could be a proper class and not a set). Let $\varepsilon_i\colon A_i \to A_1 \oplus A_2 \oplus \dots \oplus A_n$ and $\varepsilon'_j\colon B_j \to B_1 \oplus B_2 \oplus \dots \oplus B_t$ be the canonical monomorphisms, and let L be the sublattice of $\mathcal{L}$ generated by the images of the $n+t$ morphisms $\alpha\varepsilon_i$ ($i = 1, 2, \dots, n$) and ε'_j ($j = 1, 2, \dots, t$). Then L is a countable modular lattice. Let $\mathcal{P}(\{1, 2, \dots, n\})$ denote the lattice of all subsets of the set $\{1, 2, \dots, n\}$. The morphism α induces a lattice embedding $\widetilde{\alpha}\colon \mathcal{P}(\{1, 2, \dots, n\}) \to L$ defined by $\widetilde{\alpha}(S) = \bigvee_{i \in S} \operatorname{im}(\alpha\varepsilon_i)$ for every $S \subseteq \{1, 2, \dots, n\}$. Now L is a modular lattice of Goldie dimension t [Fac98, §2.6] and the $\widetilde{\alpha}(\{i\})$ ($i = 1, 2, \dots, n$) form a join-independent subset of cardinality n of L. Therefore $n \leq t$. ■

Theorem 4.1.4 (Krull-Schmidt Theorem for monogeny classes; Diracca and Facchini).
Let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ be uniform objects of an abelian category C. Then $[A_1 \oplus A_2 \oplus \dots \oplus A_n]_m = [B_1 \oplus B_2 \oplus \dots \oplus B_t]_m$ if and only if $n = t$ and there is a permutation σ of $\{1, 2, \dots, n\}$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i = 1, 2, \dots, n$.

Proof. One implication is trivial.

For the converse, let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ be uniform objects of the category C and assume that $[A_1 \oplus A_2 \oplus \dots \oplus A_n]_m = [B_1 \oplus B_2 \oplus \dots \oplus B_t]_m$, so that there exist two monomorphisms $\alpha\colon A_1 \oplus A_2 \oplus \dots \oplus A_n \to B_1 \oplus B_2 \oplus \dots \oplus B_t$ and $\beta\colon B_1 \oplus B_2 \oplus \dots \oplus B_t \to A_1 \oplus A_2 \oplus \dots \oplus A_n$. Let $\varepsilon_i\colon A_i \to A_1 \oplus A_2 \oplus \dots \oplus A_n$ and $\pi_i\colon A_1 \oplus A_2 \oplus \dots \oplus A_n \to A_i$ be the structural morphisms, that is the morphisms such that $\sum_{i=1}^n \varepsilon_i \pi_i$ is the identity morphism of $A_1 \oplus A_2 \oplus \dots \oplus A_n$, $\pi_i \varepsilon_k = 0$ for $i \neq k$, and $\pi_i \varepsilon_i$ is the identity morphism of A_i. Similarly, let $\varepsilon'_j\colon B_j \to B_1 \oplus B_2 \oplus \dots \oplus B_t$ and $\pi'_j\colon B_1 \oplus B_2 \oplus \dots \oplus B_t \to B_j$ be the structural morphisms for the biproduct $B_1 \oplus B_2 \oplus \dots \oplus B_t$. Let $\varphi_{i,j} = \pi'_j \alpha \varepsilon_i\colon A_i \to B_j$ and $\varphi'_{j,i} = \pi_i \beta \varepsilon'_j\colon B_j \to A_i$ be the composite morphisms. Let $D = D(X, Y; E)$ be the bipartite digraph having $X = \{A_1, A_2, \dots, A_n\}$ and $Y = \{B_1, B_2, \dots, B_t\}$ as disjoint sets of non-adjacent vertices, one edge from A_i to B_j for each i and j with $\varphi_{i,j}$ monic, and one edge from B_j to A_i for each i and j with $\varphi'_{j,i}$ monic.

In order to prove that the hypothesis of Lemma 4.1.1 holds, we can suppose that $T \subseteq X$ by symmetry. If $m = |T|$ and $r = |N^+(T)|$, relabeling the indices we may suppose that $T = \{A_1, A_2, \dots, A_m\}$ and $N^+(T) = \{B_1, B_2, \dots, B_r\}$. Thus the morphisms $\varphi_{i,j}$ are not monic for every $i = 1, 2, \dots, m$ and every $j = r+1, r+2, \dots, t$. Since the objects A_i are uniform, we have that $\bigcap_{j=r+1}^t \ker \varphi_{i,j} \neq 0$ for every $i = 1, 2, \dots, m$. Set $K_i = \bigcap_{j=r+1}^t \ker \varphi_{i,j}$, so that the objects K_i, $i = 1, 2, \dots, m$, are all non-zero. As every K_i is a subobject of A_i, there is a canonical monomorphism $\varepsilon\colon \oplus_{i=1}^m K_i \to \oplus_{k=1}^n A_k$.

Now for every $i = 1, 2, \dots, m$ and every $j = r+1, r+2, \dots, t$, the composite morphism

$$K_i \to A_i \xrightarrow{\varepsilon_i} \oplus_{k=1}^n A_k \xrightarrow{\alpha} \oplus_{\ell=1}^t B_\ell \xrightarrow{\pi'_j} B_j$$

is zero because $K_i \subseteq \ker \varphi_{i,j} = \ker(\pi'_j \alpha \varepsilon_i)$. Thus the image of the composite morphism

$$K_i \to A_i \xrightarrow{\varepsilon_i} \oplus_{k=1}^n A_k \xrightarrow{\alpha} \oplus_{\ell=1}^t B_\ell$$

is contained in the kernel of π'_j for every $j = r+1, r+2, \dots, t$. Since $\bigcap_{j=r+1}^t \ker \pi'_j = \oplus_{j=1}^r B_j$, it follows that there is a morphism $K_i \to \oplus_{j=1}^r B_j$ for which the diagram

$$\begin{array}{ccc} K_i & \longrightarrow & \oplus_{j=1}^r B_j \\ \downarrow & & \downarrow \\ \oplus_{k=1}^n A_k & \xrightarrow{\alpha} & \oplus_{\ell=1}^t B_\ell \end{array}$$

is commutative. By the universal property of coproducts, there is a morphism

$$\alpha'\colon \oplus_{i=1}^m K_i \to \oplus_{j=1}^r B_j$$

for which the diagram

$$\begin{array}{ccc} \oplus_{i=1}^{m} K_i & \xrightarrow{\alpha'} & \oplus_{j=1}^{r} B_j \\ \varepsilon \downarrow & & \downarrow \\ \oplus_{k=1}^{n} A_k & \xrightarrow{\alpha} & \oplus_{\ell=1}^{t} B_\ell \end{array}$$

is commutative. Here the vertical arrows denote the canonical monomorphisms, so that α' is necessarily a monomorphism. From Lemma 4.1.3 it follows that $m \leq r$, that is, $|T| \leq |N^+(T)|$. This shows that the hypothesis of Lemma 4.1.1 holds. To conclude it suffices to remark that two objects equivalent modulo the equivalence relation $\sim_s$ are in the same monogeny class. ∎

An immediate application to the category Mod-R of all right modules over an arbitrary ring R is the following result, which was proved by Zanardo [Zan88] in the case of uniform modules over a commutative ring R.

Theorem 4.1.5 (Diracca and Facchini, Zanardo).
Let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ be uniform right R-modules. Then $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_m = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_m$ if and only if $n = t$ and there is a permutation σ of $\{1, 2, \dots, n\}$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i = 1, 2, \dots, n$.

Applying Theorem 4.1.4 to the opposite category of an abelian category C we get

Theorem 4.1.6 (Krull-Schmidt Theorem for epigeny classes; Diracca and Facchini).
Let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ be couniform objects of an abelian category C. Then $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_e = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_e$ if and only if $n = t$ and there is a permutation τ of $\{1, 2, \dots, n\}$ such that $[A_i]_e = [B_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.

As a consequence of Theorems 4.1.4 and 4.1.6 we immediately get the "only if" implication in the Weak Krull-Schmidt Theorem for Biuniform Modules (Theorem 2.3.1).

Another consequence of Theorems 4.1.5, 4.1.6 and 2.3.1 is the next corollary, which was previously known in the case $n = t = 1$ only (see 2.1.7).

Corollary 4.1.7. *Let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ be biuniform right R-modules. Then $A_1 \oplus A_2 \oplus \cdots \oplus A_n \cong B_1 \oplus B_2 \oplus \cdots \oplus B_t$ if and only if $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_m = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_m$ and $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_e = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_e$.*

4.2 Examples in the category of right modules - the infinite case

In this section we give an example that shows that Theorems 4.1.5 and 4.1.6 and Corollary 4.1.7 cannot be extended to *infinite* direct sums.

Example 4.2.1. Let $\mathbb{N}$ be the set of non-negative integers. Let M_R be a uniserial right R-module, and suppose that $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\infty = M_R$ are all the submodules of M_R, so that the lattice $\mathcal{L}(M_R)$ of all the submodules of M_R is isomorphic to $\mathbb{N} \cup \{+\infty\}$. Consider the two families $\{ A_i \mid i \in \mathbf{N} \}$, $\{ B_i \mid i \in \mathbf{N} \}$ of uniform R-modules for which $A_i = M_{2i}$ and $B_i = M_{2i+1}$ for every $i \in \mathbb{N}$. As $A_i \subseteq B_i$ for every $i \in \mathbb{N}$, there is a monomorphism $\oplus_{i \in \mathbf{N}} A_i \to \oplus_{i \in \mathbf{N}} B_i$. Since $B_i \subseteq A_{i+1}$ for every $i \in \mathbb{N}$, there is a monomorphism $\oplus_{i \in \mathbf{N}} B_i \to \oplus_{i \in \mathbf{N}} A_i$, so that $[\oplus_{i \in \mathbf{N}} A_i]_m = [\oplus_{i \in \mathbf{N}} B_i]_m$. As two modules of finite length are in the same monogeny class if and only if they are isomorphic, we have that $[A_i]_m \neq [B_j]_m$ for every $i, j \in \mathbf{N}$, $i \neq j$. Hence, not only are there no bijections between the monogeny classes of the families $\{ A_i \mid i \in \mathbf{N} \}$ and $\{ B_i \mid i \in \mathbf{N} \}$, but also *no monogeny class of the family* $\{ A_i \mid i \in \mathbf{N} \}$ *is equal to any monogeny class of the family* $\{ B_i \mid i \in \mathbf{N} \}$. This shows that Theorem 4.1.5 cannot be extended to infinite families of uniform modules.

Notice that in this example we can even have *uniserial modules of finite length over a commutative ring, which may be* $\mathbf{Z}$ *or a* DVR (it suffices to take for M_R the Prüfer group $\mathbb{Z}_{p^\infty}$ or the R-module Q/R, where R is a DVR with field of fractions Q).

Now suppose that in M_R we also have that $M_i/M_1 \cong M_{i-1}$ for every $i \geq 1$ (this holds in the case $M_R = \mathbb{Z}_{p^\infty}$ or $M_R = Q/R$). There are epimorphisms $A_i = M_{2i} \to B_{i-1} = M_{2i-1}$ for every $i \geq 1$, so that there is an epimorphism $\oplus_{i \in \mathbb{N}} A_i \to \oplus_{i \in \mathbb{N}} B_i$. Since there are epimorphisms $B_i = M_{2i+1} \to A_i = M_{2i}$ for every $i \in \mathbb{N}$, there is an epimorphism $\oplus_{i \in \mathbb{N}} B_i \to \oplus_{i \in \mathbf{N}} A_i$ as well, and thus $[\oplus_{i \in \mathbf{N}} A_i]_e = [\oplus_{i \in \mathbf{N}} B_i]_e$. Again, two modules of finite length are in the same epigeny class if and only if they are isomorphic, so that $[A_i]_e \neq [B_j]_e$ for every $i, j \in \mathbb{N}$, $i \neq j$. Hence no epigeny class of the family $\{ A_i \mid i \in \mathbb{N} \}$ is equal to any epigeny class of the family $\{ B_i \mid i \in \mathbf{N} \}$. Thus Theorem 4.1.6 also cannot be extended to infinite families of couniform modules.

By 2.5.1 the two direct sums $\oplus_{i \in \mathbf{N}} A_i$ and $\oplus_{i \in \mathbf{N}} B_i$ are not isomorphic, which proves that Corollary 4.1.7 also cannot be extended to infinite families of uniserial modules. ∎

Example 4.2.2. Theorem 4.1.5 cannot be extended from the case of uniform modules to the case of indecomposable modules of finite Goldie dimension. That is, there exist indecomposable modules A_1, A_2, B_1, B_2 of finite Goldie dimension with $[A_1 \oplus A_2]_m = [B_1 \oplus B_2]_m$ and $[A_i]_m \neq [B_j]_m$ for every $i, j = 1, 2$. For this, it is sufficient to take four indecomposable, pairwise nonisomorphic, artinian modules A_1, A_2, B_1, B_2 with $A_1 \oplus A_2 \cong B_1 \oplus B_2$ (see [FHLV95, Example 1.7]). Notice that if A and B are two artinian modules in the same monogeny class and $\alpha\colon A \to B$, $\beta\colon B \to A$ are two monomorphisms, then $\beta\alpha$ and $\alpha\beta$ are injective endomorphisms, hence they are automorphisms of A and B respectively. It follows that α and β are isomorphisms, so that $A \cong B$. Thus two artinian modules are in the same monogeny class if and only if they are isomorphic.

Also notice that for every integer $n > 2$ there exists a module M of finite Goldie dimension that is the direct sum $M = A_{i,1} \oplus A_{i,2} \oplus \cdots \oplus A_{i,i}$ of i indecomposable modules $A_{i,1}, A_{i,2}, \ldots, A_{i,i}$ for every $i = 2, 3, \ldots, n$. In this case also the module M can be chosen artinian [FHLV95, Example 1.6].

Similar examples of indecomposable finitely generated modules over suitable commutative semilocal noetherian rings [Fac98, Examples 8.8 and 8.10] show that Theorem 4.1.6 cannot be extended to the case of indecomposable modules of finite dual Goldie dimension. ∎

4.3 Applications to further abelian categories

Let R be a ring, Mod-R the category of all right R-modules, R-Mod the category of all left R-modules, and ${}_R\mathrm{FP}$ the full subcategory of R-Mod whose objects are the finitely presented left R-modules. Let $C = ({}_R\mathrm{FP}, \mathbf{Ab})$ be the category of all additive functors from ${}_R\mathrm{FP}$ to the category $\mathbf{Ab}$ of abelian groups. The category C is a Grothendieck category whose injective objects are the objects isomorphic to the functors $M \otimes_R -\colon {}_R\mathrm{FP} \to \mathbf{Ab}$ in which M_R is an arbitrary pure-injective right R-module [Fac98, §1.6]. If A, B are right R-modules, we write $[A]_{pm} = [B]_{pm}$ if there are both a pure monomorphism of A into B and a pure monomorphism of B into A. Theorem 4.1.4 applied to the category C yields the following result:

Theorem 4.3.1 (Diracca and Facchini).
Let $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_t$ be right R-modules and suppose that their pure-injective envelopes are indecomposable. Then

$$[A_1 \oplus A_2 \oplus \cdots \oplus A_n]_{pm} = [B_1 \oplus B_2 \oplus \cdots \oplus B_t]_{pm}$$

if and only if $n = t$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $[A_i]_{pm} = [B_{\sigma(i)}]_{pm}$ for every $i = 1, 2, \ldots, n$.

Another abelian category that has played an important role in the study of torsion-free abelian groups of finite rank is the quotient category $\mathbf{Ab}/\mathbf{B}$, where $\mathbf{Ab}$ is the category of all abelian groups and $\mathbf{B}$ is the class of all bounded abelian groups. Essentially, an interesting result due to B. Jónsson [Jón59] says that if the notion of isomorphism is replaced by quasi-isomorphism, then one has a Krull-Schmidt Theorem for torsion-free abelian groups of finite rank. Later, Walker [Wal64] showed that two torsion-free abelian groups are quasi-isomorphic if and only if they are isomorphic in the quotient category $\mathbf{Ab}/\mathbf{B}$. Every subobject of an object G of $\mathbf{Ab}/\mathbf{B}$ can be represented by a subgroup H of G.

Since the canonical functor $J\colon \mathbf{Ab} \to \mathbf{Ab}/\mathbf{B}$ is additive and exact, if H, H' are subgroups of an abelian group G, then the intersection of H and H' in $\mathbf{Ab}$ coincides with the intersection of H and H' in $\mathbf{Ab}/\mathbf{B}$. Thus an abelian group G is a uniform object in $\mathbf{Ab}/\mathbf{B}$ if and only if it is not bounded and for every subgroup H, H' of G, $H \cap H'$ bounded implies that either H is bounded or H' is bounded. A closer examination shows that an abelian group G is a uniform object in the category $\mathbf{Ab}/\mathbf{B}$ if and only if it is isomorphic to $\mathbf{Z}_{p^\infty} \oplus T$ or $F \oplus T$ for some prime p, some bounded group T and some torsion-free group F of rank 1.

Another application of our result is the proof that the Krull-Schmidt Theorem holds for direct sums of uniform artinian modules:

Proposition 4.3.2. [Fac98, Th. 2.18] *Let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ be uniform artinian modules. Then $A_1 \oplus A_2 \oplus \cdots \oplus A_n \cong B_1 \oplus B_2 \oplus \cdots \oplus B_t$ if and only if $n = t$ and there is a permutation σ of $\{1, 2, \dots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for every $i = 1, 2, \dots, n$.*

If we remark, as we have done in Example 4.2.2, that two artinian modules are in the same monogeny class if and only if they are isomorphic and apply Theorem 4.1.5, we get an immediate proof of Proposition 4.3.2.

The standard proof of Proposition 4.3.2 (see [Fac98, Th. 2.18]) shows that uniform artinian modules have local endomorphism rings, so that it is possible to apply the classical Krull-Schmidt-Azumaya Theorem.

The dual of Proposition 4.3.2 for couniform noetherian modules holds as well, and can be treated in the same way [Fac98, Th. 2.19]. Notice, as we have already remarked in Example 4.2.2, that the Krull-Schmidt Theorem does not hold for direct sums of artinian indecomposable modules [FHLV95].

Chapter 5

Descending Chains of Modules

It is clear that there is a relation between the Krull-Schmidt Theorem (every module of finite length is a direct sum of indecomposable modules, and if $A_1 \oplus A_2 \oplus \cdots \oplus A_n = A'_1 \oplus A'_2 \oplus \cdots \oplus A'_m$ are any two such decompositions, then $n = m$ and there exists a permutation σ such that $A_i \cong A'_{\sigma(i)}$ for every $i = 1, 2, \dots, n$) and the Jordan-Hölder Theorem (every module A of finite length has a composition series, and if $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ are any two composition series, then $n = m$ and there exists a permutation σ such that $A_{i-1}/A_i \cong A'_{\sigma(i)-1}/A'_{\sigma(i)}$ for every $i = 1, 2, \dots, n$).

The relation is that what we state is equivalent to saying that some commutative monoid is free in both cases.

The Krull-Schmidt Theorem says that if $\mathcal{C}$ is the class of all right modules of finite length over a ring R, then $V(\mathcal{C})$ is a free commutative monoid. A free set of generators of $V(\mathcal{C})$ is given by the isomorphism classes of the modules indecomposable in $\mathcal{C}$.

The Jordan-Hölder Theorem says that if $\sim$ is the congruence relation on the monoid $V(\mathcal{C})$ generated by all the pairs $(\langle B \rangle, \langle A \rangle + \langle C \rangle)$ for which $A, B, C \in \mathcal{C}$ and there exists an exact sequence $0 \to A \to B \to C \to 0$, then the quotient monoid $V(\mathcal{C})/\sim$ is free. A free set of generators of $V(\mathcal{C})/\sim$ is given by the isomorphism classes of all simple R-modules.

However, as we shall see in this chapter, the relation between existence of descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules, uniqueness up to a permutation of the factors A_{i-1}/A_i, refinements of descending series, validity of Jordan-Hölder type theorems or Schreier type theorems that can be found in the mathematical literature, and freeness of the corresponding quotient monoid $V(\mathcal{C})/\sim$ is not immediate.

The results in this chapter were originally published in [DF04], a paper been born from an attempt to give a general framework to these notions. Since a number of results have been obtained recently as far as Krull-Schmidt type theorems are concerned (cf. [Bro02], [DF02], [Fac96], [Fac98], [Fac02], [Pun01a], [Wie01], [Yak00]), we hoped that we could obtain similar results for Jordan-Hölder type theorems, but the situation turned out to be more complicate than we hoped for.

The Jordan-Hölder theorem and the Schreier theorem concern partially ordered set, and in fact most of the variations on this theme that can be found in the literature pass sooner or later through the Jordan-Hölder theorem and the Schreier theorem for modular lattices [Ste75, Proposition III.3.1 and Corollary III.3.2]. For instance, both the Jordan-Hölder theorem and the Schreier theorem hold in abelian categories because the class $\mathcal{L}(A)$ of all subobjects of an object A of an abelian category is a "modular lattice" (here we write "modular lattice" in inverted commas because it is not necessarily a set). In a number of examples we have found, however, abelian categories do not appear immediately for at least two reasons. Namely, on the one hand only particular descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules are considered in some cases, for example series of pure submodules or divisible submodules or submodules with critical quotients. On the other hand, equivalences $\equiv$ weaker than isomorphism of composition factors are considered in some other cases, for instance being in the same monogeny class or in the same epigeny class.

Our input data are a class $\mathcal{C}$ of right modules over a fixed ring R, a class $\mathcal{R}$ of short exact sequences in $\mathcal{C}$, and a congruence $\equiv$ on the monoid $V(\mathcal{C})$. More precisely, suppose that we have an arbitrary class $\mathcal{C}$ of right R-modules closed under isomorphism and finite direct sums and with only a set of isomorphism classes. Then it is possible to define the monoid $V(\mathcal{C})$, which completely describes the behavior of the class as far as uniqueness of direct sum decompositions is concerned. If we fix a class $\mathcal{R}$ of exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, we can construct the quotient monoid $V(\mathcal{C})/{\sim_{\mathcal{R}}}$, where $\sim_{\mathcal{R}}$ is the congruence relation on $V(\mathcal{C})$ generated by all pairs $(\langle B\rangle, \langle A\rangle + \langle C\rangle)$ with $0 \to A \to B \to C \to 0$ in $\mathcal{R}$. If $A, B \in \mathcal{C}$ and $A \leq B$, we write $A \leq_{\mathcal{R}} B$ if the canonical exact sequence $0 \to A \to B \to B/A \to 0$ belongs to $\mathcal{R}$. Now let $\equiv$ be an arbitrary congruence on $V(\mathcal{C})$. Our aim is to study the descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$, with $A_i \leq_{\mathcal{R}} A_{i-1}$ for every i, up to the congruence $\equiv$, that is, we identify two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ if $n = m$

and there exists a permutation σ such that $\langle A_{i-1}/A_i\rangle \equiv \langle A'_{\sigma(i)-1}/A'_{\sigma(i)}\rangle$ for every $i = 1, 2, \dots, n$. In this case, we say that the two descending series are *equivalent modulo* $\equiv$. Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\equiv$ and $\sim_R$. If $A, B \in \mathcal{C}$ and there exist a descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A with $A_i \leq_{\mathcal{R}} A_{i-1}$ for every i, a descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ of submodules of B with $B_i \leq_{\mathcal{R}} B_{i-1}$ for every i and a permutation σ of $\{1, 2, \dots, n\}$ such that $\langle A_{i-1}/A_i\rangle \equiv \langle B_{\sigma(i)-1}/B_{\sigma(i)}\rangle$ for every $i = 1, 2, \dots, n$, then $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$. We study the correspondence between the existence of such descending series (*descending series in* $\mathcal{R}$) and the quotient monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$.

In the transition from the class $\mathcal{C}$ to the commutative monoid $V(\mathcal{C})$, no information about direct sum decompositions in $\mathcal{C}$ is lost (Krull-Schmidt type theorems). We show that, unluckily, the situation is not so good in the transition from descending series of submodules to the quotient monoids $V(\mathcal{C})/\sim_{\mathcal{R}}$ or $V(\mathcal{C})/\equiv_{\mathcal{R}}$ (Jordan-Hölder type theorems).

The chapter is organized as follows. In Section 5.1, we study the relation between classes $\mathcal{R}$ of short exact sequences in $\mathcal{C}$ and the corresponding congruences $\sim_{\mathcal{R}}$ on $V(\mathcal{C})$. In Section 5.2, we fix an arbitrary congruence $\equiv$ on $V(\mathcal{C})$, construct the congruence $\equiv_{\mathcal{R}}$ generated by the two congruences $\equiv$ and $\sim_R$, and consider the relation between the congruence $\equiv_{\mathcal{R}}$ and the existence of descending series up to equivalence modulo $\equiv$. In Section 5.3, we determine the conditions on the class $\mathcal{R}$ that allows us to have a reasonably good behavior of descending series in $\mathcal{R}$ as far as taking submodules and quotient modules is concerned. In Section 5.4, we see how these notions link up to give us information about the existence of refinements (Schreier type theorems) and the uniqueness of composition series (Jordan-Hölder type theorems).

Finally, in Section 5.5, we analyze some of the many examples of Jordan-Hölder type theorems existing in the mathematical literature from the point of view we have introduced. We also recall an example (critical composition series, Example 5.5.14) that falls only partially within our theory, but that we think to be very interesting.

In the literature, there are already other attempts of rationalization of the Jordan-Hölder theory, different from ours. For instance, we mention that due to Hughes [Hug60], concerning a lattice of subsystems of an algebraic system. We must remark that the construction of Grothendieck groups, in which abelian *groups* are considered instead of our monoids $V(\mathcal{C})$, $V(\mathcal{C})/\sim_{\mathcal{R}}$ and $V(\mathcal{C})/\equiv_{\mathcal{R}}$, cannot be applied in our setting, because in the construction of Grothendieck groups all information concerning cancellation from direct sums and its pathologies is lost.

5.1 Classes of exact sequences and congruences in $V(\mathcal{C})$

Let R be a fixed ring and $\mathcal{C}$ be a class of right R-modules. Let $\mathrm{Ses}(\mathcal{C})$ be the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$. If $\sim$ is a congruence on the monoid $V(\mathcal{C})$, we can construct the subclass $\mathcal{S}_\sim$ of $\mathrm{Ses}(\mathcal{C})$ whose elements are all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ and $\langle B \rangle \sim \langle A \rangle + \langle C \rangle$.

Conversely, if $\mathcal{R}$ is a subclass of $\mathrm{Ses}(\mathcal{C})$, we may consider the congruence $\sim_\mathcal{R}$ on $V(\mathcal{C})$ generated by all pairs $(\langle B \rangle, \langle A \rangle + \langle C \rangle)$ with $0 \to A \to B \to C \to 0$ in $\mathcal{R}$. We shall call $\sim_\mathcal{R}$ *the congruence associated to* $\mathcal{R}$.

Thus we have defined two correspondences $\Phi\colon \sim \mapsto \mathcal{S}_\sim$ and $\Psi\colon \mathcal{R} \mapsto \sim_\mathcal{R}$ between congruences on the monoid $V(\mathcal{C})$ and subclasses of $\mathrm{Ses}(\mathcal{C})$. Let $\mathcal{L}(V(\mathcal{C}))$ be the lattice of all congruences on $V(\mathcal{C})$. The partial order on $\mathcal{L}(V(\mathcal{C}))$ is defined by $\sim \ \leq \ \sim'$ if $\langle A_R \rangle \sim \langle B_R \rangle$ implies $\langle A_R \rangle \sim' \langle B_R \rangle$ for every $\langle A_R \rangle, \langle B_R \rangle \in V(\mathcal{C})$. Similarly, the class of all subclasses of $\mathrm{Ses}(\mathcal{C})$ is partially ordered by class inclusion $\subseteq$, and the correspondences Φ and Ψ preserve these partial orders, in the sense that $\sim \leq \sim'$ implies $\mathcal{S}_\sim \subseteq \mathcal{S}_{\sim'}$, and $\mathcal{R} \subseteq \mathcal{R}'$ implies $\sim_\mathcal{R} \ \leq \ \sim_{\mathcal{R}'}$. Moreover, $\Psi\Phi(\sim) \leq \ \sim$ for every congruence $\sim$ on $V(\mathcal{C})$, and $\mathcal{R} \subseteq \Phi\Psi(\mathcal{R})$ for every subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$. From these elementary properties, it immediately follows that $\Phi\Psi\Phi = \Phi$ and $\Psi\Phi\Psi = \Psi$, so that Φ and Ψ induce order preserving bijections, one inverse to the other, between the images of Φ and Ψ. That is, if we call *complete* the subclasses $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ of the type $\mathcal{R} = \mathcal{S}_\sim$ for some congruence $\sim$ on $V(\mathcal{C})$ (equivalently, such that $\mathcal{R} = \Phi\Psi(\mathcal{R})$), *cocomplete* the congruences $\sim$ on $V(\mathcal{C})$ of the type $\sim_\mathcal{R}$ for some subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ (equivalently, such that $\sim \ = \Psi\Phi(\sim)$), and denote by $\mathrm{Cl}(\mathcal{C})$ the set of all complete subclasses of $\mathrm{Ses}(\mathcal{C})$ and by $\mathrm{Cong}(\mathcal{C})$ the set of all cocomplete congruences on $V(\mathcal{C})$, then the partially ordered set $\mathrm{Cl}(\mathcal{C})$ is isomorphic to the partially ordered set $\mathrm{Cong}(\mathcal{C})$ via the restrictions of Φ and Ψ.

Lemma 5.1.1. *The partially ordered set* $\mathrm{Cl}(\mathcal{C})$ *is a complete lattice.*

Proof. The join of a subset $\{\, \mathcal{R}_\lambda \mid \lambda \in \Lambda \,\}$ of $\mathrm{Cl}(\mathcal{C})$ is $\Phi\Psi(\bigcup_{\lambda \in \Lambda} \mathcal{R}_\lambda)$. ∎

The class $\mathrm{Ses}(\mathcal{C})$ is the greatest element of the lattice $\mathrm{Cl}(\mathcal{C})$. The smallest element of $\mathrm{Cl}(\mathcal{C})$ is the subclass $\mathcal{S}_=$ of $\mathrm{Ses}(\mathcal{C})$ corresponding to the smallest element $=$ of $\mathrm{Cong}(\mathcal{C})$ ($=$ is the identity on $V(\mathcal{C})$). Thus $\mathcal{S}_=$ is the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ and $B \cong A \oplus C$. (Notice that $\mathcal{S}_=$ can contain sequences that are not split. An easy example can be constructed with $\mathcal{C}$ the class of all countable abelian groups and a non-split exact sequence $0 \to \mathbb{Z} \xrightarrow{2} (\mathbb{Z}/2\mathbb{Z})^{(\aleph_0)} \oplus \mathbb{Z} \to (\mathbb{Z}/2\mathbf{Z})^{(\aleph_0)} \to 0$. Particular cases in which all sequences in $\mathcal{S}_=$ are split, that is, classes of modules $\mathcal{C}$ such that if $A, B, C \in \mathcal{C}$ and $B \cong A \oplus C$, then every exact sequence $0 \to A \to B \to C \to 0$ is split, were studied in [Gur81] and [Miy67]).

Example 5.1.2. Let R be a ring and let $\mathcal{C}$ be the class of all finitely generated projective right R-modules. In this case, all exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ split. Therefore in this case $\mathrm{Cl}(\mathcal{C})$ is the lattice whose unique element is $\mathrm{Ses}(\mathcal{C})$. For instance, if D is a division ring and $\mathcal{C}$ is the class of all right vector spaces of finite dimension over D, then $V(\mathcal{C}) \cong \mathbb{N}$. In this example, the correspondence Φ maps all the congruences on $V(\mathcal{C}) \cong \mathbb{N}$ to the class $\mathrm{Ses}(\mathcal{C})$, and Ψ maps all subclasses of $\mathrm{Ses}(\mathcal{C})$ to the equality $=$ on $V(\mathcal{C})$. ∎

Proposition 5.1.3. *Let $\mathcal{S}_\sim$ be a complete class of short exact sequences. The following properties hold:*

(a) *Every sequence isomorphic to a sequence in $\mathcal{S}_\sim$ also is in $\mathcal{S}_\sim$, that is, if there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

of right R-modules and module homomorphisms in which the vertical arrows denote module isomorphisms and the sequence $0 \to A \to B \to C \to 0$ belongs to $\mathcal{S}_\sim$, then the sequence $0 \to A' \to B' \to C' \to 0$ belongs to $\mathcal{S}_\sim$ as well.

