Network Formulations of Mixed-Integer Programs*

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Abstract

We consider mixed-integer sets of the type $MIX^{TU} = \{x : Ax \geq b;\ x_i \text{ integer}, i \in I\}$, where A is a totally unimodular matrix, b is an arbitrary vector and I is a nonempty subset of the column indices of A. We show that the problem of checking nonemptiness of a set MIX^{TU} is NP-complete even in the case in which the system describes mixed-integer network flows with half-integral requirements on the nodes.

This is in contrast to the case where A is totally unimodular and contains at most two nonzeros per row. Denoting such mixed-integer sets by MIX^{2TU}, we provide an extended formulation for the convex hull of MIX^{2TU} whose constraint matrix is a dual network matrix with an integral right-hand-side vector. The size of this formulation depends on the number of distinct fractional parts taken by the continuous variables in the extreme points of $\text{conv}(MIX^{2TU})$. When this number is polynomial in the dimension of the matrix A, the extended formulation is of polynomial size. If, in addition, the corresponding list of fractional parts can be computed efficiently, then our result provides a polynomial algorithm for the optimization problem over MIX^{2TU}. We show that there are instances for which this list is of exponential size, and we also give conditions under which it is short and can be efficiently computed.

Finally we show that these results for the set MIX^{2TU} provide a unified framework leading to polynomial-size extended formulations for several generalizations of mixing sets and lot-sizing sets studied in the last few years.

1 Introduction

We study mixed-integer sets of the type

$$MIX^{TU} = \{x : Ax \geq b;\ x_i \text{ integer}, i \in I\},$$

where A is a totally unimodular (TU, for short) matrix, b is a vector that typically contains fractional components and I is a nonempty subset of the column indices of A. (Definitions

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and properties of TU matrices can be found in [14].) The study of such sets forms part of a recent effort to study and understand *simple mixed-integer* sets. In particular MIX^{TU} includes as a special case the mixing set

$$MIX = \{(s, y) \in \mathbb{R}_+ \times \mathbb{Z}^n : s + y_j \geq b_j,\ j = 1, \dots, n\},$$

as well as the more general mixed-integer edge-covering problem

$$BIP(I) = \{x \in \mathbb{R}^V_+ : x_i + x_j \geq b_{ij},\ ij \in E;\ x_i \text{ integer},\ i \in I\}$$

on a bipartite graph $G = (V, E)$ with $I \subseteq V$.

More specifically the motivation for this research was in part to understand the full scope of the proof technique based on fractionalities used in studying the mixing set and its generalizations [2, 3, 5, 18, 9, 13, 16], and also to see whether TU matrices play as important a role in mixed-integer programming as they do in pure integer programming.

From the complexity and polyhedral point of view our results are of two types. On the one hand we show that checking nonemptiness of MIX^{TU} is NP-complete, even if the system is of the form $\{x : Ax = b;\ x \geq 0;\ x_i \text{ integer},\ i \in I\}$ with A being the node-arc incidence matrix of a directed graph and b is half-integral, i.e., $2b$ is integral. This shows that finding an explicit inequality description of the polyhedron $\text{conv}(MIX^{TU})$ will most likely be an elusive task, where $\text{conv}(MIX^{TU})$ denotes the convex hull of the set MIX^{TU}.

On the other hand, if MIX^{2TU} denotes the mixed-integer set MIX^{TU} with the additional restriction that A is a TU matrix with at most two nonzero entries per row, which includes the sets MIX and $BIP(I)$, we derive an *extended formulation* for the convex hull of MIX^{2TU}, i.e. an inequality description of a polyhedron $Q = \{(x, \mu) : Ax + B\mu \geq d\}$ in a space that uses variables (x, μ) and includes the original x-space, so that $\text{conv}(MIX^{2TU})$ is the projection of Q onto the x-space.

The extended formulation of the polyhedron $\text{conv}(MIX^{2TU})$ takes explicitly into account all possible fractional parts taken by the continuous variables at the vertices of $\text{conv}(MIX^{2TU})$. If the number of these fractional parts is small, this extended formulation is *compact* (of polynomial size in n, A, b). In such cases optimizing a linear function over sets MIX^{2TU} that have this property can be carried out efficiently through linear programming if the set of fractional parts is known. On the one hand we give conditions, including many interesting cases, in which this formulation is compact and the fractional parts can be computed. On the other, we show that for sets such as $BIP(I)$ the size of our polyhedral description can be exponential.

Also using invertible linear transformations that map mixed-integer vectors into mixed-integer vectors, we show that a host of mixed-integer sets that have been investigated in the past decade can be mapped with these transformations into sets of the type MIX^{2TU} with a small number of fractional parts taken by the continuous variables. Therefore our result provides a general setting for the compact extended formulations of all these mixed-integer sets.

The outline of the paper is as follows. In Section 2 we remind the reader of some basic results on network matrices. In Section 3 we show that deciding the nonemptiness of sets MIX^{TU} is NP-complete. In Sections 4 and 5 we derive the extended formulation for $\text{conv}(MIX^{2TU})$. In Section 6 we show first that the length of the list of fractional values of all the extreme point solutions can be exponential, and then describe conditions under which the length is of polynomial size (compact). In Section 7 we give examples of mixed-integer sets that can be transformed into the form MIX^{2TU}, and we finish in Section 8 with some concluding remarks.