(b) *Every exact sequence $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ and $B \cong A \oplus C$ is in $\mathcal{S}_\sim$.*

(c) *The direct sum $0 \to A \oplus A' \to B \oplus B' \to C \oplus C' \to 0$ of two sequences $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ belonging to $\mathcal{S}_\sim$ belongs to $\mathcal{S}_\sim$ as well.*

The proof of this proposition is elementary.

5.2 Descending series

In this section, $\mathcal{C}$ will be a small class of right R-modules closed under isomorphism and finite direct sums and $\mathcal{R}$ will be a class of short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$.

By a *descending series* we mean a finite chain $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of a right module A. We call n the *length* of the series.

Definition. A *descending series in* $\mathcal{R}$ is a descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A for which all the canonical short exact sequences $0 \to A_i \to A_{i-1} \to A_{i-1}/A_i \to 0$ $(i = 1, 2, \ldots, n)$ belong to $\mathcal{R}$.

Obviously, if $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ is a descending series in $\mathcal{R}$, then $A, A_1, \ldots, A_n$ belong to $\mathcal{C}$ and $\langle A \rangle \sim_{\mathcal{R}} \langle A_0/A_1 \rangle + \langle A_1/A_2 \rangle + \cdots + \langle A_{n-2}/A_{n-1} \rangle + \langle A_{n-1} \rangle$, where $\sim_{\mathcal{R}}$ denotes the congruence associated to $\mathcal{R}$.

Let A and B be right R-modules. We shall say that two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A and $B = B_0 \geq B_1 \geq \cdots \geq B_m = 0$ of submodules of B are *isomorphic* if $n = m$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $\langle A_{i-1}/A_i \rangle \cong \langle B_{\sigma(i)-1}/B_{\sigma(i)} \rangle$ for every $i = 1, 2, \ldots, n$. In this case, we shall say that A and B *have isomorphic descending series*. Obviously, if $A, B \in \mathcal{C}$ have two descending series in $\mathcal{R}$ that are isomorphic, then $\langle A \rangle \sim_{\mathcal{R}} \langle B \rangle$.

In many examples, however, it is more useful to consider a condition on descending series weaker than isomorphism. Let $\equiv$ be an arbitrarily fixed congruence on $V(\mathcal{C})$ and let A and B be right R-modules. We shall say that two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $B = B_0 \geq B_1 \geq \cdots \geq B_m = 0$ in $\mathcal{R}$ are *equivalent modulo* $\equiv$ if $n = m$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $\langle A_{i-1}/A_i \rangle \equiv \langle B_{\sigma(i)-1}/B_{\sigma(i)} \rangle$ for every $i = 1, 2, \ldots, n$. In this case, we shall say that A and B *have descending series in* $\mathcal{R}$ *equivalent modulo* $\equiv$. Thus two descending series are isomorphic if and only if they are equivalent modulo $=$.

Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\sim_{\mathcal{R}}$ and $\equiv$. Obviously, if $A, B \in \mathcal{C}$ have descending series in $\mathcal{R}$ equivalent modulo $\equiv$, then $\langle A \rangle \equiv_{\mathcal{R}} \langle B \rangle$. More precisely, the congruence $\equiv_{\mathcal{R}}$ is the transitive closure of the relation "having descending series in $\mathcal{R}$ equivalent modulo $\equiv$", as the next theorem shows.

Theorem 5.2.1 (Diracca and Facchini).
The following conditions are equivalent for two modules $A, B \in \mathcal{C}$:

(a) $\langle A \rangle \equiv_{\mathcal{R}} \langle B \rangle$.

(b) *There exist R-modules $B_0, B_1, B_2, \ldots, B_t \in \mathcal{C}$ with $B_0 = A$, $B_t = B$ and such that B_i, B_{i-1} have descending series in $\mathcal{R}$ equivalent modulo $\equiv$ for every $i = 1, 2, \ldots, t$.*

(c) *There exist R-modules* $A_1, A_2, \dots, A_t, B_0, B_1, B_2, \dots, B_t, C_1, C_2, \dots, C_t, A'_1, A'_2, \dots, A'_t, C'_1, C'_2, \dots, C'_t \in \mathcal{C}$ *with* $B_0 = A$, $B_t = B$, $\langle A_i \rangle \equiv \langle A'_i \rangle$ *and* $\langle C_i \rangle \equiv \langle C'_i \rangle$ *for every* $i = 1, 2, \dots, t$ *and exact sequences*

$$\begin{array}{ccccccccc}
0 & \to & A_1 & \to & A & \to & C_1 & \to & 0 \\
0 & \to & A'_1 & \to & B_1 & \to & C'_1 & \to & 0 \\
0 & \to & A_2 & \to & B_1 & \to & C_2 & \to & 0 \\
0 & \to & A'_2 & \to & B_2 & \to & C'_2 & \to & 0 \\
 & & & & \vdots & & & & \\
0 & \to & A_t & \to & B_{t-1} & \to & C_t & \to & 0 \\
0 & \to & A'_t & \to & B & \to & C'_t & \to & 0
\end{array}$$

in $\mathcal{R}$.

Proof. Since (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is trivial, it is sufficient to show that (a) $\Rightarrow$ (c).

Write $\langle A \rangle \sim \langle B \rangle$ if A and B satisfy condition (c). It is easily seen that $\sim$ is an equivalence relation in $V(\mathcal{C})$ contained in $\equiv_{\mathcal{R}}$. In order to prove that $\sim$ coincides with $\equiv_{\mathcal{R}}$, it is enough to show that $\sim$ is a congruence, that for every $X, Y, Z \in \mathcal{C}$ and every $0 \to X \to Y \to Z \to 0$ in $\mathcal{R}$ one has $Y \sim X \oplus Z$, and that $\langle A \rangle \equiv \langle B \rangle$ implies $A \sim B$.

If $X \sim Y$ and Z is a module in $\mathcal{C}$, then the direct sums of the exact sequence $0 \to Z \to Z \to 0 \to 0$ of $\mathcal{R}$ and the sequences that link X to Y show that $X \oplus Z \sim Y \oplus Z$. Thus $\sim$ is a congruence in $V(\mathcal{C})$.

If $X, Y, Z \in \mathcal{C}$ and the exact sequence $0 \to X \to Y \to Z \to 0$ belongs to $\mathcal{R}$, then the two sequences

$$\begin{array}{ccccccccc}
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
0 & \to & X & \to & X \oplus Z & \to & Z & \to & 0
\end{array}$$

of $\mathcal{R}$ show that $Y \sim X \oplus Z$.

Finally, if $\langle A \rangle \equiv \langle B \rangle$, then the two sequences

$$\begin{array}{ccccccccc}
0 & \to & A & \to & A & \to & 0 & \to & 0 \\
0 & \to & B & \to & B & \to & 0 & \to & 0
\end{array}$$

of $\mathcal{R}$ show that $A \sim B$. ■

The following corollary has been suggested to us by a similar result due to Heller [Rot63, pp. 731–732].

Corollary 5.2.2. *If* $A, B \in \mathcal{C}$ *and* $\langle A \rangle \equiv_{\mathcal{R}} \langle B \rangle$, *then there exists a module* $C \in \mathcal{C}$ *such that* $A \oplus C$ *and* $B \oplus C$ *have descending series in* $\mathcal{R}$ *equivalent modulo* $\equiv$.

Proof. By Theorem 5.2.1(c), there exist R-modules $A_1, A_2, \dots, A_t, B_0, B_1, B_2, \dots, B_t, C_1, C_2, \dots, C_t, A'_1, A'_2, \dots, A'_t, C'_1, C'_2, \dots, C'_t \in \mathcal{C}$ with $B_0 = A$, $B_t = B$, $\langle A_i \rangle \equiv \langle A'_i \rangle$ and $\langle C_i \rangle \equiv \langle C'_i \rangle$ for every $i = 1, 2, \dots, t$ and exact sequences

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \to & A & \to & C_1 & \to & 0 \\ 0 & \to & A'_1 & \to & B_1 & \to & C'_1 & \to & 0 \\ 0 & \to & A_2 & \to & B_1 & \to & C_2 & \to & 0 \\ 0 & \to & A'_2 & \to & B_2 & \to & C'_2 & \to & 0 \\ & & & & \vdots & & & & \\ 0 & \to & A_t & \to & B_{t-1} & \to & C_t & \to & 0 \\ 0 & \to & A'_t & \to & B & \to & C'_t & \to & 0 \end{array}$$

in $\mathcal{R}$.

Then $A \oplus B_1 \oplus \dots \oplus B_{t-1}$ has a descending series $A \oplus B_1 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq A_1 \oplus B_1 \oplus \dots \oplus B_{t-1} \geq B_1 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq A_2 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq \dots \geq B_{t-1} \geq A_t \geq 0$ whose factors are isomorphic to $C_1, A_1, C_2, A_2, \dots, C_t, A_t$ respectively. Similarly, $B \oplus B_1 \oplus B_2 \oplus \dots \oplus B_{t-1}$ has a descending series $B \oplus B_1 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq A'_t \oplus B_1 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq B_1 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq A'_1 \oplus B_2 \oplus \dots \oplus B_{t-1} \geq \dots \geq B_{t-1} \geq A'_{t-1} \geq 0$ whose factors are isomorphic to $C'_t, A'_t, C'_1, A'_1, \dots, C'_{t-1}, A'_{t-1}$ respectively. Set $C = B_1 \oplus \dots \oplus B_{t-1} \in \mathcal{C}$. The modules $A \oplus C$ and $B \oplus C$ have descending series in $\mathcal{R}$ equivalent modulo $\equiv$. ■

Every commutative monoid can be realized as $V(\mathcal{C})/\equiv_{\mathcal{R}}$, as the next theorem shows.

Theorem 5.2.3 (Diracca and Facchini).
For every commutative monoid M, there exists a small class $\mathcal{C}$ of right R-modules, closed under isomorphism and finite direct sums, a subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ *closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and a congruence $\equiv$ on $V(\mathcal{C})$ such that $V(\mathcal{C})/\equiv_{\mathcal{R}} \cong M$.*

Proof. Recall that a commutative additive monoid N is said to be *reduced* if $a + b = 0$ implies $a = b = 0$ for every $a, b \in N$, that is, if no non-zero element a of N has an additive inverse $-a$ in N. An element u of a commutative additive monoid N is an *order-unit* if $u \neq 0$ and for any $a \in N$ there exists an element $b \in N$ and an integer $n \geq 0$ such that $a + b = nu$.

Let M be a commutative monoid and let $\varphi \colon F \to M$ be a surjective monoid homomorphism of a free commutative monoid F onto M. Let $F_{+\infty} = F \cup \{+\infty\}$ be the set obtained by adjoining a further element $+\infty$ to the set F. The addition on F extends to an associative addition on $F_{+\infty}$ by setting $a + (+\infty) = (+\infty) + a = +\infty$ for every $a \in F$. Then $F_{+\infty}$ becomes a reduced commutative monoid with order-unit $+\infty$. By Bergman and Dick's Theorem [Fac02, Theorem 2.1], there exists a ring R with $F_{+\infty} \cong V(R)$ via an isomorphism that sends the element $+\infty$ of $F_{+\infty}$ to the element $\langle R_R \rangle$ of $V(R)$. Let $\mathcal{C}$ be the class of all finitely generated projective right R-modules not isomorphic to R_R and let $\mathcal{R}$ be the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$. Notice that all sequences in $\mathcal{R}$ are split. It is easily seen that $\mathcal{C}$ and $\mathcal{R}$ have the required properties and that $V(\mathcal{C}) \cong F$. Let $\equiv$ be the congruence on $V(\mathcal{C})$ corresponding to the congruence $\ker \varphi$ on F. As $\sim_{\mathcal{R}}$ is the equality on $V(\mathcal{C})$, the congruence $\equiv_{\mathcal{R}}$ coincides with $\equiv$, so that $V(\mathcal{C})/\equiv_{\mathcal{R}} \; = V(\mathcal{C})/\equiv \; \cong F/\ker \varphi \cong M$. ■

5.3 Transitive and strongly transitive classes

In this section, $\mathcal{C}$ will be a class of right R-modules closed under isomorphism and finite direct sums and $\mathcal{R}$ will be a class of short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$.

If $A, B \in \mathcal{C}$, we shall write $A \leq_{\mathcal{R}} B$ if A is a submodule of B and the canonical short exact sequence $0 \to A \to B \to B/A \to 0$ belongs to $\mathcal{R}$. Thus descending chains in $\mathcal{R}$ are exactly the chains $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A for which $A_i \leq_{\mathcal{R}} A_{i-1}$ for every $i = 1, 2, \dots, n$. For every $B \in \mathcal{C}$, let $\mathcal{L}_{\mathcal{R}}(B)$ be the set of all submodules A of B with $A \leq_{\mathcal{R}} B$ (so that, in particular, both A and B/A must belong to $\mathcal{C}$). Then $\mathcal{L}_{\mathcal{R}}(B)$, ordered by set inclusion, is a partially ordered subset of the lattice $\mathcal{L}(B)$ of all submodules of B. The submodules 0 and B of B are the smallest element and the greatest element of $\mathcal{L}_{\mathcal{R}}(B)$ respectively.

In general, descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ in $\mathcal{R}$ do not coincide with finite descending chains in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$, as the following example shows.

Example 5.3.1. Let D be a division ring, let $\mathcal{C}$ be the class of all finite dimensional right vector spaces over D of dimension $\neq 1$, and let $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$. Then $V(\mathcal{C}) = V(\mathcal{C})/{\sim_{\mathcal{R}}} \cong \mathbf{N} \setminus \{1\}$. Let $D^5 > D^3 > D^2 > 0$ be vector spaces of dimension $5, 3, 2, 0$ respectively, each contained in the previous one. Then D^2 and D^3 belong to $\mathcal{L}_{\mathcal{R}}(D^5)$, so that $D^5 > D^3 > D^2 > 0$ is a chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(D^5)$. But $D^5 > D^3 > D^2 > 0$ is not a descending series in $\mathcal{R}$. ∎

Proposition 5.3.2. *The following conditions are equivalent for a subclass $\mathcal{R}$ of* $\mathrm{Ses}(\mathcal{C})$*:*

(a) *For every $B_0 \in \mathcal{C}$, the set of all descending series $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ in $\mathcal{R}$ coincides with the set of all finite descending chains $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$ whenever $B \in \mathcal{C}$ and $B_0 \leq_{\mathcal{R}} B$.*

(b) *For every $A', A, B \in \mathcal{C}$ with $A' \leq A$ and $A \leq_{\mathcal{R}} B$, one has that $A' \leq_{\mathcal{R}} A$ if and only if $A' \leq_{\mathcal{R}} B$.*

(c) *For every A, B in $\mathcal{C}$ with $A \leq_{\mathcal{R}} B$, the position $A' \mapsto A'$ for every $A' \in \mathcal{L}_{\mathcal{R}}(A)$ defines an injective mapping of $\mathcal{L}_{\mathcal{R}}(A) \to \mathcal{L}_{\mathcal{R}}(B)$, whose image is the interval* $[0, A]$ *of* $\mathcal{L}_{\mathcal{R}}(B)$.

PROOF. (a) $\Rightarrow$ (b) Suppose that (a) holds. Let $A' \leq A \leq B$ be modules in $\mathcal{C}$ with $A \leq_{\mathcal{R}} B$. Then $A' \leq_{\mathcal{R}} A$ if and only if $B \geq A \geq A' \geq 0$ is a descending series in $\mathcal{R}$, if and only if $B \geq A \geq A' \geq 0$ is a chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$. This happens if and only if $A' \in \mathcal{L}_{\mathcal{R}}(B)$, i.e., if and only if $A' \leq_{\mathcal{R}} B$.

(b) $\Rightarrow$ (a) Suppose that (b) holds and that $B_0 \in \mathcal{C}$. If $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ is a descending series in $\mathcal{R}$, $B \in \mathcal{C}$ and $B_0 \leq_{\mathcal{R}} B$, then $B_i \leq_{\mathcal{R}} B_{i-1}$ for every i, so that $B_i \leq_{\mathcal{R}} B$ by (b). Thus $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ is a finite descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$. Conversely, let $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ be a finite descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$ for some $B \in \mathcal{C}$ with $B_0 \leq_{\mathcal{R}} B$. Then $B_i \leq_{\mathcal{R}} B$ and $B_{i-1} \leq_{\mathcal{R}} B$ imply that $B_i \leq_{\mathcal{R}} B_{i-1}$ by (b). Thus $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ is a descending series in $\mathcal{R}$.

(c) is merely a restatement of (b). ∎

We shall say that a subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ is *transitive* if it satisfies the equivalent conditions of Proposition 5.3.2. In this case, the relation $\leq_{\mathcal{R}}$ is necessarily a transitive relation in the class $\mathcal{C}$.

Example 5.3.3. Let $\mathcal{C}$ be the class of all finite dimensional right vector spaces of dimension $\neq 1$ over a division ring D considered in Example 5.3.1. We have already seen that, in this case, $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ is not a transitive subclass. Nevertheless, it is easily seen that the relation $\leq_{\mathcal{R}}$ is a transitive relation in $\mathcal{C}$.

Here is an example that shows that the relation $\leq_{\mathcal{R}}$ can be non-transitive. Let $\mathcal{C}$ be the class of all finitely generated abelian groups, so that $V(\mathcal{C})$ is the free commutative monoid having $\langle \mathbb{Z} \rangle$ and the $\langle \mathbb{Z}/p^n\mathbb{Z} \rangle$'s ($n \geq 1$ and p a prime number) as a free set of generators. Let $\mathcal{R}$ be the complete subclass of $\mathrm{Ses}(\mathcal{C})$

generated by the canonical exact sequence $0 \to p\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$. Then $\sim_{\mathcal{R}}$ is the congruence on $V(\mathcal{C})$ generated by the pair $(\langle\mathbb{Z}\rangle, \langle\mathbb{Z}\rangle + \langle\mathbf{Z}/p\mathbb{Z}\rangle)$. Then $\langle\mathbb{Z}\rangle \not\sim_{\mathcal{R}} \langle\mathbb{Z}\rangle + \langle\mathbb{Z}/p^2\mathbb{Z}\rangle$, so that no exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^2\mathbf{Z} \to 0$ belongs to the complete class $\mathcal{R}$. Therefore $p^2\mathbb{Z} \leq_{\mathcal{R}} p\mathbb{Z}$ and $p\mathbb{Z} \leq_{\mathcal{R}} \mathbf{Z}$, but $p^2\mathbb{Z} \not\leq_{\mathcal{R}} \mathbf{Z}$. ■

Proposition 5.3.4. *The following conditions are equivalent for a transitive subclass $\mathcal{R}$ of* $\mathrm{Ses}(\mathcal{C})$*:*

(a) *For every descending chain $B_0 \geq B_1 \geq \cdots \geq B_t$ of modules of $\mathcal{C}$ with $B_t \leq_{\mathcal{R}} B_0$, one has that $B_0/B_t \geq B_1/B_t \geq \cdots \geq B_t/B_t$ is a descending chain in $\mathcal{R}$ if and only if $B_0 \geq B_1 \geq \cdots \geq B_t$ is a descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B_0)$.*

(b) *For every $A \leq C \leq B$ with $A, B \in \mathcal{C}$ and $A \leq_{\mathcal{R}} B$, one has that $C \in \mathcal{C}$ and $C \leq_{\mathcal{R}} B$ if and only if $C/A \in \mathcal{C}$ and $C/A \leq_{\mathcal{R}} B/A$.*

(c) *For every $A, B \in \mathcal{C}$ with $A \leq_{\mathcal{R}} B$, the injective mapping from the interval $[A, B]$ of $\mathcal{L}_{\mathcal{R}}(B)$ to $\mathcal{L}_{\mathcal{R}}(B/A)$, defined by the position $C \mapsto C/A$ for every $C \in [A, B]$, is well defined and surjective.*

Proof. (a) $\Rightarrow$ (c) Suppose that (a) holds. Let A, B be in $\mathcal{C}$ and $A \leq_{\mathcal{R}} B$. In order to show that the mapping is well defined, notice that if $C \in [A, B]$ (that is, $C \in \mathcal{C}$, $A \leq C \leq B$ and $C \leq_{\mathcal{R}} B$), then $B \geq C \geq A$ is a descending chain in $\mathcal{L}_{\mathcal{R}}(B)$, so by (a) $B/A \geq C/A \geq A/A$ is a descending chain in $\mathcal{R}$. In particular, $C/A \leq_{\mathcal{R}} B/A$. This shows that the mapping is well defined.

In order to prove that the mapping is surjective, fix an element C/A of $\mathcal{L}_{\mathcal{R}}(B/A)$. Then $C/A \leq_{\mathcal{R}} B/A$, and so $B/A \geq C/A \geq 0$ is a descending chain in $\mathcal{R}$. By (a), $B \geq C \geq A$ is a descending chain in $\mathcal{L}_{\mathcal{R}}(B)$. Thus $C \in [A, B]$.

(b) $\Rightarrow$ (a) Let $B_0 \geq B_1 \geq \cdots \geq B_t$ be a descending chain of modules of $\mathcal{C}$ such that $B_t \leq_{\mathcal{R}} B_0$ and $B_0/B_t \geq B_1/B_t \geq \cdots \geq B_t/B_t$ is a descending chain in $\mathcal{R}$, i.e., a chain of modules B_i/B_t of $\mathcal{C}$ with $B_i/B_t \leq_{\mathcal{R}} B_{i-1}/B_t$ for every $i = 1, 2, \ldots, t$. Apply (b) to the triple $B_t \leq B_i \leq B_{i-1}$ for every $i = 1, 2, \ldots, t$ (this is possible because $B_t \leq_{\mathcal{R}} B_{i-1}$). Thus $B_i \leq_{\mathcal{R}} B_{i-1}$ for every $i = 1, 2, \ldots, t$. By transitivity, one has that $B_i \leq_{\mathcal{R}} B_0$ for every $i = 1, 2, \ldots, t$, and so $B_0 \geq B_1 \geq \cdots \geq B_t$ is a descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B_0)$.

Conversely, let $B_0 \geq B_1 \geq \cdots \geq B_t$ be a descending chain in $\mathcal{L}_{\mathcal{R}}(B_0)$, i.e., $B_i \leq_{\mathcal{R}} B_0$ for every $i = 1, 2, \ldots, t$. Apply (b) to the triple $B_t \leq B_i \leq B_0$ for every $i = 1, 2, \ldots, t$ (which is possible because $B_t \leq_{\mathcal{R}} B_0$). Then $B_i/B_t \leq_{\mathcal{R}} B_0/B_t$ for every $i = 1, 2, \ldots, t$, i.e., $B_0/B_t \geq B_1/B_t \geq \cdots \geq B_t/B_t$ is a descending chain in $\mathcal{L}_{\mathcal{R}}(B_0/B_t)$. By transitivity, it is a descending chain in $\mathcal{R}$.

(c) is merely a restatement of (b). ■

A subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ will be called a *strongly transitive* class if it is transitive and satisfies the equivalent conditions of Proposition 5.3.4.

Remarks. (a) Notice that if $\mathcal{R}$ is a transitive subclass of $\mathrm{Ses}(\mathcal{C})$, then the set inclusion $\subseteq$ on the set $\mathcal{L}_{\mathcal{R}}(A)$ and the relation $\leq_{\mathcal{R}}$ on $\mathcal{L}_{\mathcal{R}}(A)$ coincide, and descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ in $\mathcal{R}$ coincide with finite descending chains in the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$.

(b) If, moreover, $\mathcal{R}$ is strongly transitive, then, for every $A, B \in \mathcal{C}$ with $A \leq_{\mathcal{R}} B$, one has that the partially ordered set $\mathcal{L}_{\mathcal{R}}(B/A)$ is canonically order isomorphic to the interval $[A, B]$ of $\mathcal{L}_{\mathcal{R}}(C)$ for every module $C \in \mathcal{C}$ with $B \leq_{\mathcal{R}} C$.

5.4 Refinements and composition series

In Section 5.1 we considered a completely arbitrary congruence $\equiv$ on $V(\mathcal{C})$. This choice had the advantage of a great generality, but it immediately leads to pathologies. Suppose, for instance, that there exist non-zero modules $A \in \mathcal{C}$ with $\langle A \rangle \equiv \langle 0 \rangle$. Then, not only does there exist non-zero modules $A \in \mathcal{C}$ of $\mathcal{C}$ that are zero in $V(\mathcal{C})/\equiv_{\mathcal{R}}$, but also there could be non-zero modules $B \in \mathcal{C}$ that become invertible in $V(\mathcal{C})/\equiv_{\mathcal{R}}$ (consider $\langle B \rangle$ and $\langle A/B \rangle$ for any exact sequence $0 \to B \to A \to A/B \to 0$ in $\mathcal{R}$ with $\langle A \rangle \equiv \langle 0 \rangle$). This would lead us to a theory that could be interesting, but far from the applications and the examples we have in mind. Thus we shall consider only congruences $\equiv$ with the property that $\langle A \rangle \equiv \langle 0 \rangle$ implies $A = 0$ for every $A \in \mathcal{C}$.

In this section, $\mathcal{C}$ will be a small class of right R-modules closed under isomorphism and finite direct sums, $\mathcal{R}$ will be a transitive subclass of $\mathrm{Ses}(\mathcal{C})$*, closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and $\equiv$ will be a congruence on $V(\mathcal{C})$ such that, for every $A \in \mathcal{C}$, one has $\langle A \rangle \equiv \langle 0 \rangle$ if and only if $A = 0$.*

Lemma 5.4.1. *For every $A \in \mathcal{C}$, one has that $\langle A \rangle \equiv_{\mathcal{R}} \langle 0 \rangle$ if and only if $A = 0$.*

Proof. Suppose that $A \in \mathcal{C}$ and $\langle A \rangle \equiv_{\mathcal{R}} \langle 0 \rangle$. By the hypothesis that for every $B \in \mathcal{C}$ one has $\langle B \rangle \equiv \langle 0 \rangle$ if and only if $B = 0$, the only module that has a descending series equivalent modulo $\equiv$ to a descending series of the zero module is the zero module itself. Therefore $A = 0$ by Theorem 5.2.1((a) $\Rightarrow$ (b)). ∎

Thus, under the hypotheses of this section, the commutative monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is reduced. Conversely, using the idea of the proof of Theorem 5.2.3, it is easy to see that, for every reduced commutative monoid M, there exists a small class $\mathcal{C}$ of right R-modules, closed under isomorphism and finite direct sums, a subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and a congruence $\equiv$ on $V(\mathcal{C})$ such that $V(\mathcal{C})/\equiv_{\mathcal{R}} \; \cong M$ (Apply Bergman and Dick's Theorem [Fac02, Theorem 2.1] to the reduced monoid $M \cup \{+\infty\}$ with order-unit $+\infty$, and then let $\mathcal{C}$ be the class of all finitely generated projective R-modules non-isomorphic to R_R, $\mathcal{R} = \mathcal{S}_=$, and $\equiv$ be the identity on $V(\mathcal{C})$.)

As usual, a *refinement* of a descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$ is obtained by possibly inserting further elements in the chain, and a *composition series of A in $\mathcal{R}$* is a properly descending chain in $\mathcal{L}_{\mathcal{R}}(A)$ which has no refinements except by introducing repetitions of the elements of the chain. We say that a module A is *$\mathcal{R}$-simple* if $\mathcal{L}_{\mathcal{R}}(A)$ has exactly two elements (necessarily A and 0).

Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\sim_{\mathcal{R}}$ and $\equiv$. The image in $V(\mathcal{C})/\equiv_{\mathcal{R}}$ of an element $\langle A\rangle$ of $V(\mathcal{C})$ will be denoted $\langle A\rangle_{\equiv_{\mathcal{R}}}$.

Lemma 5.4.2. *Let $\mathcal{R}$ be a transitive subclass of* $\mathrm{Ses}(\mathcal{C})$*. Suppose that for every $A, B \in \mathcal{C}$, if A is $\mathcal{R}$-simple and $\langle A\rangle \equiv \langle B\rangle$, then B is $\mathcal{R}$-simple as well.*

Then:

(1) *For every $A \in \mathcal{C}$, $\langle A\rangle_{\equiv_{\mathcal{R}}}$ is indecomposable in the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ if and only if A is $\mathcal{R}$-simple.*

(2) *For every $A, C \in \mathcal{C}$ with A $\mathcal{R}$-simple, one has $\langle A\rangle \equiv_{\mathcal{R}} \langle C\rangle$ if and only if $\langle A\rangle \equiv \langle C\rangle$.*

Proof. Implication $\Rightarrow$ of (1). If A is not $\mathcal{R}$-simple, there exists $B \in \mathcal{C}$ with $B \leq_{\mathcal{R}} A$ and $0 \neq B \neq A$. Then the canonical exact sequence $0 \to B \to A \to A/B \to 0$ belongs to $\mathcal{R}$. Thus $\langle A\rangle_{\equiv_{\mathcal{R}}} = \langle B\rangle_{\equiv_{\mathcal{R}}} + \langle A/B\rangle_{\equiv_{\mathcal{R}}}$ in $V(\mathcal{C})/\equiv_{\mathcal{R}}$, and these elements of $V(\mathcal{C})/\equiv_{\mathcal{R}}$ are not zero by Lemma 5.4.1.

(2) Suppose that $A \in \mathcal{C}$ is $\mathcal{R}$-simple, $C \in \mathcal{C}$ and $\langle A\rangle \equiv_{\mathcal{R}} \langle C\rangle$. Then there exist R-modules $A_0, A_1, \dots, A_t \in \mathcal{C}$ with $A_0 = A$, $A_t = C$ and such that A_i, A_{i-1} have descending series in $\mathcal{R}$ equivalent modulo $\equiv$ for every $i = 1, 2, \dots, t$ (Theorem 5.2.1). The module A has a unique properly descending series in $\mathcal{R}$, namely $A > 0$. Thus A_1 has a descending series in $\mathcal{R}$ equivalent to this modulo $\equiv$, that is, A_1 has a descending series of length 1 in $\mathcal{R}$ with its factor equivalent to A modulo $\equiv$. It follows that $\langle A_1\rangle \equiv \langle A\rangle$ and that A_1 is R-simple. By induction on t one proves that $\langle A_i\rangle \equiv \langle A\rangle$ and A_i is R-simple for every i. The proof of (2) follows immediately.

For the implication $\Leftarrow$ of (1), suppose A $\mathcal{R}$-simple and that $\langle A\rangle_{\equiv_{\mathcal{R}}} = \langle A'\rangle_{\equiv_{\mathcal{R}}} + \langle A''\rangle_{\equiv_{\mathcal{R}}}$ for some $A', A'' \in \mathcal{C}$, that is, $\langle A\rangle \equiv_{\mathcal{R}} \langle A' \oplus A''\rangle$. Apply (2) to the modules A and $C = A' \oplus A''$, so that one gets that $\langle A\rangle \equiv \langle C\rangle = \langle A' \oplus A''\rangle$. Thus $A' \oplus A''$ is $\mathcal{R}$-simple. It follows that either $A' = 0$ or $A'' = 0$. ∎

Proposition 5.4.3. *Let $\mathcal{R}$ be a strongly transitive subclass of* $\mathrm{Ses}(\mathcal{C})$*. Then a descending series $B = B_0 > B_1 > \cdots > B_n = 0$ in $\mathcal{R}$ is a composition series of B in $\mathcal{R}$ if and only if all the elements $\langle B_{i-1}/B_i\rangle_{\equiv_{\mathcal{R}}}$, $i = 1, 2, \ldots, n$, of the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ are indecomposable in $V(\mathcal{C})/\equiv_{\mathcal{R}}$.*

Proof. A descending series $B = B_0 > B_1 > \cdots > B_n = 0$ in $\mathcal{R}$, that is, a properly descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$ (Proposition 5.3.2), is a composition series of B in $\mathcal{R}$ if and only if all the intervals $[B_i, B_{i-1}]$ in $\mathcal{L}_{\mathcal{R}}(B)$ have exactly two elements, or, equivalently, all the sets $\mathcal{L}_{\mathcal{R}}(B_{i-1}/B_i)$ have exactly two elements, i.e., all the quotients B_{i-1}/B_i are $\mathcal{R}$-simple. By Lemma 5.4.2(1), this holds if and only if every $\langle B_{i-1}/B_i\rangle_{\equiv_{\mathcal{R}}}$ is indecomposable in $V(\mathcal{C})/\equiv_{\mathcal{R}}$. ∎

Thus if $B = B_0 > B_1 > \cdots > B_n = 0$ is a composition series of B in $\mathcal{R}$, then $\langle B\rangle_{\equiv_{\mathcal{R}}} = \langle B_0/B_1\rangle_{\equiv_{\mathcal{R}}} + \langle B_1/B_2\rangle_{\equiv_{\mathcal{R}}} + \cdots + \langle B_{n-2}/B_{n-1}\rangle_{\equiv_{\mathcal{R}}} + \langle B_{n-1}\rangle_{\equiv_{\mathcal{R}}}$ is a decomposition of $\langle B\rangle_{\equiv_{\mathcal{R}}}$ as a sum of indecomposable elements in $V(\mathcal{C})/\equiv_{\mathcal{R}}$.

For any module $A \in \mathcal{C}$, the set $\mathcal{L}_{\mathcal{R}}(A)$ is only a partially ordered subset of the lattice $\mathcal{L}(A)$. If $\mathcal{L}_{\mathcal{R}}(A)$, with the partial order induced from $\mathcal{L}(A)$, turns out to be a lattice, we shall denote by $B \vee C$ and $B \wedge C$ the join and the meet of two elements B, C of $\mathcal{L}_{\mathcal{R}}(A)$.

Recall that a monoid M is said to be a *refinement monoid* if whenever $a + b = c + d$ in M, there exist $x, y, z, t \in M$ such that $a = x + y$, $b = z + t$, $c = x + z$ and $d = y + t$.

Theorem 5.4.4 (Diracca and Facchini).
Let $\mathcal{R}$ be a strongly transitive subclass of $\mathrm{Ses}(\mathcal{C})$*. Suppose that:*

(a) *$\mathcal{L}_{\mathcal{R}}(A)$ is a modular lattice for every $A \in \mathcal{C}$.*

(b) *For every $A \in \mathcal{C}$ and every $B, C \in \mathcal{L}_{\mathcal{R}}(A)$, $\langle B \vee C/B\rangle \equiv \langle C/B \wedge C\rangle$.*

(c) *For every $A, B \in \mathcal{C}$ with $\langle A\rangle \equiv \langle B\rangle$ and every descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of A in $\mathcal{R}$, there exists a descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ of B in $\mathcal{R}$ equivalent modulo $\equiv$ to the previous one.*

Then the following statements hold:

(1) *For every $A, B \in \mathcal{C}$, $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$ if and only if A and B have descending series in $\mathcal{R}$ equivalent modulo $\equiv$.*

(2) (Schreier Refinement Theorem) *For every $A \in \mathcal{C}$, any two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ in $\mathcal{R}$ have refinements in $\mathcal{R}$ equivalent modulo $\equiv$.*

(3) *For every $A, B \in \mathcal{C}$, if A is $\mathcal{R}$-simple and $\langle A\rangle \equiv \langle B\rangle$, then B is $\mathcal{R}$-simple as well.*

(4) *$V(\mathcal{C})/\equiv_{\mathcal{R}}$ is a refinement monoid.*

Proof. (3) follows immediately form (c).

(2) By Schreier refinement theorem for modular lattices [Ste75, Proposition III.3.1], the two series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ in the lattice $\mathcal{L}_{\mathcal{R}}(A)$ have "equivalent" refinements, where "equivalent" here means that the corresponding intervals of $\mathcal{L}_{\mathcal{R}}(A)$ are projective, that is, in the transitive closure of the relation "being similar". Now two intervals of $\mathcal{L}_{\mathcal{R}}(A)$ are similar if and only if they can be written in the form $[X, X \vee Y]$ and $[X \wedge Y, Y]$ for suitable $X, Y \in \mathcal{L}_{\mathcal{R}}(A)$. By (b), the isomorphism classes of the corresponding quotient modules are equivalent modulo $\equiv$, that is, $\langle X \vee Y/X\rangle \equiv \langle Y/X \wedge Y\rangle$. It follows that the two series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ and $B = B'_0 \geq B'_1 \geq \cdots \geq B'_m = 0$ have two refinements, which are descending series in $\mathcal{R}$ equivalent modulo $\equiv$.

(1) By Theorem 5.2.1, it is sufficient to show that the relation $\approx$, defined for all $\langle A\rangle, \langle B\rangle \in V(\mathcal{C})$ by $\langle A\rangle \approx \langle B\rangle$ if A and B have descending series in $\mathcal{R}$ equivalent modulo $\equiv$, is transitive. Let A, B, C be elements of $\mathcal{C}$, let $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ be descending series in $\mathcal{R}$ equivalent modulo $\equiv$, and let $B = B'_0 \geq B'_1 \geq \cdots \geq B'_m = 0$ and $C = C_0 \geq C_1 \geq \cdots \geq C_m = 0$ be descending series in $\mathcal{R}$ equivalent modulo $\equiv$. Apply (2) to the two descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ and $B = B'_0 \geq B'_1 \geq \cdots \geq B'_m = 0$. By (2), these series have two refinements $B = B_0 = \overline{B}_0 \geq \overline{B}_1 \geq \cdots \geq \overline{B}_s = B_1 \geq \cdots \geq \overline{B}_t = 0$ and $B = B'_0 = \overline{B'}_0 \geq \overline{B'}_1 \geq \cdots \geq \overline{B'}_r = B'_1 \geq \cdots \geq \overline{B'}_t = 0$, which are descending series in $\mathcal{R}$ equivalent modulo $\equiv$. Assume that $\langle B_0/B_1\rangle \equiv \langle A_{i-1}/A_i\rangle$. As $\mathcal{R}$ is strongly transitive, we can apply hypothesis (c) and find a refinement of the series $A_{i-1} \geq A_i$ corresponding to the refinement $B_0 = \overline{B}_0 \geq \overline{B}_1 \geq \cdots \geq \overline{B}_s = B_1$ of the series $B_0 \geq B_1$. In this way, we get a refinement of the descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ in $\mathcal{R}$ equivalent modulo $\equiv$ to the descending series $B = \overline{B}_0 \geq \overline{B}_1 \geq \cdots \geq \overline{B}_s \geq \cdots \geq \overline{B}_t = 0$. Similarly, one constructs a refinement of the descending series $C = C_0 \geq C_1 \geq \cdots \geq C_m = 0$ equivalent modulo $\equiv$ to the descending series $B = \overline{B'}_0 \geq \overline{B'}_1 \geq \cdots \geq \overline{B'}_r \geq \cdots \geq \overline{B'}_t = 0$.

The proof of (4) is similar to the proof of [Bro98, Proposition 3.8]. ∎

The hypotheses of Theorem 5.4.4 are not sufficient to assure that the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is cancellative, even in the case in which $\mathcal{R}$ is the strongly transitive class $\mathrm{Ses}(\mathcal{C})$ and $\equiv$ is the identity. For instance, the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is not cancellative when $\mathcal{C}$ is the class of all right vector spaces of dimension $\leq \aleph_0$ over a division ring D. Notice that, in this example, $\mathcal{L}_{\mathcal{R}}(A) = \mathcal{L}(A)$ for every $A \in \mathcal{C}$. Here, the reason why the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is not cancellative is due to the lack of a suitable finiteness condition. We need such a condition for the Jordan-Hölder Theorem to hold.

Recall that a partially ordered set L is said to *have finite length* if there is a natural number n such that every chain in L has at most n elements. Let $\mathcal{R}$ be a transitive subclass of $\mathrm{Ses}(\mathcal{C})$ and $\equiv$ be a congruence on $V(\mathcal{C})$. If we want a Jordan-Hölder type theorem to hold, that is, if we want every properly descending series in $\mathcal{R}$ to have a refinement that is a composition series in $\mathcal{R}$, and any two composition series of A in $\mathcal{R}$ to be equivalent modulo $\equiv$, then the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$ must have finite length. Conversely, if the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$ has finite length, then (1) every $A \in \mathcal{C}$ has a composition series in $\mathcal{R}$, (2) $\mathcal{L}_{\mathcal{R}}(A)$ satisfies the acc and the dcc, and (3) every properly descending series in $\mathcal{R}$ has a refinement that is a composition series in $\mathcal{R}$. The following theorem shows that all these concepts finds their natural setting under the hypotheses of Theorem 5.4.4.

Theorem 5.4.5 (Diracca and Facchini).
Suppose that the hypotheses (a)*,* (b) *and* (c) *of Theorem* 5.4.4 *hold. Then the following conditions are equivalent:*

(1) (The Jordan-Hölder Theorem) *Any $A \in \mathcal{C}$ has a composition series in $\mathcal{R}$, and any two composition series of A in $\mathcal{R}$ are equivalent modulo $\equiv$.*

(2) *The commutative monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is free.*

(3) *The lattice $\mathcal{L}_{\mathcal{R}}(A)$ has finite length for every $A \in \mathcal{C}$.*

(4) *Any $A \in \mathcal{C}$ has a composition series in $\mathcal{R}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that (1) holds. It suffices to show that the set of all $\langle A\rangle_{\equiv_{\mathcal{R}}}$, where A ranges in the set of all $\mathcal{R}$-simple modules of $\mathcal{C}$, is a free set of generators for $V(\mathcal{C})/\equiv_{\mathcal{R}}$. In view of Lemma 5.4.2 and Proposition 5.4.3, it is clearly a set of generators.

If $B_1, \dots, B_m$ are $\mathcal{R}$-simple modules of $\mathcal{C}$ with $\langle B_1\rangle_{\equiv_{\mathcal{R}}}, \dots, \langle B_m\rangle_{\equiv_{\mathcal{R}}}$ distinct elements of $V(\mathcal{C})/\equiv_{\mathcal{R}}$, and $s_1, \dots, s_m, t_1, \dots, t_m$ are non-negative integers with $\sum_{i=1}^m s_i\langle B_i\rangle_{\equiv_{\mathcal{R}}} = \sum_{i=1}^m t_i\langle B_i\rangle_{\equiv_{\mathcal{R}}}$, then $\langle \oplus_{i=1}^m B_i^{s_i}\rangle \equiv_{\mathcal{R}} \langle \oplus_{i=1}^m B_i^{t_i}\rangle$, so that the two modules $\oplus_{i=1}^m B_i^{s_i}$ and $\oplus_{i=1}^m B_i^{t_i}$ have descending series $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{R}$ equivalent modulo $\equiv$ (Theorem 5.4.4(1)). Now the series $\mathcal{D}'_1$

$$\oplus_{i=1}^m B_i^{s_i} > B_1^{s_1-1} \oplus \oplus_{i=2}^m B_i^{s_i} > B_1^{s_1-2} \oplus \oplus_{i=2}^m B_i^{s_i} > \dots > \oplus_{i=2}^m B_i^{s_i} > \dots > 0$$

is a composition series in $\mathcal{L}_{\mathcal{R}}(\oplus_{i=1}^m B_i^{s_i})$, which has s_1 factors isomorphic to B_1, s_2 factors isomorphic to B_2, $\dots$, s_m factors isomorphic to B_m. Similarly, the module $\oplus_{i=1}^m B_i^{t_i}$ has a composition series $\mathcal{D}'_2$ with t_1 factors isomorphic to B_1, t_2 factors isomorphic to B_2, $\dots$, t_m factors isomorphic to B_m. As we have seen in the proof of Theorem 5.4.4(1), two descending series in $\mathcal{R}$ of a module always have refinements in $\mathcal{R}$ equivalent modulo $\equiv$, so that the two descending series $\mathcal{D}_2$ and $\mathcal{D}'_2$ have two refinements in $\mathcal{R}$ equivalent modulo $\equiv$. But $\mathcal{D}'_2$ has no proper refinements in $\mathcal{R}$, so that $\mathcal{D}_2$ has a refinement $\mathcal{D}''_2$ in $\mathcal{R}$ with t_1 factors equivalent modulo $\equiv$ to $B_1, \dots, t_m$ factors equivalent modulo $\equiv$ to B_m. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathcal{R}$ are equivalent, it follows that $\mathcal{D}_1$ has a refinement $\mathcal{D}''_1$ in $\mathcal{R}$ with t_1 factors equivalent to $B_1, \dots, t_m$ factors equivalent to B_m. Condition (a), applied to the two composition series $\mathcal{D}'_1$ and $\mathcal{D}''_1$ implies that $s_1 = t_1, \dots, s_m = t_m$.

(2) $\Rightarrow$ (3) Let $A \in \mathcal{C}$. Suppose that $\mathcal{L}_{\mathcal{R}}(A)$ has not finite length. Then for every $n > 0$ there is a chain in $\mathcal{L}_{\mathcal{R}}(A)$ with more than n elements. Recall that the only modules $C \in \mathcal{C}$ with $\langle C\rangle_{\equiv_{\mathcal{R}}} = \langle 0\rangle_{\equiv_{\mathcal{R}}}$ in $V(\mathcal{C})/\equiv_{\mathcal{R}}$ are the modules $C = 0$ (Lemma 5.4.1). Thus $\langle A\rangle_{\equiv_{\mathcal{R}}}$ can be written as the sum of $\geq n$ non-zero elements of $V(\mathcal{C})/\equiv_{\mathcal{R}}$ for every $n > 0$. This cannot happen in a free commutative monoid.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) follows from Theorem 5.4.4(2). ∎

5.5 Examples

Example 5.5.1. The first obvious example is that of $\mathcal{C}$ the class of all R-modules of finite composition length, $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ and $\equiv$ the identity $=$ on $V(\mathcal{C})$. Then $\mathcal{R}$ is a strongly transitive class and $\mathcal{L}_{\mathcal{R}}(A) = \mathcal{L}(A)$ for every $A \in \mathcal{C}$, so that Theorems 5.4.4 and 5.4.5 apply. The monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is a free commutative monoid.

More generally, if $\mathcal{A}$ is an arbitrary abelian category and A is an object of $\mathcal{A}$, then the class of all subobjects of A is a "modular lattice" (it can be a proper class, and not necessarily a set, but it satisfies the axioms of a modular lattice), so that the Schreirer refinement Theorem in $\mathcal{A}$ follows from the Schreirer refinement Theorem for modular lattices [Ste75, pp. 67, 91 and 92]. If $\mathcal{F}$ is the full subcategory of all objects of $\mathcal{A}$ of finite length, then the Jordan-Hölder theorem holds in $\mathcal{F}$. ∎

Example 5.5.2. Biuniform modules. Let $\mathcal{C}$ be the small class of all the R-modules which are direct sums of finitely many biuniform modules. Let $\mathcal{R}$ be the class of all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, so that $\sim_R$ coincides with the equality $=$ on $V(\mathcal{C})$, and let $\equiv$ be the congruence on $V(\mathcal{C})$ defined by $\langle A\rangle \equiv \langle B\rangle$ if $[A]_m = [B]_m$.

Proposition 5.5.3. *Let $\mathcal{C}$ be the class of all the R-modules which are direct* *sums of finitely many biuniform modules. If $A, B \in \mathcal{C}$ and A is a direct summand of B, then $B/A \in \mathcal{C}$ also.*

Proof. Induction on $n = \dim B = \operatorname{codim} B$. The cases $n = 0$ and $n = 1$ are trivial. Suppose $n \geq 2$. The case $A = 0$ also is trivial, so that we may assume $A \neq 0$. In this case A has a direct sum decomposition $A = D \oplus A'$ with D biuniform and $A' \in \mathcal{C}$. Moreover, $B = C_1 \oplus \cdots \oplus C_n$ with the C_i's biuniform, and $A \oplus A'' = B$ for a suitable submodule A'' of B. Apply [Fac96, Proposition 9.5] to the direct sum decompositions $D \oplus (A' \oplus A'') = C_1 \oplus \cdots \oplus C_n$. Then there are two distinct indices $i, j = 1, \dots, n$ and a direct sum decomposition $\overline{D} \oplus E = C_i \oplus C_j$ of $C_i \oplus C_j$ such that $D \cong \overline{D}$ and $A' \oplus A'' \cong E \oplus (\oplus_{k \neq i,j} C_k)$. As dim and codim are additive on direct sums, the module E is biuniform as well. Thus we can apply the inductive hypothesis to the direct summand A' of $A' \oplus A''$, and obtain that $A'' \in \mathcal{C}$. ∎

By this proposition, if $A, B \in \mathcal{C}$, $A \leq_{\mathcal{R}} B$ simply means that A is a direct summand of B. It easily follows that:

Corollary 5.5.4. *The class $\mathcal{R}$ is strongly transitive.*

The $\mathcal{R}$-simple modules of $\mathcal{C}$ are exactly the biuniform modules, and the composition series of a module $B \in \mathcal{C}$ with $\dim B = \operatorname{codim} B = n$ are exactly the descending series $B = B_0 > B_1 > \cdots > B_n = 0$ of direct summands B_i of B with every B_i in $\mathcal{C}$ and $\dim B_i = \operatorname{codim} B_i = n - i$.

Notice that the conclusions of Theorem 5.4.4 and the equivalent conditions of Theorem 5.4.5 hold [DF02], though $\mathcal{L}_{\mathcal{R}}(A)$ is not a modular lattice in general.

The situation can be dualized, in the sense that all we have said in this example remains true if, instead of defining $\langle A \rangle \equiv \langle B \rangle$ if $[A]_m = [B]_m$, we set $\langle A \rangle \equiv \langle B \rangle$ if $[A]_e = [B]_e$. Also note that if we take as $\mathcal{C}$ the class of R-modules that are direct sums of finitely many uniform modules, as $\mathcal{R}$ the class of all split exact sequences in $\operatorname{Ses}(\mathcal{C})$, and as $\equiv$ the congruence on $V(\mathcal{C})$ defined by $\langle A \rangle \equiv \langle B \rangle$ if $[A]_m = [B]_m$, then $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ turns out to be a free commutative monoid [DF02]. Similarly when $\mathcal{C}$ is the class of R-modules that are direct sums of finitely many couniform modules and $\equiv$ is the congruence "belonging to the same epigeny class". ∎

Remark. (and notations for the rest of this section). We saw in Example 5.5.1 that if $\mathcal{A}$ is an arbitrary abelian category and $\mathcal{F}$ is the full subcategory of $\mathcal{A}$ whose objects are all objects of $\mathcal{A}$ of finite length, then the Schreirer Refinement Theorem holds in $\mathcal{A}$ and the Jordan-Hölder Theorem holds in $\mathcal{F}$. It follows that if $\mathcal{C}$ is a class of right R-modules, viewed as a full subcategory of Mod-R, and F is a functor from $\mathcal{C}$ to $\mathcal{A}$ or $\mathcal{F}$, then the information we have about descending chain of subobjects in $\mathcal{A}$ or $\mathcal{F}$ can be lifted to get information about descending chains of subobjects in $\mathcal{C}$.

Let us apply this remark to the case in which $\mathcal{A} = \text{Mod-}R'$ for another suitable ring R'. Let R, R' be two rings and $\mathcal{C}, \mathcal{C}'$ be small classes of right R-modules and right R'-modules respectively, both closed under isomorphism and finite direct sums. View $\mathcal{C}$ and $\mathcal{C}'$ as full subcategory of Mod-R and Mod-R'. Let $\mathcal{R}'$ be a class of exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}'$ and let $\equiv'$ be a congruence on $V(\mathcal{C}')$. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be an additive functor with the following two properties:

(a) for every exact sequence $0 \to A' \to B' \to C' \to 0$ in $\mathcal{R}'$ there exists an exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Ses}(\mathcal{C})$ with $F(A) \cong A', F(B) \cong B'$ and $F(C) \cong C'$.

(b) for every $A \in \mathcal{C}$, $F(A) = 0$ implies $A = 0$.

Let $\mathcal{R}$ be the class of all exact sequences $0 \to A \to B \to C \to 0$ belonging to $\mathrm{Ses}(\mathcal{C})$ such that the corresponding sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact and belongs to $\mathcal{R}'$, and let $\equiv$ be the congruence on $V(\mathcal{C})$ defined, for every $A, B \in \mathcal{C}$, by $\langle A \rangle \equiv \langle B \rangle$ if and only if $\langle F(A) \rangle \equiv' \langle F(B) \rangle$. Then F induces a monoid isomorphism $\widetilde{F}$ of $V(\mathcal{C})/\equiv_{\mathcal{R}}$ onto $V(\mathcal{C}')/\equiv'_{\mathcal{R}'}$.

Example 5.5.5. Finitely generated modules, polyserial modules and finite-rank torsion-free modules over commutative valuation domains. A modules is *uniserial* if its submodules form a chain under inclusion. Non-zero uniserial modules are biuniform. A *valuation domain* is a commutative integral domain R with R_R uniserial. A module A_R over a valuation domain R is *polyserial* if it has a series $A = A_0 > A_1 > \cdots > A_n = 0$ of submodules with each A_i pure in A and each A_{i-1}/A_i uniserial [FS01, p. 403]. For instance, every finitely generated module over a valuation domain is polyserial [FS01, Lemma I.7.8].

Let R be a valuation domain and $\mathcal{C}$ be the class of all finitely generated R-modules. Let R' be a maximal immediate extension of R [FS01, pp. 58–60], so that R' is a flat R-algebra. If $\mathcal{C}'$ is the class of all finitely generated R'-modules, then $V(\mathcal{C}')$ is a free commutative monoid, because every finitely generated R'-module is a direct sum of cyclic R'-modules in an essentially unique way (the Krull-Schmidt Theorem holds for finitely generated modules over maximal valuation domains). Let $\mathcal{R}'$ denote the class of all sequences in $\mathrm{Ses}(\mathcal{R}')$ that are split, and let $\equiv'$ be the identity on $V(\mathcal{C}')$. Thus $\equiv_{\mathcal{R}'}$ is the identity on $V(\mathcal{C}')$ as well, and $V(\mathcal{C}')/\equiv_{\mathcal{R}'} \ = V(\mathcal{C}')$.

Let $F \colon \mathcal{C} \to \mathcal{C}'$ be the functor defined by $F(A) = A \otimes_R R'$ for every $A \in \mathcal{C}$, so that F satisfies conditions (a) and (b) above. If $0 \to A \to B \to C \to 0$ is a pure exact sequence with $A, B, C \in \mathcal{C}$, then the sequence $0 \to F(A) = A \otimes_R R' \to F(B) = B \otimes_R R' \to F(C) = C \otimes_R R' \to 0$ is pure as well. Now $C \otimes_R R'$ is a direct sum of uniserial R'-modules, hence it is pure-injective by [FS01, Theorem XIII.5.2]. Thus the exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ splits. Conversely, let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{Ses}(\mathcal{C})$ such that $0 \to F(A) \to F(B) \to F(C) \to 0$ is a splitting exact sequence of R'-modules. For every R-module M we have a commutative square

$$\begin{array}{ccc} A \otimes_R R \otimes_R M & \to & A \otimes_R R' \otimes_R M \\ \downarrow & & \downarrow \\ B \otimes_R R \otimes_R M & \to & B \otimes_R R' \otimes_R M \end{array}$$

of R-module homomorphisms. The horizontal arrows are injective because the embedding $R \to R'$ is a pure monomorphism of R-modules. As $X \otimes_R R' \otimes_R M \cong X \otimes_R R' \otimes_{R'} R' \otimes_R M = F(X) \otimes_{R'} (R' \otimes_R M)$ for every R-module X and $F(A) \to F(B)$ is a pure monomorphism of R'-modules, the vertical arrow on the right of the commutative square is injective as well. It follows that the vertical arrow on the left is injective, that is, the exact sequence $0 \to A \to B \to C \to 0$ is pure. In the notation above, we have proved that the class $\mathcal{R}$ consists exactly of all pure sequences in $\mathrm{Ses}(\mathcal{C})$.

Let us see what the congruence $\equiv$ on $V(\mathcal{C})$ is in this case. Let A be a finitely generated R-module. By [FS01, Lemma I.7.8], A has a *pure composition series with cyclic factors*, that is, there exists a descending series $A = A_0 > A_1 > \dots > A_n = 0$ of submodules with each A_i pure in A and each A_{i-1}/A_i cyclic. We have just seen that $F(A) \cong \oplus_{i=1}^n (A_{i-1}/A_i) \otimes_R R'$. Moreover, two cyclic R-modules are isomorphic if and only if they remain isomorphic when they are tensored with R'. It follows that for every $A, B \in \mathcal{C}$, $\langle F(A) \rangle = \langle F(B) \rangle$ if and only if A and B have isomorphic pure composition series with cyclic factors. Thus the congruence $\equiv$ on $V(\mathcal{C})$ is defined by $\langle A \rangle \equiv \langle B \rangle$ if and only if A and B have isomorphic pure composition series with cyclic factors. Notice that $\sim_{\mathcal{R}}$ is contained in $\equiv$, because if $0 \to A \to B \to C \to 0$ belongs to $\mathcal{R}$, then $0 \to F(A) \to F(B) \to F(C) \to 0$ splits, so that $\langle F(B) \rangle = \langle F(A) \rangle + \langle F(C) \rangle$, that is, $\langle F(B) \rangle \equiv' \langle F(A \oplus C) \rangle$, hence $\langle B \rangle \equiv \langle A \oplus C \rangle$. Therefore the congruences $\equiv$ and $\equiv_{\mathcal{R}}$ coincide, and the monoids $V(\mathcal{C})/\equiv_{\mathcal{R}} \; = V(\mathcal{C})/\equiv \; \cong V(\mathcal{C}')$ are free commutative monoids. The Jordan-Hölder Theorem holds in $\mathcal{C}$ (Salce and Zanardo, [FS01, Theorem V.5.5]).

Notice that this Jordan-Hölder Theorem in $\mathcal{C}$ does not follow from the Jordan-Hölder Theorem in the abelian category of all additive functors from finitely presented modules to abelian groups. More precisely, let ${}_R\mathrm{FP}$ denote the full subcategory of R-Mod whose objects are all finitely presented left modules over an arbitrary (not necessarily commutative) ring R. Let $({}_R\mathrm{FP},\mathrm{Ab})$ denote the category of all additive functors of ${}_R\mathrm{FP}$ into the category Ab of abelian groups. Then there is a full embedding of categories $\mathbf{\Psi}\colon \text{Mod-}R \to ({}_R\mathrm{FP},\mathrm{Ab})$ defined by $M_R \mapsto M\otimes_R-$ [Fac96, p. 26]. This embedding sends pure subobjects of Mod-R to subobjects of $({}_R\mathrm{FP},\mathrm{Ab})$, and pure-injective objects of Mod-R to injective objects of $({}_R\mathrm{FP},\mathrm{Ab})$. Thus the Jordan-Hölder Theorem in the abelian category $({}_R\mathrm{FP},\mathrm{Ab})$ yields some kind of information on the right modules M_R for which $M \otimes_R -$ is an object of finite length in $({}_R\mathrm{FP},\mathrm{Ab})$. Now if R is a commutative valuation domain and M_R is a cyclic R-module, then $\mathrm{End}(M_R)$ is a valuation ring, but not necessarily a field. Thus the corresponding object $M \otimes_R -$ is not necessarily a simple object in $({}_R\mathrm{FP},\mathrm{Ab})$, because its endomorphism ring is isomorphic to $\mathrm{End}(M_R)$. Thus the Jordan-Hölder Theorem in $\mathcal{C}$ due to Salce and Zanardo does not follow from the Jordan-Hölder Theorem in the abelian category $({}_R\mathrm{FP},\mathrm{Ab})$.

This example generalizes to the case of arbitrary polyserial modules [FS01, Theorem XII.1.6]. Namely, let $\mathcal{C},\mathcal{C}'$ be the classes of all polyserial modules over a valuation domain R and over a maximal immediate extension R' of R, respectively. Arguing as in the previous paragraph (R' is a pure R-algebra and uniserial R'-modules are pure-injective), one sees that every polyserial R'-module is a direct sum of uniserial R'-modules in an essentially unique way (the Krull-Schmidt Theorem holds because endomorphism rings of uniserial modules over commutative rings are local). Thus $V(\mathcal{C}')$ is a free commutative monoid having the set of all isomorphism classes $\langle U\rangle$ of uniserial R'-modules U as a free set of generators. Let $\mathcal{R}'$ denote the class of all splitting exact sequences in $\mathrm{Ses}(\mathcal{C}')$, $\equiv'$ be the identity on $V(\mathcal{C}')$, and $F\colon \mathcal{C} \to \mathcal{C}'$ be the functor $F\colon A \mapsto A \otimes_R R'$. As in the case of finitely generated R-modules, the class $\mathcal{R}$ of all exact sequences in $\mathrm{Ses}(\mathcal{C})$ that are mapped to split sequences via F turns out to be the class of all pure exact sequences in $\mathrm{Ses}(\mathcal{C})$. If A is a polyserial R-module and $A = A_0 > A_1 > \cdots > A_n = 0$ is a descending series of submodules with the A_i's pure in A and the A_{i-1}/A_i's uniserial, then $F(A) \cong \oplus_{i=1}^n (A_{i-1}/A_i) \otimes_R R'$, so that for every $A, B \in \mathcal{C}$, $\langle F(A)\rangle = \langle F(B)\rangle$ if and only if A and B have pure composition series with uniserial factors equivalent modulo having the same type [FS01, p. 346]. The monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}} \cong V(\mathcal{C}')$ is the free commutative monoid having the isomorphism classes of uniserial R'-modules as a free set of generators. This yields a Jordan-Hölder Theorem for $\mathcal{C}$ (cf. [FS01, Proposition XII.1.6]).

Another possible generalization is that one to finite-rank torsion-free R-modules [FS01, Theorem XV.1.7]. In this case, $\mathcal{C}$ and $\mathcal{C}'$ are the classes of all finite-rank torsion-free modules over a valuation domain R and over a maximal immediate extension R' of R, respectively. As before, every finite-rank torsion-free R'-module is a direct sum of standard uniserial R'-modules in an essentially unique way. Thus $V(\mathcal{C}')$ is a free commutative monoid having the set of all isomorphism classes $\langle U\rangle$ of standard uniserial R'-modules U as a free set of generators. Let $\mathcal{R}'$ denote the class of all splitting exact sequences in $\mathrm{Ses}(\mathcal{R}')$, $\equiv'$ be the identity on $V(\mathcal{C}')$ and $F\colon \mathcal{C} \to \mathcal{C}'$ be the functor $F\colon A \mapsto A \otimes_R R'$. If $A \in \mathcal{C}$ and $A = A_0 > A_1 > \cdots > A_n = 0$ is a descending series of submodules with the A_i's pure in A and the A_{i-1}/A_i's uniserial, then $F(A) \cong \oplus_{i=1}^n (A_{i-1}/A_i) \otimes_R R'$. The monoid $V(\mathcal{C})/\equiv_{\mathcal{R}} \cong V(\mathcal{C}')$ is the free commutative monoid having the isomorphism classes of standard uniserial R'-modules as a free set of generators, and a Jordan-Hölder Theorem for $\mathcal{C}$ holds [FS01, Proposition XV.1.7]). ■

Example 5.5.6. Artinian divisible modules over a commutative, noetherian, local, 1-dimensional, Cohen-Macaulay ring. This example is taken from [Mat73, Chapter V]. Let R be a commutative, noetherian, local, 1-dimensional, Cohen-Macaulay ring. Recall that an element of R is called *regular* if it is not a zero-divisor. An R-module A is *divisible* if $Ar = A$ for every regular element $r \in A$. Let $\mathcal{C}$ be the class of all artinian divisible R-modules. The class $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ is strongly transitive, as is easily seen. Let $\equiv$ be the congruence on $V(\mathcal{C})$ defined, for every $A, B \in \mathcal{C}$, by $\langle A\rangle \equiv \langle B\rangle$ if A and B belong to the same epigeny class (terminology as in Example 5.5.2). Recall that an R-module A is said to be a *simple divisible module* if it is a non-zero, torsion, divisible module that has no proper non-zero divisible submodules [Mat73, p. 46]. Every $A \in \mathcal{C}$ has a composition series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ in $\mathcal{R}$, and any two composition series of A are equivalent modulo $\equiv$ [Mat73, Theorem 5.10]. Thus $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is a free commutative monoid isomorphic to the free commutative monoid freely generated by the set of all epigeny classes $[A]_e$, where A ranges in the class of all simple divisible R-modules [Mat73, Theorem 5.10]. ■

Example 5.5.7. h-divisible torsion modules and complete torsion-free modules. Let R be a commutative ring, Q its total ring of fractions and $K = Q/R$. An R-module A is *h-divisible* if it is a R-homomorphic image of a Q-module [Mat73]. The *torsion submodule* $t(A)$ of A is the set of all $x \in A$ with $xr = 0$ for some regular $r \in R$. An R-module A is *torsion* if $t(A) = A$, *torsion-free* if $t(A) = 0$, and *complete* if it is Hausdorff and complete with respect to the topology on A defined by taking the submodules of the form Ar, where r is a regular element of R, as a basis of neighborhoods of 0 in A. Let $\mathcal{D}$ be the class of all h-divisible torsion R-modules and let $\mathcal{C}$ be the class of all complete torsion-free R-modules. Equivalently, $\mathcal{C}$ is the class of all torsion-free R-modules X that are *co-*

torsion, that is, have the property that $\mathrm{Hom}_R(Q, X) = 0$ and $\mathrm{Ext}^1_R(Q, X) = 0$. If we view $\mathcal{D}$ and $\mathcal{C}$ as full subcategories of Mod-R, there is a category equivalence between $\mathcal{D}$ and $\mathcal{C}$ given by the functors $\mathrm{Hom}_R(K, -)\colon \mathcal{D} \to \mathcal{C}$ and $K \otimes_R -\colon \mathcal{C} \to \mathcal{D}$ [Mat73, Corollaries 2.3 and 2.4]. Let $\mathrm{Ses}(\mathcal{C})$ be the class of all short exact sequences $0 \to X \to Y \to Z \to 0$ with $X, Y, Z \in \mathcal{C}$. This is a complete strongly transitive class, as is easily verified.