2 Dual network matrices

Our main result is an extended formulation of mixed-integer sets MIX^{2TU}, which are defined by totally unimodular matrices A having at most two nonzero entries per row. In this section, we review some basic terminology of networks and relate such TU matrices to transposes of node-arc incidence matrices of directed graphs.

The *node-arc incidence matrix* $C \in \{0, \pm 1\}^{V \times \mathcal{A}}$ of a directed graph $D = (V, \mathcal{A})$ has one row for each vertex and one column for each arc of D. The column representing the arc $e = uv$ is zero everywhere, except in the rows corresponding to u and v. These entries are $C(u, e) = 1$ and $C(v, e) = -1$ respectively. A *minimum cost flow* problem is a linear program of the form $\min\{c^T x : Cx = b,\ 0 \leq x \leq u\}$, where $C \in \{0, \pm 1\}^{m \times n}$ is a node-arc incidence matrix, c and u are in $\mathbb{R}^n$.

A $0, \pm 1$-matrix A with at most two nonzero entries per row is a *dual network matrix* if A has the following property:

$$\textit{If } a_{ij},\ a_{ik} \textit{ are both nonzero and } k \neq j, \textit{ then } a_{ij} = -a_{ik}.$$

Consequently a dual network matrix is the transpose of a node-arc incidence matrix plus some singleton rows with entries of $+1$ or -1, and such matrices arise as the constraint matrices of the linear-programming dual of a minimum cost flow problem without capacities. The problem of optimizing over such matrices is often referred to as the optimal *node-potential* or *node-label* assignment problem, see e.g. [1].

The following characterization is due to Heller and Tompkins [10], see e.g. Theorem 2.8 in [14]. Here $N = \{1, \dots, n\}$ and $\{a_j,\ j \in N\}$ denotes the set of columns of A.

Theorem 1 *Let A be a $0, \pm 1$-matrix with at most two nonzero entries per row. Then A is totally unimodular if and only if the set N can be partitioned into two sets R and B such that all entries of the vector $\sum_{j \in R} a_j - \sum_{j \in B} a_j$ are $0, \pm 1$.*

Here, we focus on mixed-integer systems defined by TU matrices which have at most two nonzero entries per row. The next corollary shows that we can restrict our attention to dual network matrices by substituting some variables x_i by $-x_i$.

Corollary 2 *Let A be a $0, \pm 1$-matrix with at most two nonzero entries per row. Then A is totally unimodular if and only if N contains a subset R such that the matrix $\widetilde{A}$, obtained from A by multiplying the columns $a_j, j \in R$ by -1, is a dual network matrix.*

Proof: Let (R, B) be a partition of the column indices of A satisfying the condition of Theorem 1. Then $\widetilde{A}$ is a dual network matrix. The converse is readily checked. □

3 Complexity of the feasibility problem for MIX^{TU}

In this section, we show that determining whether a mixed-integer set of the type MIX^{TU} is empty or not is NP-complete. More specifically, we show that feasibility of a half-integer version of the minimum cost flow problem is NP-complete.

Consider a mixed-integer set

$$\{x : Cx = b/2;\ x \geq 0;\ x_i \text{ integer},\ i \in I\}, \tag{1}$$

where C is the node-arc incidence matrix of a directed graph $D = (V, \mathcal{A})$ and $b \in \mathbb{Z}^V$ is an integral vector, which represents the requirements on the vertices of D. The set (1) is nonempty if and only if the following set is nonempty

$$\{x : Cx = b;\ x \in \mathbb{Z}_+;\ x_i \text{ even},\ i \in I\}. \tag{2}$$

This can be seen as follows. Let x be a vertex of the convex hull of (1). Then x_i is integer for each $i \in I$ and x_j is half-integer, i.e. $2x_j \in \mathbb{Z}$ for each $j \in N \setminus I$. Consequently $2 \cdot x$ is in the set (2). On the other hand, if x' is in the set (2), then $x'/2$ lies in the set (1). This discussion shows that we can decide whether the following *parity network flow* problem has a solution, given an algorithm which decides whether a mixed-integer set of type MIX^{TU} is feasible or not.

> (Parity network flow) Given a directed graph $D = (V, \mathcal{A})$, a subset $S \subseteq \mathcal{A}$ of the arcs and integral vectors $b \in \mathbb{Z}^V$, $u \in \mathbb{Z}^{\mathcal{A}}$, determine whether the network D with requirements b and capacities u has a feasible integral flow under the additional requirement that all flow-values of arcs in S are even.

Theorem 3 *The problem of deciding the nonemptiness of a mixed-integer set MIX^{TU} is NP-complete, even if the set has the form* (1).

Proof: The problem is clearly in NP. We reduce SAT to the parity network flow problem in a manner similar to that introduced by Even et al. in the proof that the edge-disjoint paths problem is NP-hard, see [7] and [12, p. 432].

Given a SAT formula over the variables $x_1, \dots, x_n$, consisting of clauses $Z_1, \dots, Z_m$, we construct an instance of the parity network flow problem as follows.