Following [dlRF86], we say that a short exact sequence $0 \to A \to B \xrightarrow{f} C \to 0$ is *h-exact* if for every homomorphism $g\colon K \to C$ there exists a homomorphism $h\colon K \to B$ such that $g = fh$. If $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathrm{Ses}(\mathcal{C})$, the corresponding sequence $0 \to K \otimes_R X \to K \otimes_R Y \to K \otimes_R Z \to 0$ is an h-exact sequence that belongs to $\mathrm{Ses}(\mathcal{D})$ (cf. [dlRF86], where this is proved for an integral domain, but the same proof holds for an arbitrary commutative ring). Conversely, if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathrm{Ses}(\mathcal{D})$ that is h-exact, than the sequence $0 \to \mathrm{Hom}_R(K, A) \to \mathrm{Hom}_R(K, B) \to \mathrm{Hom}_R(K, C) \to 0$ is exact and belongs to $\mathrm{Ses}(\mathcal{C})$. Thus the complete strongly transitive class $\mathrm{Ses}(\mathcal{C})$ corresponds, via the category equivalence $\mathcal{C} \to \mathcal{D}$, to the subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{D})$ consisting of all h-exact sequences.

Proposition 5.5.8. *The subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{D})$ consisting of all h-exact sequences is a strongly transitive subclass of $\mathrm{Ses}(\mathcal{D})$.*

The proof is straightforward. Let us show, for instance, that if $A \leq C \leq B$, $A, B, C/A \in \mathcal{D}$, $A \leq_{\mathcal{R}} B$ and $C/A \leq_{\mathcal{R}} B/A$, then $C \in \mathcal{D}$. Let c be an element of C. As $C/A \in \mathcal{D}$, the natural map $\varphi\colon K \otimes_R \mathrm{Hom}_R(K, C/A) \to C/A$ defined by $\varphi(x \otimes f) = f(x)$ is an isomorphism [Mat73, Corollary 1.2]. Thus there exist $x_1, \dots, x_n \in K$ and $f_1, \dots, f_n \in \mathrm{Hom}_R(K, C/A)$ such that $\sum_{i=1}^n f_i(x_i) = c + A$. Each f_i can be viewed as a map of K into B/A, and the sequence $0 \to A \to B \to B/A \to 0$ is h-exact because $A \leq_{\mathcal{R}} B$. Thus there exist homomorphisms $g_i\colon K \to B$ such that $\pi g_i = f_i$, where $\pi\colon B \to B/A$ denotes the canonical projection. From $f_i(K) \subseteq C/A$, it follows that $g_i(K) \subseteq C$. Moreover, $c - \sum_{i=1}^n g_i(x_i) \in A$. This implies that there exist $y_1, \dots, y_m \in K$ and $h_1, \dots, h_m \in \mathrm{Hom}_R(K, A)$ such that $\sum_{j=1}^m h_j(y_j) = c - \sum_{i=1}^n g_i(x_i)$. Thus there exists a homomorphism $K^{n+m} \to C$ whose image contains c. This proves that $C \in \mathcal{D}$. ■

Example 5.5.9. Torsion-free abelian groups of finite rank. Let $\mathcal{F}$ be the class of all torsion-free abelian groups of finite rank and $\mathcal{P} = \mathrm{Ses}(\mathcal{F})$. Then $\mathcal{P}$ is a complete strongly transitive class. A module $A \in \mathcal{F}$ is $\mathcal{P}$-simple if and only if it has torsion-free rank 1. We shall show that the monoid $V(\mathcal{F})/\sim_{\mathcal{P}}$ is not cancellative, and that there exist $A, B \in \mathcal{F}$ that do not have isomorphic descending series in $\mathcal{P}$, but $\langle A\rangle \sim_{\mathcal{P}} \langle B\rangle$.

Let $A' \subseteq \mathbb{Q}$ be a torsion-free abelian group (of rank 1) such that $pA' \neq A'$ for every prime p and A' is not isomorphic to $\mathbf{Z}$. For instance, A' could be the group of all rationals with square-free denominators. By [Rot63, Lemma 7], there exists a group E with exact sequences $0 \to A' \to E \to \mathbb{Q} \to 0$ and $0 \to \mathbb{Z} \to E \to \mathbb{Q} \to 0$. Thus $\langle A'\rangle + \langle \mathbb{Q}\rangle \sim_{\mathcal{P}} \langle E\rangle \sim_{\mathcal{P}} \langle \mathbb{Z}\rangle + \langle \mathbb{Q}\rangle$.

Suppose $\langle A'\rangle \sim_{\mathcal{P}} \langle \mathbb{Z}\rangle$. By Theorem 5.2.1, there exist $A_0, A_1, \dots, A_t \in \mathcal{F}$ with $A_0 = A'$, $A_t = \mathbb{Z}$ and such that A_i, A_{i-1} have isomorphic descending series in $\mathcal{P}$ for every $i = 1, 2, \dots, t$. As two modules with isomorphic descending series in $\mathcal{P}$ have the same torsion-free rank, all the abelian groups $A_0, A_1, \dots, A_t$ must have torsion-free rank 1. But two groups of rank 1 with isomorphic descending series in $\mathcal{P}$ are isomorphic. It follows that $A' \cong \mathbb{Z}$, a contradiction. This shows that the monoid $V(\mathcal{F})/{\sim_{\mathcal{P}}}$ is not cancellative.

Let $\varphi\colon A' \oplus \mathbb{Q} \to \mathbb{Q}$ be any group homomorphism. Then $\varphi = \varphi_{a,b}$ for suitable $a, b \in \mathbb{Q}$, where $\varphi_{a,b}\colon (x, y) \mapsto ax + by$. Thus φ is onto and $\ker\varphi \cong A'$ for $b \neq 0$, while $\varphi(A' \oplus \mathbb{Q}) \cong A'$ and $\ker\varphi = 0 \oplus \mathbb{Q}$ for $b = 0$ and $a \neq 0$. This proves that all non-trivial strictly descending series of $A' \oplus \mathbf{Q}$ in $\mathcal{P}$ are isomorphic. Similarly, all non-trivial strictly descending series of $\mathbb{Z} \oplus \mathbb{Q}$ are isomorphic. Therefore $A' \oplus \mathbb{Q}$ and $\mathbb{Z} \oplus \mathbb{Q}$ do not have isomorphic descending series in $\mathcal{P}$. Thus $A = A' \oplus \mathbb{Q}$ and $B = \mathbb{Z} \oplus \mathbb{Q}$ have the required properties.

Notice that $\mathcal{L}_{\mathcal{P}}(A)$ is a lattice of finite length for every $A \in \mathcal{F}$, and that the monoid $V(\mathcal{F})/{\sim_{\mathcal{P}}}$ is generated by all classes of torsion-free modules of rank 1, which are exactly the indecomposable elements of $V(\mathcal{F})/{\sim_{\mathcal{P}}}$. Thus all composition series in $\mathcal{P}$ of $A' \oplus \mathbb{Q}$ are isomorphic, and the same holds for $\mathbb{Z} \oplus \mathbb{Q}$, but $\langle A' \oplus \mathbf{Q}\rangle_{\sim_{\mathcal{P}}} = \langle \mathbb{Z} \oplus \mathbf{Q}\rangle_{\sim_{\mathcal{P}}}$ is an element of $V(\mathcal{F})/{\sim_{\mathcal{P}}}$ that can be written as a sum of two indecomposable elements in infinitely many different ways.

Now fix a prime p, and let J_p be the ring of p-adic integers. Let $\mathcal{F}_p$ be the class of torsion-free J_p-modules of finite rank. If Q_p denotes the field of fractions of J_p, $\mathcal{F}_p$ consists of all J_p-modules isomorphic to a submodule of Q_p^n for some $n \geq 0$. Every module in $\mathcal{F}_p$ is the direct sum of finitely many copies of J_p's and Q_p in an essentially unique way [Kel98, Theorem 12], so that $V(\mathcal{F}_p)$ is the free commutative monoid with two generators $\langle J_p\rangle$ and $\langle Q_p\rangle$. Let $\mathcal{R}'$ be the class of all split exact sequences of $\mathrm{Ses}(\mathcal{F}_p)$ and $\equiv'$ be the identity of $V(\mathcal{F}_p')$. If $F\colon \mathcal{F} \to \mathcal{F}_p$ is the functor $- \otimes_{\mathbb{Z}} J_p$, then F satisfies conditions (a) and (b) of Remark 5.5. Let us show that in the notation introduced there, the subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{F})$ coincides with the whole $\mathcal{P} = \mathrm{Ses}(\mathcal{F})$. To this end, let $0 \to A \to B \to C \to 0$ be an element of $\mathrm{Ses}(\mathcal{F})$. Then the sequence $0 \to A \to B \to C \to 0$ is pure, so that applying the functor $F = - \otimes_{\mathbb{Z}} J_p$ we get a pure exact sequence belonging to $\mathrm{Ses}(\mathcal{F}_p)$. Since J_p and Q_p are pure-injective

J_p-modules [FS01, Theorem XIII.4.6], all pure-exact sequences in $\mathrm{Ses}(\mathcal{F}_p)$ split, and therefore belong to $\mathcal{R}'$. Thus $\mathcal{R} = \mathcal{P} = \mathrm{Ses}(\mathcal{F})$. The congruence $\equiv$ on $V(\mathcal{F})$ is defined by setting, for all $A, B \in \mathcal{F}$, $\langle A \rangle \equiv \langle B \rangle$ if $\langle F(A) \rangle \equiv' \langle F(B) \rangle$, that is, if and only if A and B have the same torsion-free rank and the same p-rank. Here the *p-rank* of a torsion-free abelian group A is the number of direct summands of $A \otimes_{\mathbb{Z}} J_p$ isomorphic to J_p, or, equivalently, the number of p-reduced factor groups of any compositions series of A in $\mathcal{P}$ [Rot63, p. 730]. Thus $V(\mathcal{F})/\equiv_{\mathcal{P}} \;=\; V(\mathcal{F})/\equiv_{\mathcal{R}} \;\cong\; V(\mathcal{F}_p)$ is the free commutative monoid freely generated by two elements. ■

Example 5.5.10. Noetherian modules. Let R be an arbitrarily fixed unital ring, $\mathcal{C}$ be the class of all noetherian right R-modules, $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ be the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and $\equiv$ be the identity on $V(\mathcal{C})$. In this case, the monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ was studied by Brookfield in [Bro98], and, for R right noetherian, in [Bro00]. A commutative monoid M is said to be *strongly separative* if $a + a = a + b$ implies $a = b$ for all $a, b \in M$. This property is weaker than the cancellation property. For an arbitrary ring R, Brookfield proved that $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ is strongly separative [Bro98, Theorem 5.1]. Moreover, if $\approx$ is the smallest congruence on $V(\mathcal{C})$ with $V(\mathcal{C})/{\approx}$ cancellative, so that $\approx$ is defined by $\langle A \rangle \approx \langle B \rangle$ if $\langle A \rangle + \langle C \rangle = \langle B \rangle + \langle C \rangle$ for some $\langle C \rangle \in V(\mathcal{C})$, then $\approx$ is smaller than $\equiv_{\mathcal{R}}$. Thus the canonical projection $V(\mathcal{C}) \to V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ induces a homomorphism of the cancellative monoid $V(\mathcal{C})/{\approx}$ onto the monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$. Notice that, in this case, $\mathcal{R}$ is trivially a strongly transitive class and that the hypotheses of Theorem 5.4.4 hold. The modules $A \in \mathcal{C}$ for which the lattice $\mathcal{L}_{\mathcal{R}}(A)$ has finite length are the modules of finite composition length. ■

Example 5.5.11. Torsion-free modules in a hereditary torsion theory. Let $\mathfrak{F}$ be a right Gabriel topology on a ring R [Ste75, p. 146]. For a right R-module A, $\mathbf{Sat}_{\mathfrak{F}}(A)$ will denote the set of all $\mathfrak{F}$-*saturated submodules* of A, that is, the set of all submodules B of A with A/B $\mathfrak{F}$-torsion-free. Let Mod-$(R, \mathfrak{F})$ be the full subcategory of Mod-R whose objects are all $\mathfrak{F}$-closed modules. The partially ordered set $\mathbf{Sat}_{\mathfrak{F}}(A)$ is a complete modular lattice, isomorphic to the lattice of all subobjects of $A_{\mathfrak{F}}$ in the abelian category Mod-$(R, \mathfrak{F})$ [Ste75, Theorem IX.4.1 and Corollary IX.4.4]. Let $\mathcal{C}$ be the class of all $\mathfrak{F}$-torsion-free right R-modules A for which $\mathbf{Sat}_{\mathfrak{F}}(A)$ has finite length.

Lemma 5.5.12. *The injective envelope $E(A)$ of any module $A \in \mathcal{C}$ is an R-module of finite Goldie dimension. In particular, the class $\mathcal{C}$ is small.*

Proof. Let $A \in \mathcal{C}$ be an R-module with $E(A)$ of infinite Goldie dimension. Then A has a family of non-zero R-submodules B_i, $i \geq 0$, such that $A \supseteq \oplus_{i=0}^{\infty} B_i$. It follows that $E(A)$ has an ascending chain of direct summands $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ with $E_n = E(\oplus_{i=0}^{n} B_i)$ and $E_n/E_{n-1} \cong E(B_n)$. Consider the ascending chain of submodules

$$A \cap E_0 \subseteq A \cap E_1 \subseteq A \cap E_2 \subseteq \dots \tag{5.5.1}$$

of A. The inclusion $A \to E(A)$ induces an embedding $A/A \cap E_n \to E(A)/E_n$, and $E(A)/E_n$ is isomorphic to a direct summand of $E(A)$. Since the class of $\mathfrak{F}$-torsion-free R-modules is closed for injective envelopes and submodules, is follows that $A/A \cap E_n$ is $\mathfrak{F}$-torsion-free. Thus (5.5.1) is an ascending chain in $\mathbf{Sat}_{\mathfrak{F}}(A)$. Moreover, $A \cap E_n \supseteq B_n$ and $A \cap E_{n-1} \cap B_n = 0$, which shows that (5.5.1) is a strictly ascending chain, a contradiction to the fact that $\mathbf{Sat}_{\mathfrak{F}}(A)$ has finite length.

In particular $\mathcal{C}$ is small, because every module in $\mathcal{C}$ is isomorphic to a submodule of the injective envelope of a direct sum of finitely many cyclic R-modules. ∎

Since the functor Mod-$R \to$ Mod-$(R, \mathfrak{F})$, $A \mapsto A_{\mathfrak{F}}$, is additive, it is clear that $\mathcal{C}$ is closed for finite direct sums and isomorphism. Let $\mathcal{F}$ be the full subcategory of Mod-$(R, \mathfrak{F})$ whose objects are all objects of finite length of Mod-$(R, \mathfrak{F})$ and $F\colon \mathcal{C} \to \mathcal{F}$ be the functor localization defined by $F(A) = A_{\mathfrak{F}}$ for every $A \in \mathcal{C}$, so that F satisfies conditions (a) and (b) of Remark 5.5. Let $\mathcal{R}'$ be Ses$(\mathcal{F})$ and $\equiv'$ be the equality on $V(\mathcal{F})$. Then $\mathcal{R} = $ Ses$(\mathcal{C})$, as the following proposition shows, and $\equiv$ is the congruence on $V(\mathcal{C})$ defined, for all $A, B \in \mathcal{C}$, by $\langle A \rangle \equiv \langle B \rangle$ if $A_{\mathfrak{F}} \cong B_{\mathfrak{F}}$.

Proposition 5.5.13. *If $A, B, C \in \mathcal{C}$ and $0 \to A \to B \xrightarrow{\varphi} C \to 0$ is an exact sequence of R-modules, then the sequence $0 \to A_{\mathfrak{F}} \to B_{\mathfrak{F}} \xrightarrow{\varphi_{\mathfrak{F}}} C_{\mathfrak{F}} \to 0$ is exact in the category* Mod-$(R, \mathfrak{F})$.

Proof. Let $0 \to A \to B \xrightarrow{\varphi} C \to 0$ be an exact sequence of R-modules with $A, B, C \in \mathcal{C}$. As the functor localization F is left exact, the sequence $0 \to A_{\mathfrak{F}} \to B_{\mathfrak{F}} \xrightarrow{\varphi_{\mathfrak{F}}} C_{\mathfrak{F}}$ is an exact sequence of $R_{\mathfrak{F}}$-modules. Thus we only have to prove that the morphism $\varphi_{\mathfrak{F}}\colon B_{\mathfrak{F}} \to C_{\mathfrak{F}}$ is an epimorphism in the category Mod-$(R, \mathfrak{F})$. To this end, it suffices to show that if $\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ denotes the image of the mapping $\varphi_{\mathfrak{F}}$ (that is, the image of $\varphi_{\mathfrak{F}}$ in the category Mod-$R_{\mathfrak{F}}$), then $C_{\mathfrak{F}}/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ is an $\mathfrak{F}$-torsion R-module. Let D be the R-submodule of $C_{\mathfrak{F}}$ such that $\varphi_{\mathfrak{F}}(B_{\mathfrak{F}}) \subseteq D$ and $D/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ is the $\mathfrak{F}$-torsion submodule of $C_{\mathfrak{F}}/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$. Then the sequence $0 \to A_{\mathfrak{F}} \to B_{\mathfrak{F}} \xrightarrow{\varphi_{\mathfrak{F}}} D \to 0$ is exact in Mod-$(R, \mathfrak{F})$. Thus if

$L(X)$ denotes the length of an object $X \in \mathcal{F}$, then $L(D) = L(B_{\mathfrak{F}}) - L(A_{\mathfrak{F}})$. Since the lattice $\mathbf{Sat}_{\mathfrak{F}}(Y)$ and the lattice of all subobjects of $Y_{\mathfrak{F}}$ in the abelian category Mod-$(R, \mathfrak{F})$ are isomorphic, it follows that $L(D) = \ell(B) - \ell(A)$, where we have denoted by $\ell(Y)$ the length of the lattice $\mathbf{Sat}_{\mathfrak{F}}(Y)$ for an arbitrary $Y \in \mathcal{F}$. These are modular lattices, $\mathbf{Sat}_{\mathfrak{F}}(A)$ is isomorphic to the interval $[0, A]$ of the lattice $\mathbf{Sat}_{\mathfrak{F}}(B)$, and $\mathbf{Sat}_{\mathfrak{F}}(C)$ is isomorphic to the interval $[A, B]$ of the lattice $\mathbf{Sat}_{\mathfrak{F}}(B)$. It follows that $L(D) = \ell(C) = L(C_{\mathfrak{F}})$. Thus $C_{\mathfrak{F}}$ and its subobject D have the same length in the category Mod-$(R, \mathfrak{F})$, so that $D = C_{\mathfrak{F}}$ and $C_{\mathfrak{F}}/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ is $\mathfrak{F}$-torsion. ■

In particular, $V(\mathcal{C})/\sim_{\mathcal{R}}$ is a free commutative monoid. ■

Example 5.5.14. Critical composition series. We conclude with an example that is beyond the theory we have developed so far, but that we think interesting. Let R be an arbitrary ring and $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all noetherian right R-modules. We shall denote by $\mathrm{K.dim}(A)$ the Krull dimension of a module A. For an ordinal $\alpha \geq 0$, recall that a module A is *α-critical* if $\mathrm{K.dim}(A) = \alpha$ and $K(A/B) < \alpha$ for all non-zero submodules B of A. A module is *critical* if it is α-critical for some ordinal α. A *critical composition series* of a noetherian module A is a chain $A = A_0 \geq A_1 \geq \dots \geq A_n = 0$ of submodules of A such that each of the factors A_{i-1}/A_i is critical and such that $\mathrm{K.dim}(A_{i-1}/A_i) \geq \mathrm{K.dim}(A_i/A_{i+1})$ for all $i = 1, 2, \dots, n$ [GJ89, p. 229].

Let $\mathcal{R}$ be the class of all exact sequences $0 \to A \to B \to C \to 0$ in $\mathrm{Ses}(\mathcal{C})$ with either (1) $\mathrm{K.dim}(A) \leq \mathrm{K.dim}(C)$ and C critical, or (2) $A = 0$, or (3) $C = 0$. Notice that this class $\mathcal{R}$ does not contain all split exact sequences.

Lemma 5.5.15. (a) *Let $A, B \in \mathcal{C}$ with $0 < A < B$. Then $A \leq_{\mathcal{R}} B$ if and only if* $\mathrm{K.dim}(A) \leq \mathrm{K.dim}(B/A)$ *and B/A is critical, if and only if* $\mathrm{K.dim}(B) = \mathrm{K.dim}(B/A)$ *and B/A is critical.*

(b) *A module $A \in \mathcal{C}$ is $\mathcal{R}$-simple if and only if it is critical.*

(c) *A chain $A = A_0 \geq A_1 \geq \dots \geq A_n = 0$ of submodules of a module $A \in \mathcal{C}$ is a critical composition series of A if and only if it is a composition series of A in $\mathcal{R}$.*

Proof. (a) is clear, because $\mathrm{K.dim}(B) = \max\{\mathrm{K.dim}(A), \mathrm{K.dim}(C)\}$ for every exact sequence $0 \to A \to B \to C \to 0$.

(b) Let $A \in \mathcal{C}$ be an $\mathcal{R}$-simple module and let α be its Krull dimension. By [GJ89, Exercise 13G], A has a proper submodule B with A/B α-critical. Thus $0 \to B \to A \to A/B \to 0$ belongs to $\mathcal{R}$. As A is $\mathcal{R}$-simple, it follows that $B = 0$, so that A is critical. Conversely, if A is critical, then $\mathrm{K.dim}(A/B) <$

$\mathrm{K.dim}(A)$ for every non-zero submodule B of A, and thus the exact sequence $0 \to B \to A \to A/B \to 0$ does not belong to $\mathcal{R}$. This shows that A is $\mathcal{R}$-simple.

(c) Let $A = A_0 \geq A_1 \geq \dots \geq A_n = 0$ be a critical composition series of a module $A \in \mathcal{C}$. First of all, we shall show that this is a descending chain in $\mathcal{R}$, that is, that $A_i \leq_{\mathcal{R}} A_{i-1}$ for all i. Induction on $n - i$. The case $i = n$ is trivial. Assume that $A_{i+1} \leq_{\mathcal{R}} A_i$. Then $\mathrm{K.dim}(A_i) = \mathrm{K.dim}(A_i/A_{i+1}) \leq \mathrm{K.dim}(A_{i-1}/A_i)$, so that $A_i \leq_{\mathcal{R}} A_{i-1}$. This proves that the chain is a descending chain in $\mathcal{R}$.

In order to prove that it is a composition series in $\mathcal{R}$, suppose that $C \in \mathcal{C}$, $A_i < C < A_{i-1}$ and $A_i \leq_{\mathcal{R}} C$. As C/A_i is a non-zero submodule of the critical module A_{i-1}/A_i, we have that $\mathrm{K.dim}(A_{i-1}/C) < \mathrm{K.dim}(A_{i-1}/A_i)$, from which $\mathrm{K.dim}(A_{i-1}/C) < \mathrm{K.dim}(C/A_i)$. Now $A_i \leq_{\mathcal{R}} C$ yields $\mathrm{K.dim}(C) = \mathrm{K.dim}(C/A_i)$, so that $\mathrm{K.dim}(C) > \mathrm{K.dim}(A_{i-1}/C)$. Thus $C \not\leq_{\mathcal{R}} A_{i-1}$.

Conversely, if $A = A_0 > A_1 > \dots > A_n = 0$ is a composition series of A in $\mathcal{R}$, then $0 \to A_i \to A_{i-1} \to A_{i-1}/A_i \to 0$ belongs to $\mathcal{R}$ for every $i = 1, \dots, n$, so that A_{i-1}/A_i is critical for every $i = 1, \dots, n-1$. Moreover, the term A_{n-1} of the composition series is always $\mathcal{R}$-simple, hence critical by (b). To conclude, we must prove that $\mathrm{K.dim}(A_i/A_{i+1}) \leq \mathrm{K.dim}(A_{i-1}/A_i)$ for every $i = 1, \dots, n-1$. Now $A_i \neq 0$, so that from $A_i \leq_{\mathcal{R}} A_{i-1}$, it follows that $\mathrm{K.dim}(A_{i-1}) = \mathrm{K.dim}(A_{i-1}/A_i)$. Similarly, $A_{i+1} \leq_{\mathcal{R}} A_i$ implies $\mathrm{K.dim}(A_i) = \mathrm{K.dim}(A_i/A_{i+1})$ if $A_{i+1} \neq 0$, but the same equality holds trivially in the case $A_{i+1} = 0$ also. Now $A_i \leq A_{i-1}$ implies that $\mathrm{K.dim}(A_i) \leq \mathrm{K.dim}(A_{i-1})$. It follows that $\mathrm{K.dim}(A_i/A_{i+1}) \leq \mathrm{K.dim}(A_{i-1}/A_i)$. ■

Let $\equiv$ be the congruence on $V(\mathcal{C})$ generated by all the pairs $(\langle A\rangle, \langle B\rangle)$ with $A, B \in \mathcal{C}$ critical and with isomorphic injective envelopes $E(A) \cong E(B)$. Every module $A \in \mathcal{C}$ has a critical composition series and any two critical composition series of A are equivalent modulo $\equiv$ (Jategaonkar, Gordon [GJ89, Theorem 13.9]). Let F be the free commutative monoid freely generated by the set of the isomorphism classes $\langle E(A)\rangle$ of the injective envelopes of all critical modules $A \in \mathcal{C}$. There is a monoid homomorphism $\varphi\colon V(\mathcal{C}) \to F$ defined as follows. For every $A \in \mathcal{C}$, there is, as we have already said, a critical composition series $A = A_0 > A_1 > \dots > A_n = 0$, unique up to the congruence $\equiv$. Set $\varphi(\langle A\rangle) = \sum_{i=1}^{n} \langle E(A_{i-1}/A_i)\rangle$. The uniqueness up to equivalence of the critical composition series says that this mapping is well defined. Let us prove that the congruence $\equiv_{\mathcal{R}}$ on $V(\mathcal{C})$ generated by $\equiv$ and $\sim_{\mathcal{R}}$ is the kernel $\ker\varphi$ of the homomorphism φ. If $A, B \in \mathcal{C}$ and $(\langle A\rangle, \langle B\rangle) \in \ker\varphi$, then A and B have critical composition series equivalent modulo $\equiv$, i.e., they have composition series in $\mathcal{R}$ equivalent modulo $\equiv$. It follows that $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$. Conversely, in order to prove that $\equiv_{\mathcal{R}}$ is contained in the kernel, it suffices to show that $\equiv$ and $\sim_{\mathcal{R}}$ are both contained in the kernel, and both these facts are easily verified. Thus $V(\mathcal{C})/{\equiv_{\mathcal{R}}} \cong F$ is a free commutative monoid. ■

Chapter 6

Left-Right Symmetry in the Krull-Schmidt Monoid of a Semiperfect Ring

In this chapter monoids will always assumed to be commutative and will be written additively, with 0 as the identity element.

Recall that a monoid homomorphism $f : M_1 \to M_2$ is a *divisor homomorphism* provided whenever $f(x) + z = f(y)$ (with $x, y \in M_1$ and $z \in M_2$) there is an element $w \in M_1$ such that $x + w = y$. Also recall that a submonoid M' of a cancellative monoid M is said to be a *full submonoid* provided the inclusion $M' \hookrightarrow M$ is a divisor homomorphism, equivalently, $M' = G \cap M$, where G is a subgroup of the quotient group $Q(M) = \{x - y | x, y \in M\}$ of M. A *Krull monoid* is a commutative cancellative monoid admitting a divisor homomorphism into some free monoid $N^{(I)}$.

In the previous chapter, we investigated the cases in which the monoid $V(\mathcal{C})$ or some of its quotients are free. If $V(\mathcal{C})$ is not free, then uniqueness of direct sum decompositions into indecomposables fails, but when $V(\mathcal{C})$ is a Krull monoid, interesting things can still be said. For instance, suppose $V(\mathcal{C})$ is a Krull monoid. In this case, though uniqueness of decomposition into indecomposables can fail, there is a divisor homomorphism of $V(\mathcal{C})$ into a free commutative monoid, and this exactly means that there is a family of invariants $f_\lambda \colon \mathcal{C} \to \mathbb{N}, \lambda \in \Lambda$, such that for every $A, B \in \mathcal{C}$:

(1) $A \cong B$ if and only if $f_\lambda(A) = f_\lambda(B)$ for every $\lambda \in \Lambda$;

(2) $f_\lambda(A \oplus B) = f_\lambda(A) + f_\lambda(B)$ for every $\lambda \in \Lambda$;

(3) there exists $C \in \mathcal{C}$ such that $A \oplus C \cong B$ if and only if $f_\lambda(A) \leq f_\lambda(B)$ for every $\lambda \in \Lambda$; and

(4) for every $A \in \mathcal{C}$ we have $f_\lambda(A) = 0$ for almost all λ's.

Thus, if $V(\mathcal{C})$ is a Krull monoid, every module $A \in \mathcal{C}$ has only finitely many direct sum decompositions in $\mathcal{C}$ up to the order of summands and isomorphism, and direct sum decompositions in $\mathcal{C}$ are "regular", in the sense that the monoid $V(\mathcal{C})$ has a regular geometric pattern.

Now a sufficient condition for $V(\mathcal{C})$ to be free is that the endomorphism ring of every indecomposable module $A_R \in \mathcal{C}$ is a local ring and that every module can be written as the direct sum of indecomposable modules. In section 6.1 we are going to see what this means when $\mathcal{C}$ is proj-R and when $\mathcal{C}$ is mod-R for some ring R.

Then we want to find a "natural" sufficient condition for $V(\mathcal{C})$ to be a Krull monoid.

Finally, in section 6.2 we will investigate the left-right symmetry of these conditions. While looking for such kind of results we prove that there is an isomorphism $V(\text{mod-}R) \to V(R\text{-mod})$ whenever R is a semiperfect ring (Corollary 6.2.6).

The results of section 6.2 are original when not otherwise specified.

6.1 When $V(\mathcal{C})$ is a free or a Krull monoid

Let R be a semiperfect ring. We know (see, e.g., [Fac98, Proposition 3.14]) that the regular module R_R is a direct sum of indecomposable modules with local endomorphism rings. Thus the monoid $V(R) = V(\text{proj-}R)$ is a free monoid, which is equivalent to say that the Krull-Schmidt theorem holds in the category add-R =proj-R. Conversely if every element in add-R is a direct sum of indecomposable modules with local endomorphism rings, then the ring is semiperfect (see again [Fac98, Proposition 3.14]).