- The set V of nodes of D contains a source s, a sink t and a node z_j, $1 \leq j \leq m$, that represents the corresponding clause.
  Every variable x_i appearing in positive or negative form in clauses $Z_{i_1}, \dots, Z_{i_{p_i}}$ is represented by a "value node" v_i and nodes $x^{in}_{i,i_\ell}, x^{out}_{i,i_\ell}, \bar{x}^{in}_{i,i_\ell}, \bar{x}^{out}_{i,i_\ell}$, $1 \leq \ell \leq p_i$.
  Finally there is an additional value node v_{n+1}.
- The arcs of D that are not in the set S (unspecified capacities are unlimited) are:
  - The arcs sx^{in}_{i,i_ℓ}, $s\bar{x}^{in}_{i,i_\ell}$, $1 \leq i \leq n$, $1 \leq \ell \leq p_i$.
  - The arcs $z_j t$, $1 \leq j \leq m$, having capacity 1.
  - The arcs $x^{in}_{i,i_\ell} x^{out}_{i,i_\ell}$ and $\bar{x}^{in}_{i,i_\ell} \bar{x}^{out}_{i,i_\ell}$, $1 \leq i \leq n$, $1 \leq \ell \leq p_i$, having capacity 2.
  - If variable x_i occurs as a positive literal in clause Z_{i_ℓ}, there is an arc $x^{out}_{i,i_\ell} z_{i_\ell}$. If variable x_i occurs as a negative literal in clause Z_{i_ℓ}, there is an arc $\bar{x}^{out}_{i,i_\ell} z_{i_\ell}$.
- The following are the arcs in S and thus have to carry a flow of even value:
  - The arcs $v_i x^{in}_{i,i_1}$ and $v_i \bar{x}^{in}_{i,i_1}$, $1 \leq i \leq n$.
  - The arcs $x^{out}_{i,i_{p_i}} v_{i+1}$ and $\bar{x}^{out}_{i,i_{p_i}} v_{i+1}$, $1 \leq i \leq n$.

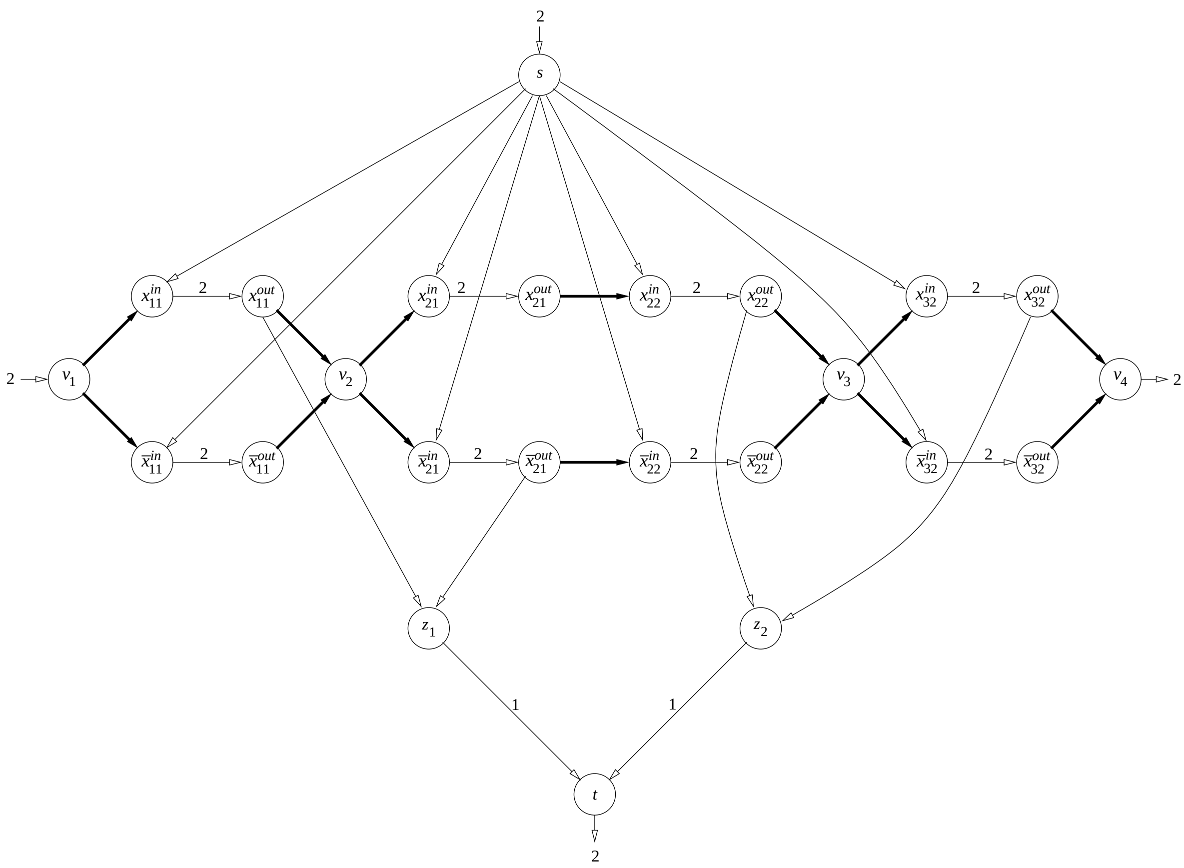

Figure 1: The network corresponding to the SAT formula $(x_1 \vee \bar{x}_2) \wedge (x_2 \vee x_3)$. Thick arcs are special arcs. Numbers on arcs are capacities.

  - The arcs $x^{out}_{i,i_\ell} x^{in}_{i,i_{\ell+1}}$ and $\bar{x}^{out}_{i,i_\ell} \bar{x}^{in}_{i,i_{\ell+1}}$, $1 \leq i \leq n$, $1 \leq \ell < p_i$.

- The requirements on the nodes are:
  - An in-flow of value m in the source s and an out-flow of value m in the sink t.
  - An in-flow of value 2 at v_1 and an out-flow of value 2 at v_{n+1}.

Figure 1 shows the network relative to the SAT formula $(x_1 \vee \bar{x}_2) \wedge (x_2 \vee x_3)$.

For $1 \leq i \leq n$, define the upper path P_i^U to be $v_i, x^{in}_{i,i_1}, x^{out}_{i,i_1}, \ldots, x^{in}_{i,i_{p_i}}, x^{out}_{i,i_{p_i}}, v_{i+1}$ and the lower path P_i^L to be $v_i, \bar{x}^{in}_{i,i_1}, \bar{x}^{out}_{i,i_1}, \ldots, \bar{x}^{in}_{i,i_{p_i}}, \bar{x}^{out}_{i,i_{p_i}}, v_{i+1}$.

Observe that the arcs in S force any feasible circulation F to satisfy the following conditions:

- For every $1 \leq i \leq n$, the arcs in S of one among P_i^U and P_i^L carry a flow of value 2, and the special arcs of the other carry a flow of value 0.

- For every $1 \leq j \leq m$, F carries a flow of value 1 along a path of the type $s, x^{in}_{i,j}, x^{out}_{i,j}, z_j, t$ and the upper path P_i^U is discharged (that is, its arcs in S carry a flow of value 0), or a flow of value 1 along a path of the type $s, \bar{x}^{in}_{i,j}, \bar{x}^{out}_{i,j}, z_j, t$ and the lower path P_i^L is discharged.

To any truth assignment T that satisfies the SAT-formula, we assign a flow value of 2 to P_i^L if $x_i = true$ in T and a flow value of 2 to P_i^U if $x_i = false$ in T. For each clause Z_j we choose any literal x_i or $\bar{x}_i$ which is true under T. Say x_i occurs as a positive literal in Z_j and $x_i = true$ in T. Then P_i^U is discharged and a flow of value 1 can be routed along $s, x_{i,j}^{in}, x_{i,j}^{out}, z_j, t$.

It is immediate to see that the converse also holds: to any feasible circulation, a truth assignment that satisfies all clauses can be derived in the above manner. □

4 The main result

Given a real number α, let $f(\alpha) = \alpha - \lfloor \alpha \rfloor$ denote its fractional part. Let $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ be an arbitrary list of decreasing fractional parts (that is, $1 > f_1 > \cdots > f_k \geq 0$), $K = \{1, \dots, k\}$ be its set of indices and $N = \{1, \dots, n\}$. Later we will make specific choices for these values. Let $X^{\mathcal{F}}$ be the set of points $x \in \mathbb{R}^n$ such that there exist μ^i, δ_ℓ^i, $i \in N, \ell \in K$, satisfying the following constraints:

$$
\begin{aligned}
x_i = \mu^i + \textstyle\sum_{\ell=1}^k f_\ell \delta_\ell^i, &\quad i \in N && (3)\\
\textstyle\sum_{\ell=1}^k \delta_\ell^i = 1,\ \delta_\ell^i \geq 0, &\quad i \in N, \ell \in K && (4)\\
x_i - x_j \geq l_{ij}, &\quad (i,j) \in N^e && (5)\\
x_i \geq l_i, &\quad i \in N^l && (6)\\
x_i \leq u_i, &\quad i \in N^u && (7)\\
\mu^i \text{ integer},\ \delta_\ell^i \text{ integer}, &\quad i \in N, \ell \in K, && (8)
\end{aligned}
$$

where $N^e \subseteq N \times N$ and $N^l, N^u \subseteq N$. In other words, $X^{\mathcal{F}}$ is the projection onto the x-space of the mixed-integer set (3)–(8). We remark that the above system may also include constraints of the type $x_i - x_j \leq u_{ij}$, as this inequality is equivalent to $x_j - x_i \geq l_{ij}$ for $l_{ij} = -u_{ij}$. In this section we give an extended formulation for the polyhedron $\mathrm{conv}(X^{\mathcal{F}})$.