Thus

every element in add-R is a direct sum of
indecomposable modules with local endomorphism (6.1.1)
rings if and only if R is semiperfect.

Let us consider the category mod-R. Obviously, if every element in mod-R is a direct sum of indecomposable modules with local endomorphism rings, then the ring is semiperfect. The converse, however, does not hold. That is to say not every indecomposable finitely presented module over a semiperfect ring has local endomorphism ring.

The rings for which every finitely presented module is a direct sum of indecomposable modules with local endomorphism rings have been characterized in

[Row86, Theorem 8]. They are exactly the *semiperfect π_∞-regular rings*. Recall that a ring is said to be π_∞-regular if $M_n(R)$ is a left π-regular ring for each n, i.e. if every matrix ring $M_n(R)$ satisfy the Descending Chain Condition on chains of the form $M_n(R)a > M_n(R)a^2 > M_n(R)a^3 > \cdots$. This condition is left-right symmetric (see [Row86, p. 2]).

Thus

every element in mod-R is a direct sum of
indecomposable modules with local endomorphism (6.1.2)
rings if and only if R is semiperfect π_∞-regular.

Now we want to find a sufficient condition for $V(\mathcal{C})$ to be a Krull monoid. The most general and most natural result we have been able to find in the literature is the following Theorem 6.1.2, which was first proved in [Fac02, Theorem 3.4]. The (still unpublished) verion we state here is the categorical approach to that result. To state the theorem, we will need some definitions.

Given an additive category $\mathcal{C}$, we will say that *idempotents split in $\mathcal{C}$* (equivalently, *$\mathcal{C}$ is amenable*) if for every object B and every idempotent $e\colon B \to B$ in $\mathcal{C}$ there exist an object $A \in \mathcal{C}$ and morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $fg = e$ and $gf = \mathrm{id}_A$. Note that idempotents split in every abelian category.

For each additive category $\mathcal{C}$, there exist a category $\widehat{\mathcal{C}}$ in which idempotents split and a functor $F\colon \mathcal{C} \to \widehat{\mathcal{C}}$, uniquely determined up to cetegorical equivalence, with the following universal property: for every functor $G\colon \mathcal{C} \to \mathcal{D}$ into an additive category $\mathcal{D}$ in which idempotents split, there exists a unique functor $H\colon \widehat{\mathcal{C}} \to \mathcal{D}$ such that $G = HF$. We will call $\widehat{\mathcal{C}}$ the *idempotent completion* of $\mathcal{C}$.

Recall that a *two-sided ideal* of an additive category $\mathcal{C}$ is a subfunctor of the two-variable functor $(\text{-},\text{-})_{\mathcal{C}}$ (see [Mit72, p.18]). We can consider the maximal ideals and compute the intersection of all maximal ideals. This intersection is called the *Jacobson radical of $\mathcal{C}$*, it turns out to be defined by $J(A,B) = \{f \in (A,B)_{\mathcal{C}} \mid \mathrm{id}_A - gf$ has a two-sided inverse for all $g \in (B,A)_{\mathcal{C}}\}$ and it turns out to be a twosided ideal (see [Mit72, p.21]).

Now, for every two-sided ideal $\mathcal{I}$ of $\mathcal{C}$ we can consider the *quotient category $\mathcal{C}/\mathcal{I}$* whose objects are those of $\mathcal{C}$ and whose morphisms are given by $(A,B)_{\mathcal{C}/\mathcal{I}} = (A,B)_{\mathcal{C}}/(A,B)_{\mathcal{I}}$.

Finally, given two additive categories $\mathcal{A}$ and $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$, we say that F is *isomorphism-reflecting* if, for every pair A, A' of objects of $\mathcal{A}$, $F(A) \cong F(A')$ implies $A \cong A'$; we say that F is *local* if, for every pair A, A' of objects of $\mathcal{A}$ and every morphism $f\colon A \to A'$ such that $F(f)\colon F(A) \to F(A')$ is an isomorphism, f is an isomorphism.

Theorem 6.1.1 (Facchini).
Let $\mathcal{A}$ be a skeletally small, additive category with the property that $\mathrm{End}_{\mathcal{A}}(A)$ is a semilocal ring for every object A of $\mathcal{A}$. Then the idempotent completion $\widehat{\mathcal{A}/J}$ of the factor category $\mathcal{A}/J$ is an amenable semisimple category and the canonical functor $G\colon \mathcal{A} \to \widehat{\mathcal{A}/J}$ is a full, isomorphism-reflecting, local functor.

Theorem 6.1.2 (Facchini).
Let $\mathcal{A}$ be a skeletally small additive category. Let F be an additive functor of $\mathcal{A}$ into an amenable semisimple category $\mathcal{B}$. If idempotents split in $\mathcal{A}$ and F is local, then $V(\mathcal{A})$ is a Krull monoid.

Again natural questions arise, such as "what does this mean when $\mathcal{C}$ is the category proj-R?" and "what does this mean when $\mathcal{C}$ is the category mod-R?"

It is clear that if all the modules in mod-R have semilocal endomorphism ring, then all the modules in proj-R have semilocal endomorphism ring, thus R is semilocal. In [FH05, Theorem 3.3] it was proved the third implication of the circle as well:

Theorem 6.1.3 (Facchini).
The endomorphism ring of a finitely presented module over a semilocal ring is a semilocal ring.

Thus

every element in mod-R is a direct sum of indecomposable modules with semilocal endomorphism rings if and only if every element in proj-R is a direct sum of indecomposable modules with semilocal endomorphism rings if and only if R is semilocal. (6.1.3)

6.2 Left-Right symmetry of the conditions above

In the previous section we characterized the rings R for which all the modules in the categories mod-R and proj-R have semiperfect or semilocal rings, thinking of these properties as natural for the Krull-Schmidt monoid of these categories to be a free or a Krull monoid. All the properties that came up are left-right-symmetric, so all the modules in the categories mod-R and proj-R have semiperfect or semilocal endomorphism rings if and only if the same happens in the corresponding category of left R-modules.

This leads to the question: is it always true that $V(\text{proj-}R) \cong V(R\text{-proj})$? Similarly: is it always true that $V(\text{mod-}R) \cong V(R\text{-mod})$?

It is well known that for any ring R, there is a duality between the category proj-R and the category R-proj. The duality, defined by $P_R \mapsto P_R^* = \mathrm{Hom}(P_R, R_R)$, induces an isomorphism of commutative monoids with order-unit $(V(R_R), \langle R_R \rangle) \to (V({}_RR), \langle {}_RR \rangle)$.

Let us now consider the categories mod-R and R-mod. From now on we are freely using notations and results from [AB69].

Let Γ be an abelian category with enough projectives, that is an abelian category such that, for every object A, there is a projective object P and an epimorphism $P \to A$. Let us define the projective stabilization $\underline{\Gamma}$ of Γ as follows.

The objects of $\underline{\Gamma}$ are the objects of Γ, and $\mathrm{Hom}_{\underline{\Gamma}}(\underline{A}, \underline{B}) = \mathrm{Hom}_\Gamma(A, B)/K_\Gamma(A, B)$, where $K_\Gamma(A, B) = \{f \in \mathrm{Hom}_\Gamma(A, B) \mid f$ factors through a projective$\}$.

There is a very useful characterization of the isomorphism classes in the stable category, namely two objects $\underline{A}, \underline{B}$ are stably isomorphic if and only if they are *projectively equivalent*, i.e. if and only if there exist projectives P and Q in Γ such that $A \oplus P \cong B \oplus Q$. Our next goal is to sketch the proof of this non-trivial characterization.

First of all we note that, for every $A \in \Gamma$, the functor $\mathrm{Hom}_{\underline{\Gamma}}(\underline{A}, -)$ is the *projective stabilization* of the functor $\mathrm{Hom}_\Gamma(A, -)$ as defined in [AB69, §1], i.e. it is the cokernel of $L_0(\mathrm{Hom}_\Gamma(A, -)) \to \mathrm{Hom}_\Gamma(A, -)$, where L_0F denotes the zero$^{\text{th}}$ derived functor of F. To prove this, set

$$\mathcal{K} = \mathrm{Coker}\Big(L_0\big(\mathrm{Hom}_\Gamma(A, -)\big) \to \mathrm{Hom}_\Gamma(A, -)\Big),$$

choose some $B \in \Gamma$ and set

$$K = \mathrm{Ker}\Big(\mathrm{Hom}(A, B) \to \mathrm{Hom}(\underline{A}, \underline{B})\Big) = \mathrm{Im}\Big(L_0\big(\mathrm{Hom}_\Gamma(A, -)\big)(B) \to \mathrm{Hom}_\Gamma(A, B)\Big).$$

To compute K, let $P_1 \to P_0 \to B \to 0$ be a projective presentation of B. By definition $L_0\big(\mathrm{Hom}(A, -)\big)(B) = \mathrm{Coker}\big(\mathrm{Hom}(A, P_1) \to \mathrm{Hom}(A, P_0)\big)$ and the transformation $L_0\big(\mathrm{Hom}(A, -)\big) \to \mathrm{Hom}(A, -)$, applied to B, renders the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Hom}(A, P_1) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A, P_0) \arrow[r] \arrow[d, equal] & L_0\big(\mathrm{Hom}(A, -)\big)(B) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}(A, P_1) \arrow[r] & \mathrm{Hom}(A, P_0) \arrow[r] & \mathrm{Hom}(A, B). &
\end{tikzcd}$$

Note that $K = \mathrm{Im}\Big(L_0\big(\mathrm{Hom}_\Gamma(A, -)\big)(B) \to \mathrm{Hom}_\Gamma(A, B)\Big) = \mathrm{Im}\Big(\mathrm{Hom}(A, P_0) \to \mathrm{Hom}(A, B)\Big)$. Thus, if $f \in K$ then f factors through

P_0. Conversely, if f factors through some projective P then we get the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ P \arrow[ur] \arrow[r, dotted] & P_0 \arrow[u, "p"'] \end{tikzcd}$$

where the dotted arrow exists because P is projective and p is surjective. Thus $f = A \to P \to P_0 \to B$, so that $f \in \operatorname{Im}\Big(\operatorname{Hom}(A, P_0) \to \operatorname{Hom}(A, B)\Big)$ and the claim is proved.

For the next step we need some more notation. Given two categories $\mathcal{C}$ and $\mathcal{D}$ and two functors $F, G \colon \mathcal{C} \to \mathcal{D}$ we denote by (F, G) the set of the natural transformations of F into G. With this notation we have $(\operatorname{Ext}^1(A, -), \operatorname{Ext}^1(B, -)) \cong (\operatorname{Hom}(\underline{A}, -), \operatorname{Hom}(\underline{B}, -))$ by [AB69, Corollary 1.18], so that $\operatorname{Ext}^1(A, -) \cong \operatorname{Ext}^1(B, -)$ if and only if $\operatorname{Hom}(\underline{A}, -) \cong \operatorname{Hom}(\underline{B}, -)$. The latter statement is equivalent, for every category, to the fact that $\underline{A} \cong \underline{B}$.

To conclude that $\underline{A} \cong \underline{B}$ if and only if there exist projectives P and Q in Γ such that $A \oplus P \cong B \oplus Q$ we only need a classical theorem by Heckmann-Hilton [EH60] and Hilton-Rees [HR61] which proves that the existence of projectives P and Q in Γ such that $A \oplus P \cong B \oplus Q$ is equivalent to the existence of a natural isomorphism $\operatorname{Ext}^1(A, -) \cong \operatorname{Ext}^1(B, -)$.

Let now A be a finitely presented R-module and let $P_1 \xrightarrow{p} P_0 \to A$ be a finite presentation of A. Define $D(M)$ to be the cokernel of the homomorphism $P_0^* \xrightarrow{p^*} P_1^*$ (here $X \mapsto X^* = \operatorname{Hom}(X, R_R)$ is the usual duality).

The module $D(M)$ is called *the Auslander-Bridger dual of* M. Note the Auslander-Bridger dual of a finitely presented module is still finitely presented.

Proposition 6.2.1. *For every ring R there are monoid isomorphisms $\varphi \colon V(R\text{–proj}) \to V(\text{proj–}R)$ and $\psi \colon V(\underline{R\text{–mod}}) \to V(\underline{\text{mod–}R})$.*

Proof. We have already seen the existence of the isomorphism $\varphi \colon V(R$-proj$) \to V($proj-$R)$.

To get the other isomorphism it is sufficient to use the Auslander-Bridger transpose. Indeed the map $\psi \colon V(\underline{R\text{–mod}}) \to V(\underline{\text{mod–}R})$ defined by $\psi(\underline{M}) = \underline{D(M)}$ is an isomorphism of monoids.

Note that the module $D(M)$ is not well-defined since it depends on the choice of the projective resolution. Nonetheless the stable isomorphism class of $D(M)$ is well-defined and depends only on the stable isomorphism class of M. Let us prove first $\psi \colon M \mapsto \underline{D(M)}$ is well defined. To see this consider two different projective resolutions

$$\begin{array}{l} \cdots \to P_1 \to P_0 \to M \to 0 \\ \cdots \to Q_1 \to Q_0 \to M \to 0 \end{array}$$

of M. They give raise to two different transposes $D(M) = \mathrm{Coker}(P_0^* \to P_1^*)$ and $D'(M) = \mathrm{Coker}(Q_0^* \to Q_1^*)$. There is, however, an homotopy equivalence between the two projective resolutions, so that there is an homotopy equivalence between

$$\begin{array}{l} 0 \to P_0^* \to P_1^* \to \cdots \quad \text{and} \\ 0 \to Q_0^* \to Q_1^* \to \cdots \end{array}$$

(projective resolutions of $D(M)$ and $D'(M)$ respectively), hence an homotopy equivalence between

$$\begin{array}{l} 0 \to \mathrm{Hom}(P_0^*, X) \to \mathrm{Hom}(P_1^*, X) \to \cdots \quad \text{and} \\ 0 \to \mathrm{Hom}(Q_0^*, X) \to \mathrm{Hom}(Q_1^*, X) \to \cdots , \end{array}$$

for every X. Therefore $\mathrm{Ext}^1(D(M), -) \cong \mathrm{Ext}^1(D'(M), -)$, so that $\underline{D}(M) \cong \underline{D'}(M)$.

If $\underline{M}, \underline{N}$ have the same image via ψ, this means their tansposes are stably isomorphic, so that the double transposes $DD(M), DD(N)$ are stably isomorphic. Since every finitely presented module K is stably isomorphic to its double transpose, M and N turns out to be stably isomorphic, i.e. $\underline{M} = \underline{N}$. Thus ψ is well defined.

Obviously the transpose of the zero module is the zero module and the transpose of the direct sum of two modules is the direct sum of the transposes (up to stable isomorphisms), so that ψ is a monoid homomorphism.

Injectivity and surjectivity are clear since the composition $\underline{M} \mapsto \underline{D(M)} \mapsto \underline{DD(M)}$ the identity. ∎

Now $V(R\text{-proj})$ is a saturated submonoid of $V(R\text{-mod})$ (it is even divisor closed), so we can define a congruence $\sim$ over $V(R\text{-mod})$ by $\langle A\rangle \sim \langle B\rangle$ if and only if there are $\langle P\rangle, \langle Q\rangle \in V(R\text{-proj})$ such that $\langle A\rangle + \langle P\rangle = \langle B\rangle + \langle Q\rangle$ i.e., if and only if A and B are stably isomorphic. We will denote the quotient by $V(R\text{-mod})/V(R\text{-proj})$ as in [AGOP98, p. 111]. In the terminology of [FHK03] the complement $P = V(R\text{-mod}) \setminus V(R\text{-proj})$ is a prime ideal of $V(R\text{-mod})$ and $V(R\text{-mod})/V(R\text{-proj})$ is the reduced localization of $V(R\text{-mod})$ at P. Note there is a natural monoid isomorphism $V(\underline{R\text{-mod}}) \cong V(R\text{-mod})/V(R\text{-proj})$.

It is natural to wonder if we can “glue together” two free (Krull) monoids as we can do with groups. That is to say, if it is sufficient for a monoid to have a free (Krull) saturated submonoid with a free (Krull) quotient in order to be free (Krull). That would be useful in view of Proposition 6.2.1, for example, to show that, given a ring R, the Krull-Schmidt theorem holds for finitely presented left R-modules if and only if it holds for finitely presented right modules. Unfortunately this is not the case, as the following examples show.

Example 6.2.2. Let M be the submonoid of the free monoid $\mathbb{N}^2$ generated by the pairs $(2,0),(1,1),(0,2)$. Let N be the submonoid of M generated by the pair $(2,0)$. Note N is a free monoid and it is divisor-closed as a submonoid of M.

The quotient M/N is a free monoid of rank 1 since the map $[(x,y)]_\sim \mapsto y$ is an isomorphism of monoids from M/N onto $\mathbb{N}$.

Nevertheless the monoid M is far from being free, since $(2,2) = (2,0) + (0,2) = (1,1) + (1,1)$.

Note however that M is a Krull monoid, since it is a saturated submonoid of $\mathbb{N}^2$.

Example 6.2.3. Let M be the set $\mathbb{N}^2$ endowed with the sum $(a,b) + (c,d) = (a+c, ac+b+d)$.

The monoid M is commutative, cancellative and reduced as it is not difficult to see.

Let N be the divisor-closed submonoid $\{(0,x) \mid x \in \mathbb{N}\}$. N is obviously a free monoid isomorphic to $\mathbb{N}$. Let $\sim$ denote the congruence associated with the submonoid N.

The quotient M/N is a free monoid of rank 1 since the map $f\colon [(x,y)]_\sim \mapsto x$ is an isomorphism of monoids from M/N onto $\mathbb{N}$.

Nevertheless the monoid M is not even a Krull monoid.

To see this suppose there exists a divisor theory $\varphi\colon M \to \mathbb{N}^{(I)}$ for some set I. Its components φ_i are essential valuations of M (see [FHK03, p. 440]).

It is not difficult to see that the only essential valuations of M are multiples of f. Indeeed if g is a valuation and $g(0,1) = x, g(1,0) = y$, then $g(n+m,0) + xnm = g(n,0) + g(m,0)$ since $(n+m,0) + (o,nm) = (n+m,nm) = (n,0) + (m,0)$. Thus we can calculate recursively $g(n,0) = ny - \frac{n(n-1)}{2}x$. If x were greater than 0, $g(n,0)$ would turn out to be negative for sufficently large n's and this cannot be. Therefore $x = 0$ and $g(n,m) = yn$. All of these are essential valuations.

Thus $\varphi(0,1) = (0)_{i\in I} \leq (1)_{i\in I} = \varphi(1,0)$. Nevertheless $(0,1) \nleq (1,0)$, contradicting the fact that φ is a divisor homomorphism. We infer that M cannot be a Krull monoid.

A couple of comments are in order. First of all, so far we do not know any example of a ring R such that the Krull-Schmidt theorem holds for finitely presented left R-modules but it does not hold for finitely presented right modules.

On the other hand, if such an example exists, then it has to be a non-semiperfect ring, as the following shows.

We start recalling the following well-known property of finitely generated modules over semiperfect rings.

Theorem 6.2.4 (Warfield).
Let M be a finitely generated module over a semiperfect ring R. Then $M = N \oplus P$, where P is projective and N has no non-zero projective summands. This decomposition is unique up to isomorphism, in the sense that if $M = N' \oplus P'$ is another decomposition with P' projective and N' without non-zero projective summands, then $N \cong N'$ and $P \cong P'$.

By the previous theorem we can define (up to isomorphism) the *projective part* $P(M)$ of a finitely presented module M over a semiperfect ring R. Thus we can decompose the Krull-Schmidt monoid $V(R\text{-mod})$ as $V(\text{proj}-R) \oplus V(\underline{\text{mod}-R})$ as the next proposition shows.

Proposition 6.2.5. *If R is a semiperfect ring and M is a finitely presented right R-module, the position $\langle M \rangle \mapsto (\langle P(M) \rangle, \langle \underline{M} \rangle)$ defines a monoid isomorphism $\varphi\colon V(\text{mod}-R) \to V(\text{proj}-R) \oplus V(\underline{\text{mod}-R})$.*

Proof. The isomorphism class of the trivial module is clearly mapped to the zero element of $V(\text{proj-}R) \oplus V(\underline{\text{mod-}R})$. Let $A = P(A) \oplus M$ and $B = P(B) \oplus N$ be two finitely presented R-modules. In order to show that φ is a monoid homomorphism one has to show that $M \oplus N$ has no projective direct summands. Indeed if $M \oplus N$ has a projective direct summand, there is an indecomposable projective direct summand P of $M \oplus N$. Now P has local endomorphism ring, thus it has the exchange property and it is isomorphic to a direct summand of, say, M. This is not possible, so $P(A) \oplus P(B) = P(A \oplus B)$ and one has $\varphi(\langle A \rangle) + \varphi(\langle B \rangle) = (\langle \underline{M} \rangle, \langle P(A) \rangle) + (\langle \underline{N} \rangle, \langle P(B) \rangle) = (\langle \underline{M} \rangle + \langle \underline{N} \rangle, \langle P(A) \rangle + \langle P(B) \rangle) = (\langle \underline{M} \oplus \underline{N} \rangle, \langle P(A) \oplus P(B) \rangle) = (\langle \underline{M \oplus N} \rangle, \langle P(A \oplus B) \rangle) = \varphi(\langle A \oplus B \rangle)$ and φ is a monoid homomorphism.

Let $A = P(A) \oplus M$ and $B = P(B) \oplus N$ be two finitely presented R-modules such that $\varphi(\langle A \rangle) = \varphi(\langle B \rangle)$. Then $P(A) \cong P(B)$ and M, N are stably isomorphic finitely presented modules without projective direct summands, thus isomorphic (cfr. for example [War75, Corollary 1.5]). Therefore $A \cong B$ and φ is injective.

Finally let A be a finitely presented R-module and P be a finitely generated projective R-module. The module A is stably isomorphic to a module M without projective direct summands (see again [War75, Corollary 1.5]) and $\varphi(\langle M \oplus P \rangle) = (\langle \underline{M} \rangle, \langle P \rangle) = (\langle \underline{A} \rangle, \langle P \rangle)$, this proving φ is surjective. ■

Corollary 6.2.6. *If R is a semiperfect ring, then there is a monoid isomorphism $\psi\colon V(R\text{-}mod) \to V(mod\text{-}R)$.*

Proof. Clear by Propositions 6.2.1 and 6.2.5. ∎

Chapter 7

The Semi Exchange Property

In Chapter 2 we defined what we mean by a module M to have the exchange property, and we proved it is equivalent to the endomorphism rings of the modules being local. These two equivalent properties are the natural property to ask to the modules belonging to a class $\mathcal{C}$ for $V(\mathcal{C})$ to be a free monoid.

In Chapter 6 we proved that a sufficient condition for $V(\mathcal{C})$ to be a Krull monoid is that every module in $\mathcal{C}$ has semilocal endomorphism ring. What about the exchange property? Is there any analogue property which is equivalent for M to the fact that $\mathrm{End}(M)$ is semilocal?

The semi exchange property was born as an attempt to give positive answers to these questions, although it is not as natural as we hoped. In particular a module M has the semi$[n]$exchange property if and only if its endomorphism ring has exactly n maximal ideals for $n = 2$, but this does not seem to happen for $n > 2$.

7.1 Definitions and main results

Definition. Let R be a ring, M be a right R-module, $\aleph$ be a cardinal and m be a positive integer. We say M has the $\aleph$-semi$[m]$exchange property if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i$$

where $M' \cong M$ and $\mid I \mid \leq \aleph$, there is a partition $I = \dot{\bigcup}_{j \in J} I_j$ with $\mid I_j \mid \leq m$ for any $j \in J$ and R-submodules B_j of $\oplus_{i \in I_j} A_i$, $j \in J$, such that $G = M' \oplus (\oplus_{j \in J} B_j)$.

Let X be a monoid, x be an element of X, $\aleph$ be a finite cardinal and m be a positive integer. We say x has the $\aleph$-semi$[m]$exchange property if whenever

$$x + y = \sum_{i \in I} a_i$$

where $| I | \leq \aleph$, there is a partition $I = \dot\bigcup_{j \in J} I_j$ with $| I_j | \leq m$ for any $j \in J$ and elements b_j of $\sum_{i \in I_j} a_i$, $j \in J$, such that $x + y = x + (\sum_{j \in J} b_j)$.

We say an R-module (an element of X) has the finite semi$[m]$exchange property if it has the $\aleph$-semi$[m]$exchange property for any finite cardinal $\aleph$.

We say an R-module has the semi$[m]$exchange property if it has the $\aleph$-semi$[m]$exchange property for any cardinal $\aleph$.

We say an R-module (an element of X) has the $\aleph$-exchange property if it has the $\aleph$-semi[1]exchange property.

Lemma 7.1.1. *An indecomposable R-module M' has the $\aleph$-semi[m]exchange property if and only if for any R-module G and any two direct sum decompositions $G = M \oplus N = \bigoplus_{i \in I} A_i$ where $| I | \leq \aleph$ and $M' \cong M$, there are indices $i_1, \dots, i_t \in I$ for some $t \leq, m$ and a submodule B of A such that $A = M \oplus B \oplus \bigoplus_{j \neq i_1, \dots, i_t} A_j$.*

Proof. Let M be an indecomposable R-module. If M has the $\aleph$-semi$[m]$exchange property and

$$M \oplus N = \bigoplus_{i \in I} A_i$$

where $| I | \leq \aleph$, there is a partition $I = \dot\bigcup_{j \in J} I_j$ with $| I_j | \leq m$ for any $j \in J$ and decompositions $\bigoplus_{i \in I_j} A_i = B_j \oplus C_j$, $j \in J$, such that $\bigoplus_{i \in I} A_i = M \oplus (\bigoplus_{j \in J} B_j)$. Therefore $M \cong \bigoplus_{j \in J} C_j$ and, since M is indecomposable, $C_j = 0$ for any j but for one index j_0. We conclude $M \oplus N = \bigoplus_{j \in J} (\bigoplus_{i \in I_j} A_i) = M \oplus B_{j_0} \oplus (\bigoplus_{i \notin I_{j_0}} A_i)$ with $| I_{j_0} | \leq m$. ∎

Lemma 7.1.2. *An indecomposable element x of a cancellative monoid X has the $\aleph$-semi[m]exchange property if and only if whenever there are $y, a_i \in X$ ($i \in I$, $| I | \leq \aleph$) such that $a = x + y = \sum_{i \in I} a_i$, there are indices $i_1, \dots, i_t \in I$ for some $t \leq m$ and a summand b of a such that $a = x + b + \sum_{j \neq i_1, \dots, i_m} a_j$.*

Proof. Let x be an indecomposable element of X. If x has the $\aleph$-semi$[m]$exchange property for some positive integers $\aleph, m$ and

$$x + y = \sum_{i \in I} a_i$$

where $\mid I \mid \leq \aleph$, there is a partition $I = \bigcup_{j \in J} I_j$ with $\mid I_j \mid \leq m$ for any $j \in J$ and decompositions $\sum_{i \in I_j} a_i = b_j + c_j$, $j \in J$, such that $\sum_{i \in I} a_i = x + (\sum_{j \in J} b_j)$. Therefore $x = \sum_{j \in J} c_j$ and, since x is indecomposable, $c_j = 0$ for any j but for one index j_0. We conclude $x + y = \sum_{j \in J}(\sum_{i \in I_j} a_i) = x + b_{j_0} + (\sum_{i \notin I_{j_0}} a_i)$ with $\mid I_{j_0} \mid \leq m$. ∎

Proposition 7.1.3. *Let M be a module and let $M = M_1 \oplus M_2$ be a decomposition of M. If M has the $\aleph$-semi[m]exchange property, then M_1 has the $\aleph$-semi[m]exchange property. If M_1, M_2 have the $\aleph$-semi[m_1]exchange property and the $\aleph$-semi[m_2]exchange property respectively, then $M_1 \oplus M_2$ has the $\aleph$-semi[$m_1 m_2$]exchange property.*

Proof. Suppose M has the $\aleph$-semi[m]exchange property and suppose $G = M_1' \oplus N = \bigoplus_{i \in I} A_i$ with $M_1' \cong M_1$ and $\mid I \mid \leq \aleph$. Then $G' = M_2 \oplus G = M' \oplus N = M_2 \oplus \bigoplus_{i \in I} A_i$ with $M' \cong M$. Let $k \in I$ be any index and define $A_i' = A_i$ for every $i \neq k$ and $A_k' = M_2 \oplus A_k$. One has $G' = M' \oplus N = \bigoplus_{i \in I} A_k'$. Thus there is a partition $I = \bigcup_{j \in J} I_j$ with $\mid I_j \mid \leq m$ and decompositions $\bigoplus_{i \in I_j} A_i = B_j \oplus C_j$, $j \in J$, such that $\bigoplus_{i \in I} A_i' = M' \oplus (\bigoplus_{j \in J} B_j)$. We will denote j_0 the index $j \in J$ such that $k \in I_{j_0}$. Since $M_2 \subseteq M_2 \oplus B_{j_0} \subseteq M_2 \oplus A$, we have by [Fac98, Lemma 2.1] that $M_2 \oplus B_{j_0} = M_2 \oplus B_{j_0}'$ where $B_{j_0}' = (M_2 \oplus B_{j_0}) \cap G \subseteq G$. Thus $M' \oplus B_{j_0} = M_1' \oplus M_2 \oplus B_{j_0} = M_1' \oplus M_2 \oplus B_{j_0}'$ and, denoting the B_j's by B_j' for every $j \neq j_0$ one has $G' = M' \oplus (\bigoplus_{j \in J} B_j')$. Note that $B_j' \subseteq G$ for every $j \in J$ and that $M_1' \subseteq G$. Thus using the modular identity we get $G = G \cap (M_2 \oplus (M_1' \oplus (\bigoplus_{j \in J} B_j'))) = (G \cap M_2) \oplus (M_1' \oplus (\bigoplus_{j \in J} B_j')) = M_1' \oplus (\bigoplus_{j \in J} B_j')$.. This shows that M_1 has the $\aleph$-semi[m]exchange property.

To see the converse set $G = M_1' \oplus M_2' \oplus N = \bigoplus_{i \in I} A_i$. Since M_1' has the $\aleph$-semi[m_1]exchange property there is a partition $I = \bigcup_{j \in J} I_j$ with $\mid I_j \mid \leq m_1$ for any $j \in J$ and decompositions $\bigoplus_{i \in I_j} A_i = B_j + C_j$, $j \in J$, such that $\bigoplus_{i \in I} A_i = M_1' \oplus (\bigoplus_{j \in J} B_j)$. By [Fac98, Lemma 2.2] there is a partition $J = \bigcup_{k \in K} J_k$ with $\mid J_k \mid \leq m_2$ for any $k \in K$ and decompositions $\bigoplus_{j \in J_k} B_j = D_k \oplus E_k$, $k \in K$, such that $\bigoplus_{i \in I} A_i = M_1' \oplus M_2' \oplus (\bigoplus_{k \in K} D_k)$. Therefore M has the $\aleph$-semi[$m_1 m_2$]exchange property. ∎

Proposition 7.1.4. *Let M be a module and let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ be a decomposition of M into indecomposable modules. If M_i has the $\aleph$-semi[m_i]exchange property for every i, then M has the $\aleph$-semi[$\sum_{i=1}^{k}(m_i - 1) + 1$]exchange property.*

Proof. Suppose

$$G = \bigoplus_{j=1}^{k} M'_j \oplus N = \bigoplus_{i\in I} A_i$$

with $M'_j \cong M_j$ for every $j = 1, \dots, k$ and with $\mid I \mid \leq \aleph$. Set $A_{0,i} = A_i$ for every $i \in I$, set $I_0 = I$, $J_0 = \emptyset$, $R_0 = \emptyset$ and $K_0 = \{A_{0,i}\}_{i\in I}$. We will define ths object recursively foe every $i = 0, \dots, k-1$.

Start substituting the M_i's one by one into

$$G = \bigoplus_{\ell\in I} A_\ell = M'_1 \oplus \dots \oplus M'_{i-1} \oplus (\bigoplus_{\ell\in I_{i-1}} A_{i-1,\ell}).$$

Since M'_i is an indecomposable module with the $\aleph$-semi$[m_i]$exchange property, by Lemma 1.1.2, for every i there is a subset $J_i \subseteq I_{i-1}$ with $\mid J_i \mid= m_i$ and a decomposition $\bigoplus_{\ell\in J_i} A_{i,\ell} = B_i + C_i$ such that

$$G = M'_1 \oplus \dots \oplus M'_{i-1} \oplus (\bigoplus_{\ell\in I_{i-1}} A_{i-1,\ell}) = M'_1 \oplus \dots \oplus M'_i \oplus B_i \oplus (\bigoplus_{\ell\notin J_i} A_{i-1,\ell}).$$

Define

$$R_i = \{A_{i,\ell}\}_{\ell\in J_i} \cup (\bigcup_{j\in\{1,\dots,i-1\}\ \textit{such that}\ B_j\in\{A_{i,\ell}\}_{\ell\in J_i}} R_j),$$

$$S_i = R_i \cap K_0, \quad T_i = R_i \setminus S_i \quad \textit{and} \quad T'_i = T_i \cup \{B_i\}$$

and rename the elements of $K_i = \{A_{i-1,\ell}\}_{\ell\in I_{i-1}\setminus J_i} \cup \{B_i\}$ as $K_i = \{A_{i,\ell}\}_{\ell\in I_i}$ for a suitable set I_i of indices.

At the step i consider the modules substituted by the B_j's. Some of the B_j's have been substituted but some other ones survived, the B_j's still in K_i. Say $B_{j_1} \dots, B_{j_n}$ are the B_j's in K_i. Each of them took the place of some of the A_ℓ's (the elements of S_{j_h}) and, possibly, of some of the B_j's (the elements of T_{j_h}). The important fact is that the sets S_{j_h} form a partition of $I \setminus K_i$. We claim each S_{j_h} has cardinality less or equal to $\sum_{h=1}^{i}(m_h - 1) + 1$. Indeed for every $j = 1, \dots, i$ it easy to see that

$$\mid S_j \mid= \sum_{\ell\in\{1,\dots,i\}\ \textit{such that}\ B_\ell\in T'_j} (m_\ell - 1) + 1.$$

Thus they all have cardinality less or equal to $\sum_{h=1}^{i}(m_h - 1) + 1$ and the equality holds if and only if $T'_i = \{B_1, \dots, B_i\}$. ∎

Proposition 7.1.5. *Let x be an element of a cancellative monoid X and let $x = x_1 + x_2$ be a decomposition of x. If x_1, x_2 have respectively the $\aleph$-semi[m_1]exchange property and the $\aleph$-semi[m_2]exchange property, then x has the $\aleph$-semi[$m_1 m_2$]exchange property.*

Proof. Suppose $a = x_1 + x_2 + y = \sum_{i\in I} a_i$. There is a partition $I = \dot\bigcup_{j\in J} I_j$ with $| I_j | \leq m_1$ for any $j \in J$ and decompositions $\sum_{i\in I_j} a_i = b_j + c_j$, $j \in J$, such that $\sum_{i\in I} a_i = x_1 + (\sum_{j\in J} b_j)$. By cancellativity of X we have $x_2 + y = \sum_{j\in J} b_j$ and by the $\aleph$-semi[m_2]exchange property of x_2 there is a partition $J = \dot\bigcup_{k\in K} J_k$ with $| J_k | \leq m_2$ for any $k \in K$ and decompositions $\sum_{j\in J_k} b_j = d_k + e_k$, $k \in K$, such that $\sum_{i\in I} a_i = x_1 + x_2 + (\sum_{k\in K} d_k)$. Therefore x has the $\aleph$-semi[$m_1 m_2$]exchange property. ∎

Remark. A directly finite monoid is free if and only if all its elements have the finite exchange property.

In fact, let F be a free monoid and let $x, y, a_1, a_2, \ldots, a_n$ be elements of F such that $x + y = \sum_{i=1}^{n} a_i$. By 7.1.5 it is sufficient to think x is indecomposable. Being F free there exist $a_{1,1}, a_{1,2}, \ldots, a_{1,t_1}, a_{2,1}, a_{2,2}, \ldots, a_{2,t_2}, \ldots, a_{n,1}, a_{n,2}, \ldots, a_{n,t_n}$ indecomposable elements of F such that $a_i = a_{i,1} + a_{i,2} + \ldots + a_{i,t_i}$ $(i = 1, 2, \ldots, t_i)$. Moreover there are k, h such that $x = a_{k,h}$, so that $x \leq a_k$ and x has the finite exchange property.

Conversely if every element $x \in F$ has the exchange property it is easy to see that, if $a = a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_m$ where the a_i's and the b_j's are indecomposable, one has $m = n$ and $a_i = b_i$ after a suitable rearrangement of the indices. This is equivalent to the fact that F is free (this is very well known, see for example [HK98, p. 7]).

Proposition 7.1.6. *If x is an element of a Krull monoid X, then x has the finite semi[m]exchange property for some m.*

Proof. Let X be a Krull monoid, let I be a set, let $\varphi\colon X \to \mathbb{N}^{(I)}$ be a divisor monoid homomorphism and let x be an element of X. Again by 7.1.5 it is sufficient to think x is indecomposable. Let n be a positive integer and let $y, a_1, a_2, \ldots, a_n \in X$ such that $x + y = a_1 + a_2 + \ldots + a_n$. Let $x_1, x_2, \ldots, x_m$ be indecomposable elements of $\mathbb{N}^{(I)}$ such that $\varphi(x) = x_1 + x_2 + \ldots + x_m$. Since x_i has the finite exchange property for every i, one has $x_i \leq \varphi(a_{j[i]})$ for some $j[i]$, so that $\varphi(x) \leq \sum_{i=1}^{m} \varphi(a_{j[i]})$. Since φ is a divisor homomorphism one has $x \leq \sum_{i=1}^{m} a_{j[i]}$. ∎

Example 7.1.7. There exist a non-Krull monoid all elements of which have the finite semi$[m]$exchange property for some integer m.

Proof. Consider the indecomposable elements of the monoid descibed in the example 6.2.3. They are $(0,1)$, $(n,0)$. It's easy to see that $(0,1)$ has the finite exchange property and $(n,0)$ has the finite semi$[n]$exchange property. ∎

Proposition 7.1.8. *It does not make sense to consider the semi[0]exchange property, i.e. no module (element of a monoid) has the $\aleph$-semi[0]exchange property. Every module (element of a monoid) has the m-semi[m]exchange property. If a module (element of a monoid) has the $m+1$-semi[m]exchange property, then it has the finite semi[m]exchange property.*

Proof. Obviously no element has the m-semi[0]exchange property and every element has the m-semi$[m]$exchange property. We will show that, for every $n > m$, if x has the n-semi$[m]$exchange property then it has the $n+1$-semi$[m]$exchange property. In fact if

$$x + y = \sum_{i=1}^{n+1} a_i,$$

then $x + y = \sum_{i=1}^{n} b_i$ where $b_i = a_i$ for $i = 1, 2, \ldots, n-1$ and $b_n = a_n + a_{n+1}$. Thus there is a partition $\{1, 2, \ldots, n\} = \dot\bigcup_{j\in J} I_j$ with $\mid I_j \mid \leq m$ for any $j \in J$ and decompositions $\sum_{i\in I_j} b_i = c_j + c'_j$, $j \in J$, such that

$$x + y = x + (\sum_{j\in J} c_j).$$

One has $b_n \in I_{j_0}$ for some index j_0. Set $I'_j = I_j$ for every $j \neq j_0$ and $I'_{j_0} = I_{j_0} \cup \{n+1\}$. If $\mid I_{j_0} \mid < m$ we are done. If $\mid I_{j_0} \mid = m$, then $\mid I'_{j_0} \mid = m+1$. Since $c'_{j_0} \leq x$, it has the n-semi$[m]$exchange property and, since $n > m$, it has the $m+1$-semi$[m]$exchange property. Now $\sum_{i\in I'_{j_0}} a_i = c_{j_0} + c'_{j_0}$, so that there is a partition $I'_{j_0} = \dot\bigcup_{j\in J'} I'_j$ with $\mid I'_j \mid \leq m$ for any $j \in J'$ and decompositions $\sum_{i\in I'_j} a_i = d_j + d'j$, $j \in J'$, such that

$$\sum_{i\in I'_{j_0}} a_i = c_{j_0} + c'_{j_0} = c_{j_0} + \sum_{j\in J'} d_j$$

so that

$$x + y = x + (\sum_{j\in J} c_j) = x + (\sum_{j\in J\setminus\{j_0\}} c_j) + (\sum_{j\in J'} d_j)$$

and we are done.

The same proof works for modules as well. ■

7.2 Examples

Proposition 7.2.1. *The regular module $\mathbb{Z}_{\mathbb{Z}}$ does not have the finite semi[m]exchange property for any m.*

Proof. Let $G = A_1 \oplus \ldots \oplus A_{m+1}$ where $A_i = \mathbb{Z}$ for every i, let $p_1, p_2, \ldots, p_{m+1}$ be $m+1$ distinct primes, let $\pi = \Pi_{i=1}^{m+1} p_i$ be their product, let $t_i = \pi/p_i$ for every $i = 1, 2, \ldots, m+1$ and let M be the cyclic submodule of G generated by the element $(t_1, t_2, \ldots, t_{m+1}) \in G$. Obviously M is isomorphic to $\mathbb{Z}$. We want to show that M is a direct summand of G and that there are no index $i = 1, 2, \ldots, m+1$ and submodule $B \leq G$ such that $G = M \oplus B \oplus A_i$.

To show that M is a direct summand of G, it is sufficient to show that there are integers $a_{1,1}, a_{1,2}, \ldots, a_{1,m+1}, a_{2,1}, a_{2,2}, \ldots, a_{2,m+1}, \ldots, a_{m,1}, a_{m,2}, \ldots, a_{m,m+1}$ such that

$$det \begin{pmatrix} t_1 & t_2 & \ldots & t_m & t_{m+1} \\ a_{1,1} & a_{1,2} & \ldots & a_{1,m} & a_{1,m+1} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m} & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,m} & a_{m,m+1} \end{pmatrix} = 1, \qquad (*)$$

i.e. that $(t_1, t_2, \ldots, t_m, t_{m+1})$ is the first row of an invertible matrix.

By [Rot79, Theorem 4.51] $\mathbb{Z}$ has the unimodular row property. Now $(t_1, t_2, \ldots, t_m, t_{m+1})$ is a unimodular row so that it is the first row of an invertible matrix.

Given a matrix A we will denote by A^I_J the minor obtained from A taking out the i-th column for every $i \in I$ and the j-th row for every $j \in J$. Since $GCD(t_1, t_2, \ldots, t_{m+1}) = 1$, there are integers $x_1, x_2, \ldots, x_{m+1}$ such that $x_1 t_1 + x_2 t_2 + \ldots + x_{m+1} t_{m+1} = 1$. Computing the determinant using the first row, it is easy to see that to show $(*)$ it is sufficient to show that for every $x_1, x_2, \ldots, x_{m+1}$ there is an $m \times (m+1)$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,m+1} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,m+1} \end{pmatrix}$$

such that

$$det(A^1) = det \begin{pmatrix} a_{1,2} & a_{1,3} & \dots & a_{1,m+1} \\ a_{2,2} & a_{2,3} & \dots & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,2} & a_{m,3} & \dots & a_{m,m+1} \end{pmatrix} = x_1, \qquad (*_1)$$

$$det(A^2) = det \begin{pmatrix} a_{1,1} & a_{1,3} & \dots & a_{1,m+1} \\ a_{2,1} & a_{2,3} & \dots & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,3} & \dots & a_{m,m+1} \end{pmatrix} = x_2 \qquad (*_2)$$

$$\vdots$$

$$det(A^{m+1}) = det \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,m} \end{pmatrix} = x_{m+1}. \qquad (*_{m+1})$$

This can be done as follows: by induction we will show there are such $a_{i,j}$ with $a_{i,j} = 0$ for every $i < j \leq m$. The cases $m = 1, 2$ are easy. Suppose we proved it for $m = n$. We will compute all determinants using the last row.

Define $a_{n+1,n+1} = GCD(x_1, x_2, \dots, x_n, x_{n+2})$ and define $x'_i = x_i / a_{n+1,n+1}$ for every $i = 1, 2, \dots, n, n+2$. By the inductive hypothesis there is a matrix

$$B = \begin{pmatrix} b_{1,1} & 0 & \dots & 0 & b_{1,n+2} \\ b_{2,1} & b_{2,2} & \dots & 0 & b_{2,n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} & b_{n,n+2} \end{pmatrix}$$

such that

$$det(B^1) = x'_1, \qquad (*'_1)$$

$$det(B^2) = x'_2, \qquad (*'_2)$$

$$\vdots$$

$$det(B^n) = x'_n, \qquad (*'_n)$$

$$det(B^{n+2}) = x'_{n+2}. \qquad (*'_{n+2})$$

Note we used a nonstandard numeration for the columns: this is in order to simplify the notation later on.

Computing the determinants, we can rewrite the equations $(*_1)$,$(*_2)$,…,$(*_n)$ and $(*_{n+2})$ as

$$a_{n+1,n+1}det(A_{n+1}^{1,n+1}) = -x_1, \tag{$*_1$}$$

$$a_{n+1,n+1}det(A_{n+1}^{2,n+1}) = -x_2, \tag{$*_2$}$$

$$\vdots$$

$$a_{n+1,n+1}det(A_{n+1}^{n,n+1}) = -x_n, \tag{$*_n$}$$

$$a_{n+1,n+1}det(A_{n+1}^{n+2,n+1}) = x_{n+2}, \tag{$*_{n+2}$}$$

and $(*_{n+1})$ as

$$\begin{gathered}(-1)^{n+2}a_{n+1,1}det(A_{n+1}^{n+1,1}) + \ldots + (-1)^{2n+1}a_{n+1,n}det(A_{n+1}^{n+1,n}) + \\ +(-1)^{2n+2}a_{n+1,n+2}det(A_{n+1}^{n+1,n+2}) = x_{n+1}.\end{gathered} \tag{$*_{n+1}$}$$

Choosing $a_{i,j} = b_{i,j}$ whenever $b_{i,j}$ is defined, the equations $(*_i')$'s turn out to be equivalent to the $(*_i)$'s for every $i = 1, 2, \ldots, n, n+2$, since $B^i = A_{n+1}^{i,n+1}$, so that they are the same equation up to $a_{n+1,n+1}$. Thus the $a_{i,j}$'s satisfy $(*_i)$ for every $i = 1, 2, \ldots, n, n+2$

Now we need to choose $a_{n+1,1}, a_{n+1,2}, \ldots, a_{n+1,n}, a_{n+1,n+2}$ in such a way that the $a_{i,j}$'s satisfy $(*_{n+1})$. We know that $GCD(x_1', x_2', \ldots, x_n', x_{n+2}') = 1$, so that there are $a_{n+1,1}', a_{n+1,2}', \ldots, a_{n+1,n}', a_{n+1,n+2}'$ such that

$$\begin{gathered}(-1)^{n+2}a_{n+1,1}'x_1' + (-1)^{n+3}a_{n+1,2}'x_2' + \ldots \\ +(-1)^{2n+1}a_{n+1,n}'x_n' + (-1)^{2n+2}x_{n+2}'a_{n+1,n+2}' = 1.\end{gathered}$$

Thus

$$\begin{gathered}(-1)^{n+2}a_{n+1,1}x_1' + (-1)^{n+3}a_{n+1,2}x_2' + \ldots \\ +(-1)^{2n+1}a_{n+1,n}x_n' + (-1)^{2n+2}x_{n+2}'a_{n+1,n+2} = x_{n+1}\end{gathered}$$

with $a_{n+1,i} = x_{n+1}a_{n+1,i}'$ for every $i = 1, 2, \ldots, n, n+2$. To conclude it is sufficient to substitute equations $(*_i')$ $(i = 1, 2, \ldots, n, n+2)$ into this last equation. In this way we see that the $a_{i,j}$'s satisfy $(*_{n+1})$. Thus the matrix A we defined is the matrix we were looking for to show M is a direct summand of G.

To conclude we will show that there is no submodule B of G such that $G = M \oplus B \oplus A_{m+1}$, the other cases being similar. Indeed if this were true there would be integers $a_{1,1}, a_{1,2}, \ldots, a_{1,m+1}, \ldots, a_{m-1,1}, a_{m-1,2}, \ldots, a_{m-1,m+1}$

such that

$$det \begin{pmatrix} t_1 & t_2 & \dots & t_m & t_{m+1} \\ a_{1,1} & a_{1,2} & \dots & a_{1,m} & a_{1,m+1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} & a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \dots & a_{m-1,m} & a_{m-1,m+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = 1.$$

Computing the determinant using the last row, we see it is equal to $det(A_{m+1}^{m+1})$, which is divisible by p_{m+1} since the t_i's ($i \neq m+1$) are so. This leads to a contraddiction and completes the proof. ∎

Proposition 7.2.2. *Let m be a positive integer, let $p_1, p_2, \dots, p_m$ be m distinct primes and let $R = \mathbb{Z}_{p_1,p_2,\dots,p_m}$ be the localization of the ring of the integers at the primes $p_1, p_2, \dots, p_m$, i.e. the ring of quotients $\mathbb{Z}S^{-1}$ with respect to the set $S = \{n \in \mathbb{Z} \mid p_i$ does not divide n for every $i = 1, 2, \dots, m\}$. The regular module R_R has the finite semi[m]exchange property but it does not have the finite semi[$m-1$]exchange property.*

Proof. The proof of the fact that R_R does not have the semi[$m-1$]exchange property is analogous to the proof of Proposition 7.2.1.

To prove it has the semi[m]exchange property we procede similarly. Let $M \cong R$ be a direct summand of $G = A_1 \oplus \dots \oplus A_{m+1}$.

First of all note we can assume, without loss of generality, that the A_i's are indecomposable. In fact if R has the semi[m]exchange property "only for the indecomposables" and $A_i = B_{i,1} \oplus B_{i,2} \oplus \dots \oplus B_{i,k_i}$ for every i, then $G = \bigoplus_{i=1}^{m+1}(\bigoplus_{j=1}^{k_i} B_{i,j})$ and there are indices $i_1, \dots, i_m = 1, 2, \dots, m+1$ and $j_1 \leq k_{i_1}, \dots, j_m \leq k_{i_m}$ and a direct summand B of $B_{i_1,j_1} \oplus \dots \oplus B_{i_m,j_m}$ such that $G = M \oplus B \oplus (\bigoplus_{(i,j)\neq(i_\ell,j_\ell),\ell=1,\dots,m} B_{i,j})$. Consider now the direct summand $B' = B \oplus (\bigoplus_{j\neq j_1} B_{i_1,j}) \oplus \dots \oplus (\bigoplus_{j \neq j_m} B_{i_m,j})$ of $A_{i_1} \oplus \dots \oplus A_{i_m}$. One has $G = M \oplus B' \oplus (\bigoplus_{i\neq i_1,\dots,i_m} A_i)$ and R has the "general" semi[m]exchange property.

The finitely generated indecomposable R-modules are, up to isomorphism, R and $\mathbb{Z}/p_i^k\mathbb{Z}$ for every $i = 1, 2, \dots, m$ and every $k > 0$. Consider $G = M \oplus N = A_1 \oplus \dots \oplus A_n$ with $A_1 \cong \mathbb{Z}/p_i^k\mathbb{Z}$ for some $i = 1, 2, \dots, m$ and some $k > 0$. Let $\varepsilon_M : M \to G$, $\pi_M : G \to M$, $\varepsilon_i : A_i \to G$ and $\pi_i : G \to A_i$ be the structural morphisms associated to the direct sum decompositions. Since there is no nonzero morphism $\mathbb{Z}/p_i^k\mathbb{Z} \to R$, one has $\pi_M \varepsilon_1 = 0$. Thus $1_M = \pi_M(\sum_{i=1}^n \varepsilon_i \pi_i)\varepsilon_M = \pi_M(\sum_{i=2}^n \varepsilon_i\pi_i)\varepsilon_M$ and $G = \mathrm{Im}\,\varepsilon_M \oplus \ker \pi_M(\sum_{i=2}^n \varepsilon_i\pi_i) = M \oplus \ker \pi_M(\sum_{i=2}^n \varepsilon_i\pi_i)$. Now $\pi_i(A_1) = 0$ for every $i = 2, 3, \dots, n$ so that $A_1 \subseteq$

$\ker \pi_{\mathrm{M}}(\sum_{\mathrm{i}=2}^{\mathrm{n}} \varepsilon_{\mathrm{i}}\pi_{\mathrm{i}}) \subseteq \mathrm{A}_1 \oplus (\bigoplus_{\mathrm{i}=2}^{\mathrm{n}})\mathrm{A}_{\mathrm{i}}$ and, by Lemma 1.1.1, $\ker \pi_{\mathrm{M}}(\sum_{\mathrm{i}=2}^{\mathrm{n}} \varepsilon_{\mathrm{i}}\pi_{\mathrm{i}}) = \mathrm{A}_1 \oplus \mathrm{B}$. Thus $G = M \oplus B \oplus A_1$ and we can, using Lemma 1.1.2, "take A_1 out of the game".

Without loss of generality we shall think that all the A_i's are isomophic to R from now on. Let $(t_1, \ldots, t_{m+1})$ be a generator of M. Since it is not difficult to deal with the invertibles, we can think $M = (p_1^{k_{1,1}} p_2^{k_{1,2}} \ldots p_m^{k_{1,m}}, \ldots, p_1^{k_{m+1,1}} p_2^{k_{m+1,2}} \ldots p_m^{k_{m+1,m}})R$. As M is a direct summand of G, the t_i's must be coprime, so that for every $i = 1, 2, \ldots, m$ there is an index $j_i = 1, 2, \ldots, m+1$ such that p_i does not divide t_{j_i}. Because of the cardinality, there is an index $\bar{j}$ such that $\bar{j} \neq j_i$ for every $i = 1, 2, \ldots, m$. Suppose, without loss of generality, $m+1$ is such a $\bar{j}$.

As we did in the proof of Proposition 7.2.1, we can find an $(m-1) \times m$ matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,m} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \ldots & a_{m-1,m} \end{pmatrix}$$

such that

$$det \begin{pmatrix} t_1 & t_2 & \ldots & t_m \\ a_{1,1} & a_{1,2} & \ldots & a_{1,m} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \ldots & a_{m-1,m} \end{pmatrix} = 1.$$

The fact that

$$det \begin{pmatrix} t_1 & t_2 & \ldots & t_m & t_{m+1} \\ a_{1,1} & a_{1,2} & \ldots & a_{1,m} & 0 \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \ldots & a_{m-1,m} & \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} = \pm 1$$

shows we can substitute $A_1 \oplus \ldots \oplus A_m$ with $M \oplus B$ for some B. Thus R_R has the semi[m]exchange property. ■

Proposition 7.2.3. *Let R be a principal ideal domain. If R has at least m maximal ideals, then it has the semi[m]exchange property. If R has less then m maximal ideals, then it does not have the semi[m] exchange property.*

Proof. The proof is perfectly analogous to the proofs of Propositions 7.2.1 and 7.2.2, since a principal ideal domain has the unimodular row property by [Rot79, Theorem 4.51]. ■

Proposition 7.2.4. *Let M be a direct summand of G which endomorphism ring has two right maximal ideals.*

Then M has the semi[2]exchange property.

Proof. Let $G = X_1 \oplus X_2 \oplus \ldots \oplus X_n$ and let M' be a direct summand of G isomorphic to M. Let $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{M'}, \pi_1, \ldots, \pi_n, \pi_{M'}$ be the inclusions and projections associated to the direct sum decomposition, let S be the endomorphism ring of G, let $J = J(S)$ be its Jacobson radical and let $F\colon \textit{mod-}R \to \textit{proj-}S$ be the category equivalence. Let $p_\ell = F(\pi_\ell)$ and $e_\ell = F(\varepsilon_\ell)$ for every index ℓ and finally, for every homomorphism $f\colon P \to Q$, let $\overline{f}\colon P/PJ \to Q/QJ$ be the map induced by f.

Since $\sum_{j=1,\ldots,n}^{n} \pi_{M'}\varepsilon_j\pi_j\varepsilon_{M'} = id_{M'}$, then $\sum_{j=1,\ldots,n}^{n} p_{M'}e_jp_je_{M'} = id_{F(M')}$ and $\sum_{j=1,\ldots,n}^{n} \bar{p}_{M'}\bar{e}_j\bar{p}_j\bar{e}_{M'} = id_{F(M')/F(M')J}$.

Therefore either there is an index i such that $\bar{p}_{M'}\bar{e}_i\bar{p}_i\bar{e}_{M'}$ is an automorphism or there are two indices h, k such that $\bar{p}_{M'}\bar{e}_h\bar{p}_h\bar{e}_{M'} + \bar{p}_{M'}\bar{e}_k\bar{p}_k\bar{e}_{M'}$ is an automorphism.

From now on we procede as in Proposition 2.2.1. ■

Corollary 7.2.5. *Biuniform modules of type* 2 *have the semi[2]exchange property but they do not have the semi[*1*] exchange property.*

Chapter 8

Cohomological reduction by split pairs

The study of the direct sum decompositions of a module is very natural, since most of the properties of a module are preserved in such a decomposition. It is known, for example, that the projective and the injective dimension of a module are preserved under finite direct sum decomposition. Indeed all the Ext^n functors behave well with respect to the finite direct sum decompositions.

Consider now the global dimension of a ring, which is defined as the sup of all the projective dimensions of its modules. Also this property behaves well with respect to the direct sum decomposition of the ring, and in particular if R and S are two rings such that S is a direct summand of R i.e., $R \cong S \times T$ for some ring T, then $gl.dim(R) \geq gl.dim(S)$. The aim of this chapter is to find some weaker condition on R and S which gives us the same inequality. We will see that, given two rings R and S, if S is a *split quotient* of R, which is to say if S is a subring of R (via an embedding ε sending the unit of S to that of R) and there exists a surjective homomorphism $\pi \colon R \twoheadrightarrow S$, such that the composition $\pi \circ \varepsilon$ is the identity on S, then $gl.dim(R) \geq gl.dim(S)$.

More generally, suppose we are given two rings R and S and a ring homomorphism $f : R \to S$ and we would like to compare the cohomology in the category of, say, left R-modules with that of left S-modules. In general, nothing can be said. There are, however, some situations, which have been studied intensively and successfully.

Assume that f is an embedding of rings (sending the unit of R to that of S). Still nothing can be said - either ring could be semisimple without the other one being so. Assume also conditions like S being a projective R-module (via f). This is perfectly reasonable, for example, in representation theory of finite groups, where we could have $R = kH$ and $S = kG$ for a subgroup H of a finite group G (and a field k). Then the machinery of induction and restriction functors will allow to compare cohomology, for example by a Mackey formula.

Another customary assumption is that f is surjective. Again this is not enough - either ring could have finite global dimension without the other one having so. But we can consider some additional condition like the kernel of f being a projective R-module (at least on one side). This makes sense, for example in representation theory of algebraic groups or of Lie algebras, when defining quasi-hereditary or stratified algebras. Then projective resolutions behave well under inflating S-modules to R-modules and in good cases one gets a full embedding of derived categories $D^b(S\text{-mod}) \hookrightarrow D^b(R\text{-mod})$.

In this chapter we will develop, and apply, a new method of *comparing cohomology*, combining a subring situation with a quotient ring situation, but without assuming any of the strong conditions normally used in either of these situations. In particular, this method can be used to show the *non-vanishing of cohomology* in certain situations.

One feature of this approach is that it usually does not lead to isomorphisms in cohomology, but to surjective (or injective) maps between extension groups over the two rings involved. Thus, on the level of derived categories we do not get embeddings in the usual sense (that is, injective on objects and bijective on morphisms). Instead we get exact (triangle preserving) functors, which are injective on objects and injective on morphisms (or, going in the opposite direction, surjective on objects and surjective on morphisms).

Having developed the general machinery, we then collect some evidence for this method to be practical and useful, both when dealing with abstract problems - we recover and extend a number of results in the literature, in particular on the *strong no loops conjecture* and on trivial extensions of abelian categories - and when studying algebras occuring in nature - we relate the cohomology of *Brauer algebras* with that of various *symmetric groups*.

We refer the reader to [AF92] and [BD68] for background material on rings and categories of modules, to [Rot79] for homological algebra and to [Kel96, Kel98] for an introduction into derived categories.

8.1 Definitions and basic properties

We begin this section by defining the basic structure we are going to use throughout the chapter, the structure of an (exact) split pair of functors between two categories.

Exact split pairs will be used to compare the cohomology of two categories of modules. Indeed, an exact split pair of functors between two abelian categories induces a split pair of functors between the derived categories and hence relates the cohomology of the two categories; there are induced surjections and injections between the Ext groups in the two abelian categories. This allows to compare these Ext groups and numerical invariants associated, such as projective dimensions.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. A pair (F, G) of additive functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ is a *split pair of functors* (between $\mathcal{A}$ and $\mathcal{B}$) if the composition $F \circ G$ is an autoequivalence of the category $\mathcal{B}$. If the categories are equipped with exact structures, and if the two functors are exact with respect to these exact structures, the split pair is called an *exact split pair of functors* (between $\mathcal{A}$ and $\mathcal{B}$).

Note that in this definition the pairs $(\mathcal{A}, \mathcal{B})$ and (F, G) are ordered.

The definition of exact split pair of functors between two categories of modules can be reformulated, as for Morita equivalences, in terms of the existence of two bimodules.

Lemma 8.1.1. *Let A and B be two rings. Denote the categories A-mod and B-mod (of left modules) by $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the existence of an exact split pair of functors (F, G) between $\mathcal{A}$ and $\mathcal{B}$ is equivalent to the existence of two bimodules ${}_BT_A$ and ${}_AS_B$, each projective on the right, such that ${}_BT\underset{A}{\otimes} S$ is a projective generator, that is, it is projective and $B \in add(T\underset{A}{\otimes} S)$, as a left B-module.*

Proof. If there are two such bimodules, then $F = {}_BT\underset{A}{\otimes} -$ and $G = {}_AS\underset{B}{\otimes} -$ are exact by right projectivity of S and T, respectively, and they form a split pair of functors by the assumption on $T \otimes S$.

Conversely, if (F, G) is an exact split pair of functors between $\mathcal{A}$ and $\mathcal{B}$, then by Watts's theorem (see for example [Rot79, Theorem 3.33]), the right exact functors F, G are taking tensor products by bimodules ${}_BT_A$ and ${}_AS_B$, respectively. That is, $F = {}_BT\underset{A}{\otimes} -$ and $G = {}_AS\underset{B}{\otimes} -$. The functors also being left exact, the bimodules T and S must be projective as right A and B modules, respectively.

Since the composition $F \circ G$ is an equivalence, it sends the projective generator B to a projective generator (as left B-module), which is ${}_BT\underset{A}{\otimes} S\underset{B}{\otimes} B \cong {}_BT\underset{A}{\otimes} S$. ∎

Note that $F \circ G$ being an equivalence implies that $T \otimes_A S$ also is a projective generator on the right. The example of Morita equivalences shows that in general $T \otimes_A S$ need not be projective as a bimodule.

Proposition 8.1.2. *If (F, G) is a split pair of functors between two additive categories $\mathcal{A}$ and $\mathcal{B}$, then F is surjective on the isomorphism classes of objects and G is injective on the isomorphism classes of objects.*

Moreover, given objects $M, N \in \mathcal{B}$, the functor F induces an epimorphism of abelian groups $\mathrm{Hom}_{\mathcal{B}}(G(M), G(N)) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(FG(M), FG(N)) \cong \mathrm{Hom}_{\mathcal{A}}(M, N)$ on the morphism groups and the functor G induces a monomorphism of abelian groups $\mathrm{Hom}_{\mathcal{B}}(M, N) \rightarrowtail \mathrm{Hom}_{\mathcal{A}}(G(M), G(N))$ on the morphism groups.

Proof. Since $F \circ G$ is an equivalence, any object M of $\mathcal{B}$ is isomorphic to an object of the form $(F \circ G)(N)$, that is, it is of the form $F(K)$ for $K = G(N)$. Moreover if M, N are two non-isomorphic objects in $\mathcal{B}$, then $(F \circ G)(M)$ and $(F \circ G)(N)$ are non-isomorphic (cf. [AF92, 21.1] for the abelian case; their proof works for additive categories as well), thus $G(N)$ and $G(M)$ can't be isomorphic.