Consider the following unimodular transformation:

$$\mu_0^i = \mu^i,\ \ \mu_\ell^i = \mu^i + \sum_{j=1}^{\ell} \delta_j^i,\ \ i \in N, \ell \in K. \tag{9}$$

Define $f_0 = 1$ and $f_{k+1} = 0$. For fixed $i \in N$, an equation in (3) becomes:

$$x_i = \sum_{\ell=0}^{k} \mu_\ell^i (f_\ell - f_{\ell+1}) \tag{10}$$

and the inequalities in (4) become:

$$\mu_k^i - \mu_0^i = 1,\ \ \mu_\ell^i - \mu_{\ell-1}^i \geq 0,\ \ \ell \in K. \tag{11}$$

4.1 Modeling $x_i \geq l_i$, $x_i \leq u_i$ and $x_i - x_j \geq l_{ij}$

Let ℓ_{l_i} be the highest index $\ell \in \{0, \dots, k\}$ such that $f_\ell \geq f(l_i)$. Now if x_i, δ_ℓ^i, μ_ℓ^i satisfy (3), (4), (8), (9), then $x_i \geq l_i$ if and only if

$$\mu_{\ell_{l_i}}^i \geq \lfloor l_i \rfloor + 1. \tag{12}$$

Similarly if ℓ_{u_i} is the highest index such that $f_\ell > f(u_i)$, then constraint $x_i \le u_i$ is satisfied if and only if

$$\mu^i_{\ell_{u_i}} \le \lfloor u_i \rfloor . \tag{13}$$

Now we consider the constraint $x_i - x_j \ge l_{ij}$. Define k_{ij} to be the highest index $\ell \in \{0, \ldots, k\}$ such that $f_\ell + f(l_{ij}) \ge 1$. Given an index $t \in K$, define k_t to be the highest index $\ell \in \{0, \ldots, k\}$ such that $f_\ell \ge f(f_t + f(l_{ij}))$.

Lemma 4 *Assume x_i, x_j, δ^i_ℓ, δ^j_ℓ, μ^i_ℓ, μ^j_ℓ satisfy (3), (4), (8), (9). Then $x_i - x_j \ge l_{ij}$ if and only if the following set of inequalities is satisfied:*

$$\mu^i_{k_t} - \mu^j_t \ge \lfloor l_{ij} \rfloor + 1, \quad 1 \le t \le k_{ij} \tag{14}$$

$$\mu^i_{k_t} - \mu^j_t \ge \lfloor l_{ij} \rfloor , \qquad k_{ij} < t \le k. \tag{15}$$

Proof: Substituting for x_i and x_j, the inequality $x_i - x_j \ge l_{ij}$ becomes

$$\mu^i + \sum_{\ell=1}^{k} f_\ell \delta^i_\ell \ge \mu^j + \sum_{\ell=1}^{k} f_\ell \delta^j_\ell + \lfloor l_{ij} \rfloor + f(l_{ij}).$$

First we show that the inequality (15) is valid for $t > k_{ij}$. As $\sum_{\ell > k_t} f_\ell \delta^i_\ell \le f_{k_t+1}$ and $\sum_{\ell=1}^{k} f_\ell \delta^j_\ell \ge \sum_{\ell \le t} f_\ell \delta^j_\ell \ge f_t \sum_{\ell \le t} \delta^j_\ell$, we obtain the valid inequality

$$\mu^i + \sum_{\ell \le k_t} f_\ell \delta^i_\ell \ge \mu^j + f_t \sum_{\ell \le t} \delta^j_\ell + \lfloor l_{ij} \rfloor + f(l_{ij}) - f_{k_t+1}.$$

Adding the valid inequality $(1 - f_t) \ge (1 - f_t) \sum_{\ell \le t} \delta^j_\ell$ gives

$$\mu^i + \sum_{\ell \le k_t} f_\ell \delta^i_\ell + 1 - f_t \ge \mu^j + \sum_{\ell \le t} \delta^j_\ell + \lfloor l_{ij} \rfloor + f(l_{ij}) - f_{k_t+1}.$$

By definition $f(l_{ij}) + f_t > f_{k_t+1}$. Since $\delta^i_\ell, \delta^j_\ell \ge 0$ for all $\ell \in K$, Chvátal-Gomory rounding gives

$$\mu^i + \sum_{\ell \le k_t} \delta^i_\ell \ge \mu^j + \sum_{\ell \le t} \delta^j_\ell + \lfloor l_{ij} \rfloor , \quad \text{or}$$

$$\mu^i_{k_t} \ge \mu^j_t + \lfloor l_{ij} \rfloor .$$

The argument when $t \le k_{ij}$ is the same, except that $f(l_{ij}) - f_{k_t+1} + f_t > 1$.

To establish the converse, we consider the case in which $\delta^j_t = 1$. Then $\mu^j_t = \mu^j_0 + 1$, $\mu^j_{t-1} = \mu^j_0$ and

$$x_j = \mu^j_0 + \sum_{\ell=1}^{k} (\mu^j_\ell - \mu^j_{\ell-1}) f_\ell = \mu^j_0 + f_t.$$

Inequality $\mu^i_{k_t} \ge \mu^j_t + \lfloor l_{ij} \rfloor$ implies that either $\mu^i_0 \ge \mu^j_0 + 1 + \lfloor l_{ij} \rfloor$ or that $\mu^i_0 = \mu^j_0 + \lfloor l_{ij} \rfloor$ and $\sum_{\ell \le k_t} \delta^i_\ell = 1$. This implies that $x_i \ge \mu^j_0 + \lfloor l_{ij} \rfloor + f_{k_t}$. Now, assuming $t > k_{ij}$,

$$\begin{aligned} x_i - x_j &\ge \mu^j_0 + \lfloor l_{ij} \rfloor + f_{k_t} - \mu^j_0 - f_t \\ &= \lfloor l_{ij} \rfloor + f_{k_t} - f_t \\ &\ge \lfloor l_{ij} \rfloor + f(l_{ij}), \end{aligned}$$

as $f_{k_t} \ge f(f_t + f(l_{ij}))$ and $f_t + f(l_{ij}) < 1$. Again the other case with $t \le k_{ij}$ is similar. □

4.2 Integrality of the extended formulation

Let $Q^{\mathcal{F}}$ be the polyhedron on the space of the variables $\{(x_i, \mu^i_\ell), i \in N, \ell \in K \cup \{0\}\}$ defined by the inequalities

$$\begin{array}{rl} (10), (11), & i \in N \\ (12), & i \in N^l \\ (13), & i \in N^u \\ (14), (15), & (i,j) \in N^e. \end{array}$$

Theorem 5 *The polyhedron* $\mathrm{conv}(X^{\mathcal{F}})$ *is the projection onto the space of the x-variables of the polyhedron $Q^{\mathcal{F}}$.*

Proof: Since, for $i \in N$, variable x_i is determined by the corresponding equation (10), we only need to show that the polyhedron defined by inequalities

$$\begin{array}{rl} (11), & i \in N \\ (12), & i \in N^l \\ (13), & i \in N^u \\ (14), (15), & (i,j) \in N^e. \end{array}$$

is integral. Let A_μ be the constraint matrix of the above system. By construction A_μ is a dual network matrix. Since dual network matrices are totally unimodular and the right-hand-sides of the above inequalities are all integer, the statement follows from the theorem of Hoffman and Kruskal [11]. □

5 An extended formulation for conv(MIX^{2TU})

Let $X = \{x : Ax \geq b;\ x_i \text{ integer},\ i \in I\}$ be a mixed-integer set, where $(A \mid b)$ is a rational matrix and I is a nonempty subset of the column indices of A. A list $\mathcal{F} = \{f_1 > f_2 > \cdots > f_k\}$ of fractional parts is *complete* for X if the following property is satisfied:

> Every minimal face F of $\mathrm{conv}(X)$ contains a point $\bar{x}$ such that for each $i \in N$, $f(\bar{x}_i) = f_j$ for some $f_j \in \mathcal{F}$ and for each $i \in I$, $f(\bar{x}_i) = 0$.

In our applications, minimal faces are vertices and the above condition becomes:

> If $\bar{x}$ is a vertex of $\mathrm{conv}(X)$, then for each $i \in N$, $f(\bar{x}_i) = f_j$ for some $f_j \in \mathcal{F}$.

Since I is nonempty, every complete list $\mathcal{F}$ must include the value 0, thus $f_k = 0$.

We now consider a mixed-integer set $MIX^{DN} = \{x : Ax \geq b;\ x_i \text{ integer},\ i \in I\}$, where A is a dual network matrix. That is, the system $Ax \geq b$ is constituted by inequalities of the type (5)–(7). We assume that we are given a list $\mathcal{F} = \{f_1 > f_2 > \cdots > f_k\}$ which is complete for MIX^{DN}. In order to obtain an extended formulation for $\mathrm{conv}(MIX^{DN})$, we consider the following mixed-integer set:

$$
\begin{aligned}
x_i = \mu^i + \textstyle\sum_{\ell=1}^k f_\ell \delta_\ell^i, &\quad i \in N &(16)\\
\textstyle\sum_{\ell=1}^k \delta_\ell^i = 1,\ \delta_\ell^i \geq 0, &\quad i \in N, \ell \in K &(17)\\
\delta_\ell^i = 0, &\quad i \in I,\ \ell \in K \setminus \{k\} &(18)\\
x_i - x_j \geq l_{ij}, &\quad (i,j) \in N^e &(19)\\
x_i \geq l_i, &\quad i \in N^l &(20)\\
x_i \leq u_i, &\quad i \in N^u &(21)\\
\mu^i \text{ integer},\ \delta_\ell^i \text{ integer}, &\quad i \in N, \ell \in K, &(22)
\end{aligned}
$$

where inequalities (19)–(21) constitute the system $Ax \geq b$.

Let $MIX^{\mathcal{F}}$ be the set of vectors x such that there exist μ^i, δ_ℓ^i, $i \in N, \ell \in K$ satisfying the above constraints. Note that equations (18) force variables $x_i, i \in I$ to be integer valued in $MIX^{\mathcal{F}}$.

Lemma 6 $\text{conv}(MIX^{DN}) = \text{conv}(MIX^{\mathcal{F}})$.

Proof: If $\bar{x} \in MIX^{\mathcal{F}}$ then $\bar{x}$ satisfies the system $Ax \geq b$ (i.e. the inequalities (19)–(21)). Furthermore equations (18) force $x_i, i \in I$ to be integer. So $\bar{x} \in MIX^{DN}$. This shows $MIX^{\mathcal{F}} \subseteq MIX^{DN}$ and therefore $\text{conv}(MIX^{\mathcal{F}}) \subseteq \text{conv}(MIX^{DN})$.