Now let M, N be objects in $\mathcal{B}$. Consider the group homomorphisms induced by F and G:

$$\mathrm{Hom}_{\mathcal{B}}(M, N) \xrightarrow{G^*} \mathrm{Hom}_{\mathcal{A}}(GM, GN) \xrightarrow{F^*} \mathrm{Hom}_{\mathcal{B}}(FGM, FGN).$$

Since $F^* \circ G^*$ is a group isomorphism ([BD68, Proposition 1.15]), the morphism F^* induced on the Hom groups by the functor F is surjective and the morphism G^* induced on the Hom groups by the functor G is injective. ∎

Proposition 8.1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. An exact split pair of functors (F, G) between $\mathcal{A}$ and $\mathcal{B}$ induces a split pair of functors (F_*, G_*) between the derived categories $D^b(\mathcal{A})$ and $D^b(\mathcal{B})$.*

Proof. The exact functors F, G induce functors $F_* \colon D^b(A\text{-mod}) \to D^b(B\text{-mod})$ and $G_* \colon D^b(B\text{-mod}) \to D^b(A\text{-mod})$ between the derived categories.

Moreover, $F \circ G$ being an equivalence, there exists an inverse equivalence $\Phi \colon \mathcal{B} \to \mathcal{B}$ which is also exact. It induces a functor $\Phi_* \colon D^b(\mathcal{B}) \to D^b(\mathcal{B})$ at the derived level which turns out to be the inverse equivalence of $F_* \circ G_* = (F \circ G)_*$. The claim follows. ∎

Corollary 8.1.4. *An exact split pair of functors (F, G) between two abelian categories $\mathcal{A}$ and $\mathcal{B}$ induces, for $n \geq 0$, for M, N objects in $\mathcal{B}$, surjections $\mathrm{Ext}^n_{\mathcal{A}}(GM, GN) \twoheadrightarrow \mathrm{Ext}^n_{\mathcal{B}}(M, N)$ and injections $\mathrm{Ext}^n_{\mathcal{B}}(M, N) \rightarrowtail$*

$\mathrm{Ext}^n_{\mathcal{A}}(GM, GN)$. *For $M = N$, these maps are ring homomorphisms of Yoneda algebras.*

In the terminology to be introduced in the next Section, the last statement means that an exact split pair induces split quotients of Yoneda algebras.

PROOF. Use that $\mathrm{Ext}^n_{A}(M, N) \cong \mathrm{Hom}_{D^b(A)}(M[0], N[n])$, where $L[t]$ is the complex with the module L in degree t and with zero elsewhere. Then the claim follows from Proposition 8.1.3 and Proposition 8.1.2. ■

Therefore homological properties in one abelian category can be compared to such properties in another, possibly smaller, abelian category, which may have 'less' cohomology. For example, we have the following inequalities between homological dimensions:

Corollary 8.1.5. *With the previous notations one has $pd(M) \leq pd(G(M))$, $id(M) \leq id(G(M))$, for any B-module M, and $gl.dim(\mathcal{A}) \geq gl.dim(\mathcal{B})$.*

PROOF. Let M, N be objects in $\mathcal{B}$. By proposition 8.1.2 there exist objects H, K in $\mathcal{A}$ such that $F(H) \cong M, F(K) \cong N$. For every natural number n, if $\mathrm{Ext}^n_{\mathcal{B}}(M, N) \neq 0$, then $\mathrm{Ext}^n_{\mathcal{A}}(H, K) \neq 0$ since there exists an injective homomorphism $\mathrm{Ext}^n_{\mathcal{B}}(M, N) \rightarrowtail \mathrm{Ext}^n_{\mathcal{A}}(H, K) \neq 0$.

Since $pd(M) = sup\{n \mid \exists N : \mathrm{Ext}^n(M, N) \neq 0\}$, $id(M) = sup\{n \mid \exists N : \mathrm{Ext}^n(N, M) \neq 0\}$ and $gl.dim(\mathcal{A}) = sup\{n \mid \exists M, N : \mathrm{Ext}^n(A, B) \neq 0\}$, the claim follows. ■

Example 8.1.6. The 'composition' of split pairs need not be a split pair. Indeed, let R be any ring with a ring endomorphism f, which is not surjective. Let $A := R \oplus R$ be the sum of two copies of R and $B := R$. The map $(1, f)$ is an embedding of B into A, which composed with the projection onto the first summand gives the identity. This induces an exact split pair $F = {}_BA\underset{A}{\otimes} -$ and $G = {}_AB\underset{B}{\otimes} -$ of functors (it is an example of a split quotient as defined in the next section).

Swapping the two summands of A, that is, multiplying by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is an automorphism, which induces an autoequivalence H. The pair (H, Id) also is a split pair.

However, the composition $F \circ H \circ Id \circ G$ sends R to an R-module, which as a set also is R, but with R-action given by f. This module may, for example, decompose. (To get an explicit example, choose R to be $k[x]/(x^3)$ for some field k. Then let f send x to x^2. Then under the action via f, R decomposes into a two-dimensional summand and a one-dimensional summand.)

8.2 Some exact split pairs

We will describe three classes of examples of exact split pairs. In the next section we will show that all exact split pairs are made up (in a sense to be made precise) of these three types of examples.

8.2.1 Split Quotients

Let A and B be two rings. Then we call B a *split quotient* of A, if B is a subring of A (via an embedding ε sending the unit of B to that of A) and there exists a surjective homomorphism $\pi\colon A \twoheadrightarrow B$, such that the composition $\pi \circ \varepsilon$ is the identity on B. The homomorphisms π and ε induce two exact functors $F = {}_BA\underset{A}{\otimes} -$ and $G = {}_AB\underset{B}{\otimes} -$ between the categories A-mod and B-mod. The composition $F \circ G$ is the identity on B-mod.

The bimodules T, S generating the functors as in Proposition 8.1.1 are ${}_BA_A$ and ${}_AB_B$ respectively, with the B-action on A given by $b \cdot a = \varepsilon(b)a$ and the A-action on B given by $a \cdot b = \pi(a)b$.

The embedding G (which is just inflation along π) maps simple B-modules to simple A-modules. The functor F then restricts their A-structure back to the B-structure.

(In the following we usually will assume that ε is just an inclusion.)

Split quotients are *retracts* of rings. They also appear under the name of *cleft extensions*, for example in [Bel00], where they also have been used to compare cohomology of two module categories.

8.2.2 Centralizer subrings eAe

Let A be a ring, e an idempotent in A. Let B be the centralizer subring eAe, and let ${}_BT_A$ be the bimodule ${}_BeA_A$ and ${}_AS_B$ the bimodule ${}_AAe_B$. Assume S to be eAe-projective. Then the functors $F = {}_BT_A\underset{A}{\otimes} -\colon A\text{-mod}\to B\text{-mod}$ and $G = {}_AS_B\underset{B}{\otimes} -\colon B\text{-mod}\to A\text{-mod}$ form an exact split pair of functors between A-mod and B-mod.

The functor F sends simple A-simples to simple B-modules or to zero. But the functor G in general does not send simples to simples; it may add composition factors of type $1 - e$.

Sometimes, centralizer subrings eAe are also called *corner rings*.

8.2.3 Morita equivalences

Let A and B be two Morita equivalent rings and let $\varphi\colon A\text{-mod}\to B\text{-mod}$, $\psi\colon B\text{-mod}\to A\text{-mod}$ be two reciprocally inverse equivalences of categories. Obviously, both (φ, ψ) and (ψ, φ) are split exact pairs of functors since $\varphi\psi \cong id_{\mathcal{B}}$ and $\psi\varphi \cong id_{\mathcal{A}}$. Both the functor F and the functor G send simple modules to simple modules.

8.3 All exact split pairs

In this section we describe the exact split pairs in general. Combining the examples of split quotients and centralizer rings and relaxing the condition in the latter case, we get a more general example of an exact split pair. We then show that up to certain Morita equivalences this is the general case.

DEFINITION. Let A be a ring, e an idempotent, and B a split quotient of eAe (viewed as a subring of eAe). Then we call B a *corner split quotient* if there is an A-eAe-bimodule S, which is projective as a right B-module (via the embedding of B into eAe) and which satisfies $eS \simeq B$ as left B-modules.

Note that every B-module is an eAe-module via the quotient map. Thus, in the definition, we may equivalently require S to be a right B-module.

Lemma 8.3.1. *Let B be a corner split quotient of A. Then the functors $F = eA \otimes_A -$ and $G = S \otimes_{eAe} B \otimes_B -$ form an exact split pair.*

PROOF. The functor F is exact by construction and G is so by assumption. The composition $F \circ G$ is tensoring with ${}_BeA \otimes_A S = eS \simeq {}_BB$, hence it is an autoequivalence of B-mod. ∎

As Example 8.1.6 has shown, composing exact split pairs (or even just split quotients) with Morita equivalences in general need not result in a split pair. There are, however, more restricted options of composing split pairs with equivalences in order to produce new split pairs.

Lemma 8.3.2. *Let A and B be two rings. Let (F, G) be an exact split pair of functors between A-mod and B-mod.*

$$A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \qquad F \circ G \textit{ autoequivalence of } B\text{-}mod$$

(a) Let $E_1 : A$-mod $\to A'$-mod and $E_2 : A'$-mod $\to A$-mod be two mutually inverse equivalences. Then $(F \circ E_2, E_1 \circ G)$ is an exact split pair.

(b) Let $E_3 : B$-mod $\to B'$-mod and $E_4 : B'$-mod $\to B$-mod be any two equivalences. Then $(E_3 \circ F, G \circ E_4)$ is an exact split pair.

Note that in part (b) we may as well assume $B' = B$ (and hide the equivalence inside B-mod).

If we would use a more restricted definition of split pairs, requiring $F \circ G$ to be the identity, then the composition of split pairs always would be a split pair.

PROOF. We note that equivalences are automatically exact. Then (a) follows from the equality $F \circ E_2 \circ E_1 \circ G = F \circ G$, whereas (b) follows from $E_3 \circ F \circ G \circ E_4$ being an equivalence. ∎

Now we can show that the sufficient conditions for exact split pairs given in the previous two Lemmas are also necessary; that is, any exact split pair is obtained from a split corner quotient as in Lemma 8.3.1 by composing with admissible Morita equivalences as in Lemma 8.3.2.

Theorem 8.3.3 (Diracca and Koenig).
Let A and B be two rings. Let (F, G) be an exact split pair of functors between A-mod and B-mod.

$$A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \qquad (F \circ G \; autoequivalence\; of\; B\text{-}mod)$$

Then there exists a ring A', an idempotent $e \in A'$, a bimodule ${}_{A'}S_{eA'e}$ and a pair of mutually inverse equivalences ($E_1\colon A$-mod $\to A'$-mod, $E_2\colon A'$-mod $\to A$-mod) such that the following properties are satisfied:

The ring B is a split corner quotient of A' with respect to the bimodule S. In particular, B is a split quotient of $eA'e$.

Setting $E_3 = Id : B$-mod $\to B$-mod and $E_4 = (F \circ G)^{-1}$, the following diagram describes the situation:

$$A'\text{-}mod \underset{E_1}{\overset{E_2}{\rightleftarrows}} A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \underset{E_4}{\overset{E_3}{\rightleftarrows}} B\text{-}mod \qquad \begin{gathered} F' = E_3 \circ F \circ E_2,\; G' = E_1 \circ G \circ E_4 \\ F' \circ G' = id_{B\text{-}mod} \end{gathered}$$

Here, $F' = E_3 \circ F \circ E_2$ and $G' = E_1 \circ G \circ E_4$ are the functors ${}_BeA' \underset{A'}{\otimes} -$ and ${}_{A'}S \underset{eA'e}{\otimes} B \underset{B}{\otimes} -$ respectively, both ${}_BeA'_{A'}$ and ${}_{A'}S \underset{eA'e}{\otimes} B_B$ being right projective and $eS_B \cong B$.

Conversely, any such situation describes an exact split pair.

PROOF. Since the functors F and G are exact, there exist two modules ${}_BT'_A$ and ${}_AS'_B$, both right projective, such that $F = {}_BT' \underset{A}{\otimes} -$ and $G = {}_AS' \underset{B}{\otimes} -$.

Moreover, $F \circ G$ being an equivalence, we have that ${}_BT' \underset{A}{\otimes} S'$ is a projective generator in B-mod.

Since T'_A is projective, there exists a ring A', Morita equivalent to A, and mutually inverse equivalences $A'\text{-}mod \underset{E_1}{\overset{E_2}{\rightleftarrows}} A\text{-}mod$ such that the module $T = E_1(T')$ is of the form eA' for some idempotent $e \in A'$.

Recall we have set $E_3 := Id : B\text{-mod} \to B\text{-mod}$ and $E_4 := (F \circ G)^{-1}$, so that we have the diagram as claimed.

$$A'\text{-}mod \underset{E_1}{\overset{E_2}{\rightleftarrows}} A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \underset{E_4}{\overset{E_3}{\rightleftarrows}} B\text{-}mod \qquad \begin{array}{c} F' = E_3 \circ F \circ E_2,\ G' = E_1 \circ G \circ E_4 \\ F' \circ G' = id_{B\text{-}mod} \end{array}$$

Hence we have moved into the following situation. We have exact functors $F' = E_3 \circ F \circ E_2$ and $G' = E_1 \circ G \circ E_4$ with $F' \circ G' = id_{B\text{-}mod}$. These functors can be written $F' = {}_BT \underset{A'}{\otimes} -$ and $G' = {}_{A'}S \underset{B}{\otimes} -$, with ${}_BT_{A'} = eA'$ and ${}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$. The last isomorphism ${}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$ also identifies right B-module structures. Indeed, tensoring on the left does not affect right module structures and the equivalence $F' \circ G'$ transports the right module structure over the endomorphism ring, which is B, into a right module structure over the image of the endormorphism ring, which is again B.

It remains to check that B is a split quotient of $eA'e$, in a natural way:

Since T is a B-A'-bimodule, there exists a ring homomorphism $\varepsilon\colon B \to \operatorname{End}(T_{A'}) = eA'e$.

Since S is an A'-B-bimodule, there is an $eA'e$-B-bimodule structure on eS. Hence there is a ring homomorphism $\pi\colon eA'e \to \operatorname{End}(eS_B) = \operatorname{End}(eA' \underset{A'}{\otimes} S_B) = \operatorname{End}(B_B) = B$.

We claim that $\pi\varepsilon = id_B$. Given any $b_1 \in B$ we can write it as eae for some $a \in A'$. We have to show that in

$$\begin{array}{ccccc} B & \overset{\varepsilon}{\longrightarrow} & eA'e & \overset{\pi}{\longrightarrow} & B \\ b_1 & \mapsto & eae & \mapsto & b_2 \end{array},$$

there is an equality $b_1 = b_2$.

Now $eae = \varepsilon(b_1)$ means $eaet = b_1t$ for every $t \in T$, while $b_2 = \pi(eae)$ means that for every $s \in S$ one has $eae \bullet (e \otimes s) = b_2(e \otimes s)$. Here the action of eae on $eA' \otimes S$, denoted by $\bullet$, is given by the action of A' on S under the isomorphism $eS \to eA' \underset{A'}{\otimes} S$ $(es \mapsto e \otimes s)$. Hence it is given by $eae \bullet (e \otimes s) = e \otimes aes$ and then extended by linearity. The action of B on $eA' \underset{A'}{\otimes} \cong {}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$ is given by considering B as the endomorphism ring of B_B. Therefore B is acting on the left with the usual action on the regular module.

Therefore for every $ea' \otimes s \in {}_BT\underset{A'}{\otimes} S_B \cong {}_BB_B$ we have $b_1(ea' \otimes s) = (b_1ea')\otimes s = (eaea')\otimes s = e\otimes(aea's) = eae\bullet(e\otimes a's) = b_2(e\otimes a's) = b_2(ea'\otimes s)$. Thus $b_1 = b_2$ since they act equally on every element of the regular module.

To get the converse of this characterization we just combine Lemma 8.3.1 and Lemma 8.3.2. ■

The following examples show that the theorem cannot be strengthened any further:

Example 8.3.4. The module S in the definition of split corner quotient need not be projective as a left A-module. In particular, S need not be isomorphic to Ae.

Let k be a field. Let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ and let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $B = eAe = k$ is a split corner quotient when setting $S = Ae$, but also when using the simple A-module $S = Ae/rad(Ae)$.

While the composition $F \circ G$ is the same in both cases, the images of B-modules under G are different.

Example 8.3.5. It may even happen that $S = Ae$ does not satisfy our assumptions, but still there is a split corner quotient for a different choice of S.

Let k be a field and $B := k[x]/(x^2)$. Let $A = \begin{pmatrix} k & k \\ 0 & B \end{pmatrix}$ (where multiplication uses the action of B on its simple quotient k). Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $B = eAe = k$ is a split corner quotient when setting $S = Ae/rad(Ae)$, but Ae is not projective as a right B-module.

Example 8.3.6. Often, a good candidate for S is $S = Ae\otimes_{eAe}B$ for some split quotient B of eAe. In this situation, the condition ${}_BeS = {}_BB$ is automatic, and it remains to check that S is right B-projective.

In the context of Brauer algebras (Section 6), we will see examples where S takes this form $Ae \otimes_{eAe} B$ for some B strictly smaller than eAe. Note that $Ae \otimes_{eAe} B$ is (in general, and usually in these examples) not isomorphic to the restriction of the right eAe-module structure of Ae to B.

8.4 Examples of exact split pairs

In this section we are collecting a number of situations in the literature to which our machinery applies in a natural way. In some cases, we thus reprove known results, in other cases we get something new.

We first list some well-known examples of split quotients.

A semidirect product of finite groups fits into a split quotient situation relating the group algebra of the quotient subgroup with that of the semidirect product.

Let R be a ring and $R[x]$ be the polynomial ring over R in the indeterminate x. (That is, x is like a loop.)

The ring homomorphisms $\varepsilon\colon R \to R[x]$ (the canonical embedding) and $\pi\colon R[x] \to R$ (the canonical projection on the zero-degree term) show that R is a split quotient of $R[x]$.

Thus there exists a split exact pair of functors between $R[x]$-mod and R-mod. Similar split pairs exist for various 'twisted' polynomial rings.

In a similar way one can show there exists a split exact pair of functors between $R\big[[x]\big]$-mod and R-mod, where $R\big[[x]\big]$ is the ring of formal power series over R.

Note than the generator does not need to be torsion-free. The ring R is a split quotient of $R[x]/\big(f(x)\big)$ as well, if $f(x)$ has no zero-degree term.

8.4.1 Tensor products and twisted tensor products

Our technology applies both to tensor products of algebras and to algebras which are tensor products of other algebras, but with slightly twisted multiplicative structure.

First we deal with the classical case:

Proposition 8.4.1. *Let A and B be finite dimensional algebras over a perfect field k. Then there exists a split pair relating $A \otimes_k B$ and A.*

Proof. Let $S(B)$ be a maximal subalgebra of B. Then there is a split quotient situation $S(B) \hookrightarrow B \twoheadrightarrow S(B)$, which induces a split quotient situation $A \otimes_k S(B)$. Another split quotient situation relates the semisimple algebra $S(B)$ with k by combining a Morita equivalence to a product of copies of k with projection onto one component. ∎

A similar argument also works in a much more general situation, thus for example covering algebras, which have attracted some recent interest within the theory of quasi-hereditary algebras.

Proposition 8.4.2. *Let A be a finite-dimensional algebra over a perfect field k, which has a vector space decomposition $A = B \otimes_S C$ such that B and C are k-algebras, S is a semisimple algebra contained in a maximal semisimple subalgebra $S(B)$ of B and also in a maximal semisimple subalgebra $S(C)$ of C*

and B and C are subalgebras of A via $B \simeq B \otimes_S S \subset B \otimes_S S(C) \subset B \otimes_S C = A$ and $C \simeq S \otimes_S C \subset S(B) \otimes_S C \subset B \otimes_S C = A$.

Then there are split pairs (A, B) and (A, C).

PROOF. As in the previous proof there are split quotient situations $B \otimes_S S(C) \subset A$ and $S(B) \otimes_S C \subset A$, which can be combined with split quotients relating $S(B)$, or $S(C)$, and S and then Morita equivalences $B - mod \simeq B \otimes_S S - mod$ and $C - mod \simeq S \otimes_S C - mod$. ∎

Algebras of this kind include Xi's dual extension algebra [Xi95] and the twisted doubles of Deng and Xi [DX95]; these constructions (imposing additional conditions on B and C) were defined to produce examples of quasi-hereditary algebras, which then could be related to subalgebras B and C. In the situations studied mostly, much stronger statements are true than what our machinery is producing, but in more general situations (still covered by these definitions), we get new results.

8.4.2 Trivial extensions of algebras and of categories

A familiar construction in ring theory, analogous to taking semidirect products of groups, is the trivial extension of a ring along a bimodule. The input is a ring R and an R-R-bimodule M and the output is a new ring $T(R, M)$, which as an abelian group is $R \oplus M$, but with multiplication $(r, m) \cdot (r', m') := (rr', rm' + mr')$, that is by multiplication of 2×2-matrices of the form $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ with $r \in R$ and $m \in M$.

There is a split quotient $T(R, M) \twoheadrightarrow R$ (whatever M is).

Using the above matrix interpretation, a module X over $T(R, M)$ is an R-module with an R-map $M \otimes_R X \to X$, which 'squares' to zero, that is $M \otimes M \otimes X \to M \otimes X \to X$ composes to zero.

Generalizing this situation, Fossum, Griffith and Reiten defined in [FGR75] trivial extensions of abelian categories.

According to [FGR75] Fossum, Griffith and Reiten the *trivial extension* $\mathcal{A} \ltimes \Phi$ of an abelian category $\mathcal{A}$ by an additive endofunctor $\Phi \colon \mathcal{A} \to \mathcal{A}$ is defined as follows:

- an object in $\mathcal{A} \ltimes \Phi$ is a morphism $\alpha \colon \Phi A \to A$ for an object A in $\mathcal{A}$ such that $\alpha \circ \Phi\alpha = 0$;
- if $\alpha \colon \Phi A \to A$ and $\beta \colon \Phi B \to B$ are objects in $\mathcal{A} \ltimes \Phi$, then a morphism

$\gamma\colon \alpha \to \beta$ is a morphism $\gamma\colon A \to B$ such that the diagram

$$\begin{tikzcd} \Phi A \arrow[r, "\Phi\gamma"] \arrow[d, "\alpha"] & \Phi B \arrow[d, "\beta"] \\ A \arrow[r, "\gamma"] & B \end{tikzcd}$$

is commutative;

- composition in $\mathcal{A} \ltimes \Phi$ is just composition in $\mathcal{A}$.

Fossum, Griffith and Reiten proved ([FGR75, Proposition 1.1]) the category $\mathcal{A} \ltimes \Phi$ is abelian if the functor Φ is right exact. In this case there is a full embedding $G\colon \mathcal{A} \to \mathcal{A} \ltimes \Phi$ given by $A \mapsto (0\colon \Phi A \to A)$ and a functor $F\colon \mathcal{A} \ltimes \Phi \to \mathcal{A}$ given by $(\alpha\colon \Phi A \to A) \mapsto A$ (the actions of F and G on the morphisms being the natural ones). The functor G is obviously exact.

To show the functor F is exact, let us see how kernels and cokernels are defined in the category $\mathcal{A} \ltimes \Phi$. Given two objects $\alpha\colon \Phi A \to A$ and $\beta\colon \Phi B \to B$ in $\mathcal{A} \ltimes \Phi$ and a morphism $\gamma\colon \alpha \to \beta$ the kernel (the cokernel) of γ is the kernel (the cokernel) of $\gamma\colon A \to B$ when considered as a morphism in $\mathcal{A}$ (see [FGR75, Proposition 1.1]). The functor F preserves kernels and cokernels and this is equivalent (see e.g. [Kel98]) to exactness of the functor F.

It is straightforward, finally, that $FG = \mathrm{id}_{\mathcal{A}}$.

Thus we are in a split pair situation. Some of the basic homological results in [FGR75] can be deduced by our framework, although the machinery they use in that specific situation is more precise, so that they can retrive more detailed informations.

8.4.3 Brauer algebras

In this subsection we find split corner quotients ‘in nature’, relating cohomology of Brauer algebras (which occur in representation theory of algebraic groups of type B and C) with cohomology over symmetric groups (which occur in type A). On the ring theoretic level, Brauer algebras have been related to symmetric groups in [KX01] in terms of cellular structures. On a module theoretic level, such connections have been found and used in [HP05], where also some cohomological statements can be found. Our results add another cohomological aspect of this connection between types A and B, C.

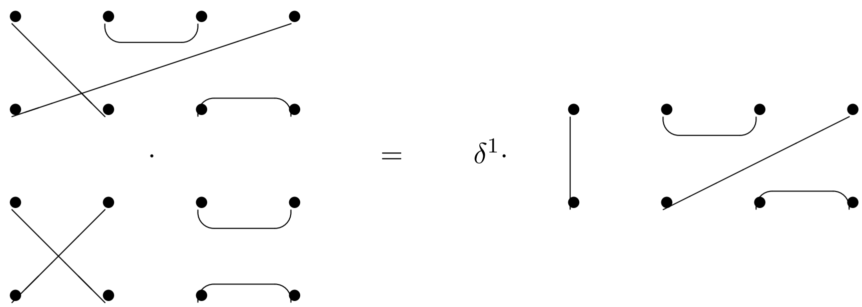
Schur–Weyl duality relates the representation theory of the infinite group $GL_n(k)$ (where k is an infinite field of arbitrary characteristic) with that of the symmetric group Σ_r via the mutually centralising actions of the two groups on the space $(k^n)^{\otimes r}$. Brauer defined the algebras which are now called 'Brauer algebras' by an analogous situation where GL_n is replaced by either an orthogonal or a symplectic group (types B and C) and the group algebra of the symmetric group is replaced by a Brauer algebra. More precisely, for a fixed integer r and a given base ring k (a field in Brauer's case), a whole family of Brauer algebras $B_k(r, \delta)$ is defined, depending on a parameter $\delta \in k$, which has to be specialised to certain integers to cover the situation Brauer was interested in.

More recently Brauer algebras and their generalisations, especially the Birman-Murakami-Wenzl algebras, have been looked at in the context of quantum groups and low–dimensional topology. Other closely related algebras, such as Temperley-Lieb algebras and partition algebras are also of interest in statistical mechanics.

Definition. Fix a commutative noetherian domain k, an element $\delta \in k$ and a natural number r. Then the *Brauer algebra* $B_k(r, \delta)$ as a k–vector space has a basis consisting of diagrams of the following form: a diagram contains $2r$ vertices, r of them called 'top vertices' and the other r called 'bottom vertices' such that the set of vertices is written as a disjoint union of r subsets each of them having two elements; these subsets are called 'edges'. Two diagrams x and y are multiplied by concatenation, that is, the bottom vertices of x are identified with the top vertices of y, thus giving rise to edges from the top vertices of x to the bottom vertices of y, hence defining a diagram z. Then $x \cdot y$ is defined to be $\delta^{m(x,y)}z$ where $m(x, y)$ counts those connected components of the concatenation of x and y which do not appear in z, that is, which neither contain a top vertex of x nor a bottom vertex of y.

(Note that in this definition and for the rest of this section, k need not be a field, but just any commutative noetherian domain.)

Let us illustrate this definition by an example, multiplying two elements in $B_k(4, \delta)$:



Brauer algebras are cellular algebras [GL96, KX01]; in particular, they have cell modules, which play a role analogous to that of Specht modules for symmetric groups.

An easy observation is:

Proposition 8.4.3. *The group algebra $k\Sigma_r$ is a split quotient of $B(r, \delta)$.*

Proof. Those diagrams which just consist of through strings (that is, strings going from the top row to the bottom row) define permutations, and the k-space generated by them is a subalgebra of $B(r, \delta)$, which is isomorphic to $k\Sigma_r$. Those diagrams, which have at least one horizontal edge (in the top row and thus also in the bottom row) are the k-basis of a two-sided ideal, the quotient by which again is $k\Sigma_r$, and this quotient map restricts to an isomorphism on the algebra generated by through strings. ■

At this point we get for free the following known corollary:

Corollary 8.4.4. *[KX99] Let k be a field, $\delta \neq 0$ and $char(k) = p$. Then the Brauer algebra $B(r, \delta)$ has finite global dimension if and only if $p > r$.*

Proof. (Note that $B(r, \delta)$ is rarely symmetric or self-injective, unlike $k\Sigma_r$. Thus the problem is non-trivial.) If $p > n$ then the cell chain given in [KX01] is a heredity chain. Thus $B(r, \delta)$ is a quasi-hereditary algebra and hence of finite global dimension. (See [KX01, KX99] for details.) If $char(k) = p \leq n$, then the known cell chain is not a heredity chain, and at this point one may invoke the main theorem of [KX99] to conclude that $B(r, \delta)$ must have infinite global dimension. Alternatively, and more easily, this follows from Proposition 8.1.5 by using that for $char(k) = p \leq r$ the group algebra $k\Sigma_r$ has infinite global dimension. ■

A more interesting application, which makes full use of our split pair technology, providing non-trivial examples of split corner quotients, is a formalization of the observation that the Brauer algebra $B(r, \delta)$ is related not just to the symmetric group algebra $k\Sigma_r$, but also to many smaller symmetric groups. Indeed, in [KX01], Theorem 5.6, the Brauer algebra has been written as follows:

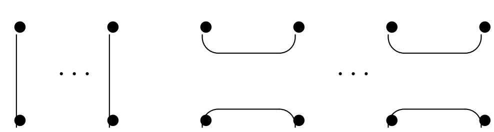
As a free k–module, $A = B(r, \delta)$ is equal to

$$k\Sigma_r \oplus (V_{r-2} \otimes V_{r-2} \otimes k\Sigma_{r-2}) \oplus (V_{r-4} \otimes V_{r-4} \otimes k\Sigma_{r-4}) \oplus \dots$$

(ending with indices 0 or 1 when r is even or odd), where V_l is a free k-space, whose k–rank equals the number of possibilities to draw $(r-l)/2$ edges between $r - l$ out of r vertices. This decomposition produces a chain of ideals (which can be refined to a cell chain) of $B(r, \delta)$, where the ideals are defined by adding up any right hand part $(V_l \otimes V_l \otimes k\Sigma_l) \oplus (V_{l-2} \otimes V_{l-2} \otimes k\Sigma_{l-2}) \oplus \dots$ in this decomposition.

Each layer $V_l \otimes V_l \otimes k\Sigma_l$ (which is a subquotient of two ideals in the above chain of ideals) has a basis consisting of diagrams with $(r - l)/2$ horizontal edges in top and bottom row each (recorded in the first and second copy of V_l) and the remaining edges being through strings (recorded as elements of the symmetric group Σ_l).

From now on let us assume δ is invertible in k. Then we define (as in [KX01] or [HP05]) an idempotent element e_l in $A = B(r, \delta)$ by $e_l = \delta^{-(r-l)/2} d_l$ where d_l is the diagram obtained by putting l vertical through strings at the beginning and then putting horizontal edges relating a vertex with a direct neighbour, that is, d_l is of the form:



The corner ring $e_l A e_l$ is isomorphic to the Brauer algebra $B(l, \delta)$. It has, of course, a split quotient $A_l \simeq k\Sigma_l$.

Proposition 8.4.5. *Using the above notation (in particular, δ is invertible), the algebra $A_l \simeq k\Sigma_l \subset e_l A e_l$ is a corner split quotient of the Brauer algebra $B(r, \delta)$.*

Proof. This proof is very similar to arguments used in [HP05], where a general theory of Young modules for Brauer algebras is developed.

We know already that $A_l \simeq k\Sigma_l$ is a split quotient of the small Brauer algebra $B(l,\delta)$, which is isomorphic to the corner ring e_lAe_l of the big Brauer algebra $A = B(r,\delta)$ and we also know the ring homomorphisms used in this context. It remains to prove that the module $Ae_l \otimes_{e_lAe_l} A_l$ is projective as a right A_l-module.

By the multiplication rule in the Brauer algebra, and by the definition of e_l, the projective A-module Ae_l has a basis consisting of diagrams with at least $m = (r-l)/2$ horizontal edges, where the bottom row has at least the horizontal edges occuring in e_l. Similarly, e_lAe_l has a basis consisting of diagrams with at least m horizontal edges, where both in the top and in the bottom row at least the horizontal edges used in e_l do occur. The algebra e_lAe_l acts on A_l via the quotient map α, which has in its kernel all diagrams in e_lAe_l with more than m loops; that is, if a diagram has loops not already in e_l, then the diagram is in the kernel of α.