To prove the reverse inclusion, we show that all rays and minimal faces of $\text{conv}(MIX^{DN})$ belong to $\text{conv}(MIX^{\mathcal{F}})$. If $\bar{x}$ is a ray of $\text{conv}(MIX^{DN})$ then the vector defined by

$$x_i = \bar{x}_i,\ \ \mu_i = \bar{x}_i,\ \ \delta_\ell^i = 0,\ \ i \in N,\ \ell \in K$$

is a ray of the polyhedron which is the convex hull of the vectors satisfying (16)–(22). This implies that $\bar{x}$ is a ray of $\text{conv}(MIX^{\mathcal{F}})$.

Since the list $\mathcal{F}$ is complete, every minimal face F of $\text{conv}(MIX^{DN})$ contains a point $\bar{x} \in MIX^{\mathcal{F}}$. Furthermore F is an affine subspace which can be expressed as $\{x : x = \bar{x} + \sum_{t=1}^h \lambda_t r_t,\ \lambda_t \in \mathbb{R}\}$ for some subset of rays $r_1, \ldots, r_h$ of $\text{conv}(MIX^{DN})$. Since $\bar{x} \in MIX^{\mathcal{F}}$ and $r_1, \ldots, r_h$ are all rays of $\text{conv}(MIX^{\mathcal{F}})$, then $F \subseteq \text{conv}(MIX^{\mathcal{F}})$. □

Applying the unimodular transformation (9), inequalities (16)–(17) become inequalities (10)–(11), while inequalities (19)–(21) become inequalities (12)–(15). Let Q be the polyhedron on the space of the variables $\{(x_i, \mu_\ell^i), i \in N, \ell \in K \cup \{0\}\}$ defined by the inequalities (10)–(15) corresponding to inequalities (16), (17), (19), (20), (21) under transformation (9) and let Q^I be the face of Q defined by equations

$$\mu_\ell^i - \mu_{\ell-1}^i = 0,\ \ i \in I,\ \ell \in K \setminus \{k\},$$

which are equivalent to equations (18) under transformation (9).

Theorem 7 *The polyhedron* $\text{conv}(MIX^{DN})$ *is the projection onto the space of the x-variables of the face Q^I of Q.*

Proof: Theorem 5 shows that every minimal face of Q contains a vector $(\bar{x}, \bar{\mu})$ with integral $\bar{\mu}$. So the same holds for Q^I, which is a face of Q. By applying the transformation which is the inverse of (9), this shows that every minimal face of the polyhedron defined by (16)–(21) contains a point $(\bar{x}, \bar{\mu}, \bar{\delta})$ where $(\bar{\mu}, \bar{\delta})$ is integral. So the projection of this polyhedron onto the x-space coincides with $\text{conv}(MIX^{\mathcal{F}})$ and by Lemma 6 we are done. □

We now consider a mixed-integer set $MIX^{2TU} = \{x : Ax \geq b;\ x_i \text{ integer},\ i \in I\}$, where A is a TU matrix with at most two nonzero entries per row. By Corollary 2, A can be transformed into a dual network matrix by changing signs of some of its columns. Then MIX^{2TU} is transformed into a set of the type MIX^{DN}. Notice that if $\mathcal{F} = \{f_1 > \cdots > f_k\}$ is a list which is complete for MIX^{2TU}, then the list $\{0;\ f_\ell, 1 - f_\ell,\ 1 \leq \ell < k\}$ is complete for the transformed set MIX^{DN}. Theorem 7 has the following implication:

> If a mixed-integer set MIX^{2TU} admits a complete list $\mathcal{F}$ whose size is polynomial in the size of its description (given by the system $Ax \geq b$), the extended formulation of the corresponding set MIX^{DN} given by the inequalities that define Q^I is compact. Therefore the problem of optimizing a linear function over such sets MIX^{2TU} can be solved in polynomial time provided that the list of fractionalities can be efficiently computed.

This implies the next corollary, since in this case a complete list of fractional parts is $\{0, 1/D, \ldots, (D-1)/D\}$:

Corollary 8 *Let $A \in \{0, \pm 1\}$ be a totally unimodular matrix, which has at most two nonzero entries per row, $I \subseteq N$, $b \in \mathbb{Z}^m$, $D \in \mathbb{Z}_+ \setminus \{0\}$ and $c \in \mathbb{R}^n$. The mixed-integer optimization problem*

$$\max\{c^T x : Ax \geq (b/D);\ x_i \text{ integer},\ i \in I\}$$

can be solved in time polynomial in m, n and D.

6 On the length of a complete list

In this section we first show that a complete list of fractional parts can have exponential length for a set of the type MIX^{2TU} and then describe conditions ensuring that the list (and thus the extended formulation) is compact.

6.1 A non-compact example

We prove here the following result:

Theorem 9 *In the set of vertices of the polyhedron Q defined by the inequalities*

$$\sigma_i + r_j \geq \tfrac{3^{(j-1)n+i}}{3^{n^2+1}}, \quad i, j \in N \tag{23}$$

$$\sigma_i \geq 0,\ r_j \geq 0, \qquad i, j \in N \tag{24}$$

the number of distinct fractional parts taken by variable σ_n is exponential in n.