The tensor product $Ae_l \otimes_{e_lAe_l} A_l$ is generated (over k) by tensors of the form $x \otimes y$ where x is a diagram sharing the m horizontal edges in its bottom row with e_l, but possibly having more of them, and y is an element in the symmetric group Σ_l. If x has more than m horizontal edges, then we can write $x = x \cdot e_q$ for some $q < l$, with $e_q \in e_lAe_l$. Here, e_q is an idempotent in a lower layer of the cell chain, having more than m horizontal edges in each row. Therefore, $\alpha(e_q) = 0$. Thus $x \otimes y = x \cdot e_q \otimes y = x \otimes e_q y = 0$. Therefore, $Ae_l \otimes_{e_lAe_l} A_l$ is generated (over k) by elements $x \otimes y$ with x having precisely m horizontal edges in each row, those in the bottom row being the same as in e_l, whereas those in the top row can be arranged freely. Rewriting, by a slight abuse of notation, $x \otimes y$ as $xy \otimes 1$ (with 1 the unit in the symmetric group Σ_l) it follows that $Ae_l \otimes_{e_lAe_l} A_l$ is just a direct sum of copies of A_l (the number given by the number of possibilities to arrange the m horizontal edges in the top row of x, that is, by the dimension of V_l). Indeed, let $J \subset Ae_l$ be the left ideal generated (over k) by diagrams with more than m edges. (This is the intersection with Ae of an ideal in the cell chain.) Then we have just shown that $J \otimes_{e_lAe_l} A_l$ vanishes. Hence, $Ae_l \otimes_{e_lAe_l} A_l$ is isomorphic to $(Ae_l/J) \otimes_{e_lAe_l} A_l$. Those elements in e_lAe_l, which have more than m horizontal edges in each row, act trivially both on A_l and on Ae/J. Thus the tensor product $A/J \otimes_{e_lAe_l} A_l$ over e_lAe_l is isomorphic to the tensor product $A/J \otimes_{A_l} A_l$ over A_l, which leaves us with A/J. This has a k-basis consisting of diagrams which in the bottom row have m horizontal edges arranged in the same way as in e_l and the m horizontal edges in the top arranged freely. The algebra A_l acts on the right by the symmetric group's action on the through strings. ∎

We note that in this situation Ae_l need not be projective as as right e_lAe_l-module. (The case $r = 4$, $l = 2$ produces already a counterexample.)

We refer the reader to [HP05] for more details on comparing A-modules with A_l-modules, especially cell modules and Young modules.

Finally we note that similar situations occur for other diagram algebras, such as partition algebras or Birman-Murakami-Wenzl algebras.

8.5 Homological reductions and the strong no loops conjecture

In this section we work with finite dimensional algebras $A = kQ/I$ given by a quiver Q and a relation ideal $I = \langle R \rangle$. We are collecting some reduction methods relating cohomology in A-mod to that in module categories of smaller algebras, defined by removing parts of the quiver Q. The aim is to get lower bounds for cohomology of A-modules. At the end of the section we apply these lower bounds to obtain the validity of the strong no loops conjecture for certain classes of algebras.

The setup is the following: let k be a field, $Q = (\Delta_0, \Delta_1)$, a quiver and kQ the path algebra. Let R be a set of relations (linear combinations of paths) in kQ and $I = \langle R \rangle$ the relation ideal in kQ generated (as a two-sided ideal) by R. Let $A = kQ/I$ be the path algebra of Q over k with relations R.

8.5.1 Removing vertices, keeping cohomology

This subsection does not use our machinery of split pairs. We just quote and then apply results from the literature, about isomorphisms in cohomology.

The context is that of an algebra A and a quotient algebra B modulo an ideal J satisfying strong properties. The following theorem is due to Cline, Parshall and Scott (cf. [CPS88], Theorem 3.1). It has been crucial in the development of the theory of quasi-hereditary algebras and, more generally, of stratified algebras.

Theorem 8.5.1 (Cline and Parshall and Scott).
Let A be a ring, J be an ideal of A and $B = A/J$. Let $\mathcal{A}$ be the category of all left A-modules and $\mathcal{B}$ the category of all left B modules. The full embedding $i\colon \mathcal{B} \to \mathcal{A}$ induces a functor between the derived categories $i_*\colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$. This functor is a full embedding if:

(a) $\mathrm{Ext}^n_A({}_AB, {}_AB) = 0$ *for every* $n > 0$ *and*

(b) $\mathrm{pdim}({}_AB) < \infty$.

We are applying this result in a very concrete situation; our aim is to remove certain vertices v from the quiver Q of $A = kQ/I$.

Let v be a vertex in Q. We will consider the quiver $Q^v = (\Delta_0^v, \Delta_1^v)$ obtained from Q by removing v and the arrows starting from v or ending in v. In the path algebra kQ^v we define the ideal $I^v = \{r \in I \mid \text{no summand of } r \text{ passes through } v\}$. Let us denote by A^v the algebra kQ^v/I^v.

If the vertex v is a source or a sink, the algebra A^v is isomorphic to the algebra A/J where J is the ideal Ae_vA generated by the idempotent e_v associated to the vertex v. (Note that in our notation a projective module Ae is k-generated by all paths ending at the vertex e. Thus a source e has a simple projective module Ae.)

In order to apply the theorem, it is sufficient to show that the ideal J is projective and that $\mathrm{Ext}^1_A({}_AB, {}_AB) = 0$.

If v is a sink, then $J = Ae_vA = Ae_v$ (since no non-trivial path leaves v) is projective and there are no nonzero homomorphisms ${}_AJ \to {}_AB$. This implies that $\mathrm{Ext}^1_A(B, B) = 0$ since it is a quotient of $\mathrm{Hom}_A(J, B) = 0$.

If v is a source, then $J = Ae_vA$ is the trace of the simple projective A-module Ae_v, hence it it is a semisimple projective module. Finally $\mathrm{Ext}^1_A({}_AB, {}_AB) = 0$, since it is a quotient of $\mathrm{Hom}({}_AJ, {}_AB)$ which vanishes by definition of J and of B.

Therefore, the functor i_* is a full embedding, thus giving isomorphisms, not just epimorphisms, between Ext groups in A-mod and B-mod.

This gives the first two statements in the following proposition (which also could be proved directly, not using [CPS88], without much effort):

Proposition 8.5.2. *Keep the above notation for $A = kQ/I$ and A^v.*

(a) Suppose v is a sink and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.

(b) Suppose v is a source and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.

(c) Suppose v is a sink but for loops (that is, all arrows ending in v are loops at v) and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.

(d) Suppose v is a source but for loops (that is, all arrows leaving v are loops at v) and L is a simple B-module. Then L has non-vanishing self-

extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.

PROOF. The first two statements have already been shown. In case (c), the two-sided ideal $J = Ae_vA = Ae_v$ is again projective, as a left A-module, and the same proof works as for (a).

Denote by A^{op} the opposite algebra of A and by L' the simple A^{op}-module corresponding to L under the duality $\mathrm{Hom}_k(-, k)$. Using the isomorphism $\mathrm{Ext}^n_A(L, L) \simeq \mathrm{Ext}^n_{A^{op}}(L', L')$ and noting that v is a sink in the quiver of A^{op}, claim (d) follows from (c). ∎

We remark that in the situation of the Proposition stronger statements are true. In parts (a), (b) and (c) it is true that $\mathrm{Ext}^n_A(X, Y) \simeq \mathrm{Ext}^n_B(X, Y)$ for all n and all B-modules X and Y. Since J in these three cases is projective, we can also say that the projective dimension of a B-module X is finite if and only if X has finite projective dimension over A.

The proposition does, however, not give any information about the cohomology between a simple B-module S_1 and a simple A-module S_2, which is not defined over B. Therefore, in part (d) it also does not relate the projective (or in part (c) the injective) dimension of S_1 over B to the same dimension over A. The following example shows that these dimensions can be rather different.

Example 8.5.3. Let k be a field and $B := k[x]/(x^2)$. Let $A = \begin{pmatrix} B & k \\ 0 & k \end{pmatrix}$ (where multiplication uses the action of C on its simple quotient k). Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and let v be the vertex associated with e; v is a source but for loops. Here, $A^v = k$. But over A, the simple A^v-module S has infinite projective dimension.

8.5.2 Removing arrows, reducing cohomology

As before, we are given an algebra $A = kQ/I$ by quiver and relations. We are using split quotients to remove arrows from the quiver Q.

Let α be an arrow in Q. We consider the quiver $Q^\alpha = (\Delta_0, \Delta_1 \setminus \{\alpha\})$ obtained from Q by removing α. In the path algebra kQ^α we consider the ideal $I^\alpha = \{r \in I \mid \text{no summand of } r \text{ has } \alpha \text{ as a subpath }\}$. Let us call A^α the algebra kQ^α/I^α. For every A-module M associated to the representation $(V_i, \varphi_\beta)_{i\in\Delta_0,\beta\in\Delta_1}$ of the quiver Q which respects the relations in $I = (R)$, we can consider the representation of the quiver Q^α given by $(V_i, \varphi_\beta)_{i\in\Delta_0^\alpha,\beta\in\Delta_1^\alpha}$. Obviously this representation respects the relations in I^α and therefore it is associated to an A^α module M^α. The map $M \to M^\alpha$ (extended to morphisms in the obvious way) defines a functor between the abelian categories A-mod and A^α-mod. We will denote it by F^α. This is an exact functor.

Proposition 8.5.4. *Keep the above notations.*

(a) The algebra A^α is isomorphic to the subalgebra of A, which is generated by the set $\{p + I \in A = kQ/I \mid p$ a path in Q and α not a subpath of $p\}$.

(b) Assume that the arrow α is involved only in monomial relations. Then $A/A\alpha A \cong A^\alpha$ and the surjective homomorphism $A \to A^\alpha$ induces a full embedding $G\colon A^\alpha$-mod $\to A$-mod. Then (F^α, G) is an exact split pair of functors.

Note that we may remove more than one such arrow at a time, since the composition of split quotients is again a split quotient.

Proof. In the path algebra kQ, the two-sided ideal $kQ \cdot \alpha \cdot kQ$ is generated (over k) by all paths going through α. The subalgebra kQ^α in whose quiver α is missing, is a split quotient of kQ via the projection $kQ \to kQ/(kQ \cdot \alpha \cdot kQ)$.

We can write $I^\alpha = I \cap kQ^\alpha$. Indeed, I^α is contained in the right hand side by definition. Conversely, an element $r \in I - I^\alpha$ has a summand, which has the arrow α as a subpath. Hence $r \notin kQ^\alpha$.

The subalgebra A^α of A is a quotient of kQ^α. In fact, A^α is the image of kQ^α under the projection from kQ to A. The relation ideal of A^α is $I \cap kQ^\alpha = I^\alpha$.

If α is involved only in monomial relations, then any of the generating relations r is either a path containing α or a linear combination of paths, none of which contains α. Hence we can decompose the relation ideal I into a direct sum $I = I^\alpha \oplus ((kQ)\alpha(kQ) \cap I)$, where I^α as above is k-generated by all relations not involving α. Then $A^\alpha \simeq kQ^\alpha/I^\alpha \simeq (kQ^\alpha \oplus (kQ \cdot \alpha \cdot kQ))/(I^\alpha \oplus (kQ \cdot \alpha \cdot kQ)) \simeq (kQ^\alpha \oplus (kQ \cdot \alpha \cdot kQ))/(I + (kQ \cdot \alpha \cdot kQ)) \simeq A/A^\alpha$. ∎

Note that if α is involved in non-monomial relations, then the algebra A^α exists, but it may not be isomorphic to a quotient of A any more. The functor F^α as explicitly constructed is still an exact functor between the categories A-mod and A^α-mod. It may, however, lack a right inverse and it may not be surjective on the morphisms.

8.5.3 Removing vertices and reducing cohomology

In order to remove vertices, which are neither sinks nor sources (and all arrows and loops attached to these vertices), we will use corner rings.

Let $\Delta_0 = \{1, 2, \ldots, i, \ldots, n\}$ be the set of vertices of Q, let e_j, for every j in Δ_0, be the trivial (idempotent) path starting and ending at the vertex j

and let $e = e_1 + e_2 + \ldots + e_{i-1} + e_{i+1} + \ldots + e_n = 1 - e_i$.

Let Q' be the quiver with vertices $\Delta'_0 = \Delta_0 \setminus \{i\}$ and arrows $\Delta'_1 = \{\alpha \in \Delta_1 \mid e(\alpha) \neq i \neq s(\alpha)\} \cup \{p = \alpha_n \ldots \alpha_1$ path in $Q \mid s(\alpha_1) \neq i \neq e(\alpha_n) \wedge s(\alpha_n) = \ldots = s(\alpha_2) = i = e(\alpha_{n-1}) = \ldots = e(\alpha_1)\}$. Thus $kQ' = ekQe$.

Given R a set of relations over kQ such that $A = kQ/\langle R\rangle$, we define a set of relations R' such that $\langle R\rangle = \langle R'\rangle \cap kQ'$: Write $R = R_0 \cup R_e \cup R_s \cup R_{se}$ where R_0 is the set of the relations in R neither starting nor ending at the vertex i, R_s is the set of the relations starting, but not ending, at i, R_e is the set of the relations ending, but not starting, at i, and R_{se} is the set of the relations both starting and ending at the vertex i. Consider the set of relations $R' = R'_0 \cup R'_e \cup R'_s \cup R'_{se}$ on the path algebra kQ', where $R'_0 = R_0$, $R'_s = \{\alpha r \mid r \in R_s, \alpha$ arrow $e(\alpha) = i, s(\alpha) \neq i\}$, $R'_e = \{r\alpha \mid r \in R_e, \alpha$ arrow $, s(\alpha) = i, e(\alpha) \neq i\}$, $R'_{se} = \{\alpha r \beta \mid \alpha, \beta$ arrows $, r \in R_{se}, e(\alpha) = i = s(\beta)\}$. We are stretching the notation since the path $p = \alpha_n \ldots \alpha_1$ has length $\ell > 1$ when considered as an element of R while it can be an arrow when considered as an element of R'; thus some of the new relations may not be admissible (that is, they may involve paths of length one).

Proposition 8.5.5. *Keep the above notation.*

(a) There is an algebra isomorphism $A' = kQ'/ < R' > \cong eAe$.

(b) Suppose $R_s e = \emptyset$*, that is, no relation ending in a vertex* $j \neq i$ *is starting at* i*. Then* A *and* eAe *are related by an exact split pair (with bimodule* Ae*).*

(c) Suppose $eR_e = \emptyset$*, that is, no relation starting at a vertex* $j \neq i$ *is ending at* i*. Then* A *and* eAe *are related by an exact split pair (with bimodule* eA*) for their categories of right modules.*

PROOF. Statement (a) is clear by construction.

For claim (b) we need to show that Ae is projective over B. Decompose the right B-module Ae into a projective summand eAe and another summand e_iAe. Then $e_iAe = \oplus_{j\neq i} e_iAe_j$ is k-generated by all paths starting at i and ending in vertices different from i. Fix a vertex $j \neq i$ and let $p_1, \ldots p_l$ be the shortest possible paths from i to j; that is, they form a generating set of non-zero paths from i to j, which consist only of loops at i composed with arrows from i to j. Then multiplying with $(p_1, \ldots, p_l)$ is an injective B-module map from $(e_jAe)^l$ into e_iAe, since there is no relation involving any of the $p_1, \ldots, p_l$. Varying j produces disjoint images under these maps and adding them all up shows e_iAe is projective as a B-module.

(c) follows from (b) by considering opposite algebras. ∎

Setting $B = A'$, we can describe the functor $F = {}_BeA_A \underset{A}{\otimes} -$: A-

mod$\rightarrow$ B-mod explicitly in terms of representations. Given a representation $V = (V_j, \varphi_\alpha)_{j\in\Delta_0,\alpha\in\Delta_1}$ of the quiver Q over the field k respecting the relations in R (that is, given an A-module), we get $F(V) = (V_j, {\varphi'}_\alpha)_{j\in{\Delta'}_0,\alpha\in{\Delta'}_1}$ where ${\varphi'}_\alpha = \varphi_\alpha$ if $\alpha \in \Delta_1$, $e(\alpha) \neq i \neq s(\alpha)$ and $\varphi_\alpha = \varphi_{\alpha_n} \circ \ldots \circ \varphi_{\alpha_1}$ if $\alpha = \alpha_n \ldots \alpha_1$ with $s(\alpha_1) \neq i \neq e(\alpha_n) \wedge s(\alpha_n) = \ldots = s(\alpha_2) = i = e(\alpha_{n-1}) = \ldots = e(\alpha_1)$.

8.5.4 Removing parts of the quiver, reducing cohomology

Combining the previous methods allows to cut larger parts of the quiver:

Proposition 8.5.6. *Given $A = kQ/I$ with $I = \langle R\rangle$ and e an idempotent. Let $A' = eAe$. Denote the vertices involved in e by Δ_1 and the others by Δ_2.*

(a) Suppose no relation ending at vertices in Δ_1 is starting in a vertex in Δ_2. Then A and $A' = eAe$ are related by an exact split pair.

(b) Suppose the arrows from Δ_2 to Δ_1 are involved in monomial relations only. Then A and $A' = eAe$ are related by a sequence of exact split pairs (from A to a split quotient $\tilde{A}$ and then from $\tilde{A}$ to A').

Both claims have right module analogues as well.

Proof. (a) has the same proof as part (b) of Proposition 8.5.5.

(b) First we apply Proposition 8.5.4 to remove all arrows from Δ_2 to Δ_1. This can be done by just one split quotient (which is the composition of the split quotients used to remove one such arrow at a time). Afterwards we can apply (a). ∎

The relations of A' can be described in an analogous way as in the case of e being primitive.

8.5.5 Some cases of the Strong No Loops Conjecture

The reduction methods developed so far can be used to prove the strong no loops conjecture for certain classes of algebras.

The strong no loops conjecture (SNLC) states the following: Let A be an artinian algebra and S a simple A-module. Suppose, $Ext^1_A(S, S) \neq 0$. Then S has infinite projective dimension. (This is open problem (7) in the list of open problems in [ARS97].)

An algebra A whose underlying quiver has a loop at a vertex v, has infinite global dimension. This statement, the 'no loops conjecture' (proved by Igusa [Igu90] and implicitly by Lenzing [Len69]) means that at least one of the simple A-modules has infinite projective dimension; the strong no loops conjecture says the simple module associated to the vertex v has infinite projective dimension.

Igusa proved in [Igu90] that the strong no loops conjecture holds for monomial algebras, i.e. for finite dimensional algebras which are quotients of a path algebra kQ of a quiver Q over a field k modulo a relation ideal which is generated by a set of paths.

We first list two classes of algebras, for which SNLC is true by the methods from subsection 8.5.1; these algebras may serve as input for part (a) of Theorem 8.5.8.

We call an algebra A *quasi-directed*, if its primitive idempotents can be ordered in such a way that $Hom_A(Ae_i, Ae_j) \neq 0$ implies $i \leq j$. (Note that there is no condition on endomorphisms of indecomposable projective modules.) Such an algebra has no oriented cycles except possibly loops.

We use the definition of *standardly stratified* algebras given in [ADL98]. (Note that the term 'stratified algebra' is not completely unified in the literature.)

Proposition 8.5.7. *(a) SNLC holds true for local algebras.*

(b) SNLC holds true for quasi-directed algebras.

(c) SNLC holds true for standardly stratified algebras.

Proof. A local algebra is either simple or of infinite global dimension, hence (a).

In the cases (b) and (c), we can inductively apply the methods of subsection 8.5.1. In case (b) we use Proposition 8.5.2. Part (c) follows from a fundamental property of standardly stratified algebras, which itself is a consequence of Theorem 8.5.1: The derived category of a standardly stratified algebra A has a stratification (a sequence of recollements) by derived categories of local algebras, and the functors involved identify the simple A-modules, and their self-extensions, with simple modules, and their self-extensions, over these local algebras. ∎

The main result in this Section generalizes Igusa's result for monomial algebras. It implies the validity of SNLC for relatively large classes of algebras, and at the same time it constructs more algebras with SNLC from known ones. We call an algebra A a *monomial union of corner rings* e_iAe_i if the following conditions are satisfied:

1. the idempotents e_i are pairwise disjoint and orthogonal and they add up to the unit of A;

2. the arrows from vertices in e_i to vertices in e_j, for any $i < j$, are involved only in monomial relations.

We call an algebra *quasi-monomial* if its arrows except possibly the loops are involved in non-monomial relations only.

Theorem 8.5.8 (Diracca and Koenig).
(a) Let A be a monomial union of corner rings e_iAe_i and suppose SNLC holds true for each e_iAe_i. Then SNLC holds true for A itself.
(b) SNLC holds true for monomial algebras.
(c) SNLC holds true for quasi-monomial algebras.

Proof. Part (a) is proved by inductively applying Proposition 8.5.6. Parts (b) and (c) are special cases of (a). ∎

While the previous results describe globally defined classes of algebras, the next result is local, allowing to single out certain loops.

Proposition 8.5.9. *In $A = kQ/I$ choose an idempotent e. Assume that e is not involved in oriented cycles outside of e, that is, assume that the multiplication map $eA(1-e) \otimes_k (1-e)Ae \to eAe$ is zero. Then there is a split pair relating A and eAe.*
In particular, if e is primitive and its simple module L satisfies $Ext^1_A(L, L) \neq 0$, then L has infinite projective dimension, that is, it satisfies SNLC.

Proof. We set up a split corner quotient situation with $B = eAe$. Setting $S = Ae$ does not work in general. However, the assumption implies that $(1-e)Ae$ is an A-submodule of Ae; in fact, $eA(1-e)Ae = 0$ implies $A(1-e)Ae \subset (1-e)Ae$. Thus $S = Ae/(1-e)Ae$ equals eS and it is a left A-module, which as right and left eAe-module is just $B = eAe$ itself. ∎

This can be generalized further: For simplicity we give only a statement for a primitive idempotent e.

Theorem 8.5.10 (Diracca and Koenig).
In $A = kQ/I$ choose a primitive idempotent e, corresponding to the simple module S. Denote the loops at e by $\alpha_1, \ldots, \alpha_l$ ($l \geq 1$)and the oriented cycles at e, which are not loops, by $\beta_1, \ldots, \beta_m$ ($m \geq 0$). Suppose for some $p \geq 1$ the first p loops, $\alpha_1, \ldots, \alpha_p$ are not involved in relations of the form $a = b$, where a is a linear combination of products of these p loops and b involves also loops $\alpha_{p+1}, \ldots, \alpha_l$ or cycles $\beta_1, \ldots, \beta_m$. Then SNLC is true for the simple module S.

PROOF. Let B be the subalgebra of A generated by the loops $\alpha_1, \dots, \alpha_p$. The assumption guarantees that B is a split quotient of eAe. Let X be the A-submodule of Ae, which is generated by the trace of all Af with f not equivalent to e. Let Y be the A-submodule of Ae, which is generated by the loops $\alpha_{p+1}, \dots, \alpha_l$. Let $S = Ae/(X+Y)$. By definition, Ae is generated by all paths ending at e. The quotient Ae/X is generated by all paths ending at e and not going through any vertex different from e, that is, by all loops at e. Thus S is generated by all paths, which are just products of the loops $\alpha_1, \dots, \alpha_p$. Hence $S = eS$ and as a left and right B-module it is isomorphic to B.

■

Bibliography

[AB69] M. Auslander and M. Bridger, *Stable module theory*, American Mathematical Society, Providence, R.I., 1969.

[ADL98] I. Agoston, V. Dlab, and E. Lukacs, *Stratified algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **20** (1998), 22–28.

[AF92] F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, second ed., Springer-Verlag, 1992.

[AGOP98] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105–137.

[Arn82] David M. Arnold, *Finite rank torsion free abelian groups and rings*, Lecture Notes in Mathematics, vol. 931, Springer-Verlag, Berlin-New York, 1982.

[ARS97] M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge University Press, Cambridge, 1997.

[BD68] I. Bucur and A. Deleanu, *Introduction to the theory of categories and functors*, Pure and Applied Mathematics, vol. XIX, Interscience Publishers, New York-London, 1968.

[Bel00] A. Beligiannis, *Cleft extensions of abelian categories and applications to ring theory*, Comm. Algebra **28** (2000), 4503–4546.

[Bro98] G. Brookfield, *Direct sum cancellation of noetherian modules*, J. Algebra **200** (1998), 207–224.

[Bro00] ———, *The Grothendieck group and the extensional structure of noetherian module categories*, Algebra and its applications (Athens, OH, 1999), Contemp. Math. 259, Amer. Math. Soc., Providence, R.I., 2000, pp. 111–131.

[Bro02] ———, *A Krull-Schmidt theorem for noetherian modules*, J. Algebra **251** (2002), 70–79.

[CPS88] E. Cline, B. Parshall, and L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[CR62] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, vol. XI, Interscience Publishers, New York-London, 1962.

[DF97] N.V. Dung and A. Facchini, *Weak Krull-Schmidt for infinite direct sums of uniserial modules*, J. Algebra **193** (1997), 102–121.

[DF02] L. Diracca and A. Facchini, *Uniqueness of monogeny classes for uniform objects in abelian categories*, J. Pure Appl. Algebra **172** (2002), 183–191.

[DF04] ———, *Descending chains of modules and Jordan-Hölder Theorem*, Semigroup Forum **68** (2004), 373–399.

[dlRF86] B. de la Rosa and L. Fuchs, *On h-Divisible torsion modules over domains*, Comment. Math. Univ. St. Paul. **35** (1986), 53–57.

[DX95] B. M. Deng and C. C. Xi, *Quasi-hereditary algebras which are twisted double incidence algebras of posets*, Beitraege Algebra Geom. **36** (1995), 37–71.

[EH60] B. Eckmann and P. J. Hilton, *Homotopy groups of maps and exact sequences*, Comment. Math. Helv. **34** (1960), 271–304.

[Fac96] A. Facchini, *Krull-Schmidt fails for serial modules*, Trans. Amer. Math. Soc. **348** (1996), 4561–4575.

[Fac98] ———, *Module Theory. endomorphism rings and direct sum decompositions in some classes of modules*, Birkhäuser Verlag, 1998.

[Fac02] ———, *Direct sum decompositions of modules, semilocal endomorphism rings, and Krull monoids*, J. Algebra **256** (2002), 280–307.

[Fai73] C. Faith, *Algebra I: rings, modules and categories*, Grundlehren der Mathematischen Wissenschaften, vol. 190, Springer-Verlag, Berlin-New York, 1973.

[FGR75] R. M. Fossum, P. A. Griffith, and I. Reiten, *Trivial extensions of abelian categories*, Springer-Verlag, Berlin-New York, 1975.

[FH05] A. Facchini and D. Herbera, *Local morphisms and modules with a semilocal endomorphism ring*, 2005.

[FHK03] A. Facchini and F. Halter-Koch, *Projective modules and divisor homomorphisms*, J. Algebra Appl. **2** (2003), 435–449.

[FHLV95] A. Facchini, D. Herbera, L. S. Levy, and P. Vamos, *Krull-Schmidt fails for artinian modules*, Proc. Amer. Math. Soc. **123** (1995), 3587–3592.

[FS01] L. Fuchs and L. Salce, *Modules over non-noetherian domains*, American Mathematical Society, Providence, 2001.

[Fuc73] L. Fuchs, *Infinite abelian groups*, vol. 2, Academic Press, New York, 1973.

[GJ89] K. R. Goodearl and R. B. Warfield Jr., *An introduction to noncommutative noetherian rings*, London Mathematical Society Student Texts, vol. 16, Cambridge University Press, Cambdridge, 1989.

[GL96] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[Goo91] K. R. Goodearl, *Von Neumann regular rings*, second ed., Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991.

[Gur81] R. M. Guralnick, *Roth's theorems and decomposition of modules*, Linear Algebra Appl. **39** (1981), 155–165.

[HK98] F. Halter-Koch, *Module theory. Ideal systems. An introduction to multiplicative ideal theory.*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 211, Marcel Dekker, Inc., New York, 1998.

[HP05] R. Hartmann and R. Paget, *Young modules and filtration multiplicities for brauer algebras*, 2005.

[HR61] P. J. Hilton and D. Rees, *Natural maps of extension functors and a theorem of r. g. swan*, Proc. Cambridge Philos. Soc. **57** (1961), 489–502.

[Hug60] N. J. S. Hughes, *The Jordan-Hölder-schreier theorem for general algebraic systems*, Compositio Math. **14** (1960), 228–236.

[Igu90] K. Igusa, *Notes on the No Loops Conjecture*, J. Pure Appl. Algebra **69** (1990), 161–176.

[Jón59] B. Jónsson, *On direct decomposition of torsion-free abelian groups*, Math. Scand. **7** (1959), 361–371.

[Kel96] B. Keller, *Derived categories and their uses*, Handbook of algebra, Vol. 1 (M. Hazewinkel, ed.), Elsevier, 1996.

[Kel98] ______, *Introduction to abelian and derived categories*, Representations of Reductive Groups (R. W. Carter and M. Geck, eds.), Cambridge University Press, 1998, pp. 41–62.

[KX99] S. Koenig and C. C. Xi, *When is a cellular algebra quasi-hereditary?*, Math. Ann. **315** (1999), 281–293.

[KX01] ______, *A characteristic free approach to Brauer algebras*, Trans. Amer. Math. Soc. **353** (2001), 1489–1505.

[Lad74] E. L. Lady, *Summands of finite rank torsion free abelian groups*, J. Algebra **32** (1974), 51–52.

[Len69] H. Lenzing, *Nilpotente elemente in ringen von endlicher globaler dimension*, Math. Z. **108** (1969), 313–324.

[Mat73] E. Matlis, *1-dimensional Cohen-Macaulay rings*, Lecture Notes in Math., vol. 327, Springer-Verlag, Berlin, 1973.

[Mit72] B. Mitchell, *Rings with several objects*, Advances in Math. **8** (1972), 1–161.

[Miy67] T. Miyata, *Note on direct summands of modules*, J. Math. Kyoto Univ. **7** (1967), 65–69.

[Pří05] P. Příhoda, *A version of the weak Krull-Schmidt theorem for infinite direct sums of uniserial modules*, 2005.

[Pop73] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, 1973.

[Pun01a] G. Puninsky, *Some model theory over a nearly simple uniserial domain and decomposition of serial modules*, J. Pure Appl. Algebra **163** (2001), 319–337.

[Pun01b] ______, *Some model theory over an exceptional uniserial ring and decompositions of serial modules*, J. London Math. Soc. **64** (2001), 311–326.

[Rot63] J. Rotman, *The Grothendieck group of torsion-free abelian groups of finite rank*, Proc. London Math. Soc. **13** (1963), 724–732.

[Rot79] J. J. Rotman, *An introduction to homological algebra*, Academic Press, Inc., New York-London, 1979.

[Row86] L.H. Rowen, *Finitely presented modules over semiperfect rings*, Proc. Amer. Math. Soc. **97** (1986), 1–7.

[Ste75] B. Stenström, *Rings of quotients*, Grundlehren der math. Wissenschaften, vol. 217, Springer-Verlag, Berlin, 1975.

[Wal64] E. A. Walker, *Quotient categories and quasi-isomorphisms of abelian groups*, Proc. Colloq. Abelian Groups, Tihany (Hungary), 1963, Akadémiai Kiadó, Budapest, 1964, pp. 147–162.

[War75] R. B. Warfield, *Serial rings and finitely presented modules*, J. Algebra **37** (1975), 187–222.

[Wes01] D. B. West, *Introduction to graph theory*, second ed., Prentice Hall, Upper Saddle River, NJ 07458, 2001.

[Wie01] R. Wiegand, *Direct-sum decompositions over local rings*, J. Algebra **240** (2001), 83–97.

[Xi95] C. C. Xi, *Global dimensions of dual extension algebras*, Manuscripta Math. **88** (1995), 25–31.

[Yak00] A. V. Yakovlev, *On direct decompositions of p-adic groups*, Algebra i Analiz **12** (2000), 217–223.

[Zan88] P. Zanardo, *Quasi-isomorfisms of finitely generated modules over valuation domains*, Annali Mat. Pura Appl. (4) **151** (1988), 109–123.