Observation 1 *Since the constraint matrix of inequalities* (23)–(24) *is a TU matrix with at most two nonzero entries per row, there exists a mixed-integer set M of the type MIX^{2TU} which is defined on continuous variables $\sigma_i, r_j, i, j \in N$ and integer variables $y_h, h \in I$ such that the polyhedron* $\mathrm{conv}(M) \cap \{(\sigma, r, y) : y_h = 0, h \in I\}$ *is a nonempty face of* $\mathrm{conv}(M)$ *described by inequalities* (23)–(24). *Therefore Theorem 9 shows that any extended formulation of* $\mathrm{conv}(M)$ *that explicitly takes into account a list of all possible fractional parts of the continuous variables will not be compact in the description of M.*

Now let b_{ij} be as in the theorem, i.e. $b_{ij} = \frac{3^{(j-1)n+i}}{3^{n^2+1}}$, $i, j \in N$.

Observation 2 *$b_{ij} < b_{i'j'}$ if and only if $(j,i) \prec (j',i')$, where $\prec$ denotes the lexicographic order. Thus $b_{11} < b_{21} \cdots < b_{n1} < b_{12} < \cdots < b_{nn}$.*

Lemma 10 *(i) Suppose that $\alpha \in \mathbb{Z}_+^q$ with $\alpha_t < \alpha_{t+1}$ for $1 \le t \le q-1$, and $\Phi(\alpha) = \sum_{t=1}^q (-1)^{q-t} 3^{\alpha_t}$. Then $\frac{3}{2}3^{\alpha_q} > \Phi(\alpha) > \frac{1}{2}3^{\alpha_q}$.*
(ii) Suppose that α is as above and $\beta \in \mathbb{Z}_+^{q'}$ is defined similarly. Then $\Phi(\alpha) = \Phi(\beta)$ if and only if $\alpha = \beta$.

Proof: (i) $\sum_{j=0}^{\alpha_q-1} 3^j = \frac{3^{\alpha_q}-1}{3-1} < \frac{1}{2}3^{\alpha_q}$. Now $\Phi(\alpha) \ge 3^{\alpha_q} - \sum_{j=0}^{\alpha_q-1} 3^j > 3^{\alpha_q} - \frac{1}{2}3^{\alpha_q} = \frac{1}{2}3^{\alpha_q}$, and $\Phi(\alpha) \le 3^{\alpha_q} + \sum_{j=0}^{\alpha_q-1} 3^j < 3^{\alpha_q} + \frac{1}{2}3^{\alpha_q} = \frac{3}{2}3^{\alpha_q}$.
(ii) Suppose $\alpha \ne \beta$. Wlog we assume $q \ge q'$. Assume first $(\alpha_{q-q'+1}, \dots, \alpha_q) = \beta$. Then $q > q'$ (otherwise $\alpha = \beta$) and, after defining $\bar\alpha = (\alpha_1, \dots, \alpha_{q-q'})$, we have $\Phi(\alpha) - \Phi(\beta) = \Phi(\bar\alpha) > 0$ by (i). Now assume $(\alpha_{q-q'+1}, \dots, \alpha_q) \ne \beta$. Define $h = \min\{\tau : \alpha_{q-\tau} \ne \beta_{q'-\tau}\}$ and suppose $\alpha_{q-h} > \beta_{q'-h}$ (the other case is similar). If we define the vectors $\bar\alpha = (\alpha_1, \dots, \alpha_{q-h})$ and $\bar\beta = (\beta_1, \dots, \beta_{q'-h})$, (i) gives $\Phi(\alpha) - \Phi(\beta) = \Phi(\bar\alpha) - \Phi(\bar\beta) > \frac{1}{2}3^{\alpha_{q-h}} - \frac{3}{2}3^{\beta_{q'-h}} \ge 0$, as $\alpha_{q-h} > \beta_{q'-h}$. □

We now give a construction of an exponential family of vertices of Q such that at each vertex variable σ_n takes a distinct fractional part. Therefore this construction proves Theorem 9.

Let $(i_1, \dots, i_m)$ and $(j_1, \dots, j_{m-1})$ be two increasing subsets of N with $i_1 = 1$ and $i_m = n$. For $i, j \in N$, let $p(i) = \max\{t : i_t \le i\}$ and $q(j) = \max\{t : j_t \le j\}$, with $q(j) = 0$ if $j < j_1$.

Consider the following system of equations:

$$\begin{aligned}
\sigma_{i_1} &= 0 \\
\sigma_{i_t} + r_{j_t} &= b_{i_t j_t}, && 1 \le t \le m-1 \\
\sigma_{i_{t+1}} + r_{j_t} &= b_{i_{t+1} j_t}, && 1 \le t \le m-1 \\
\sigma_{i_{q(j)+1}} + r_j &= b_{i_{q(j)+1} j}, && j \notin \{j_1, \dots, j_{m-1}\} \\
\sigma_i + r_{j_{p(i)}} &= b_{i j_{p(i)}}, && i \notin \{i_1, \dots, i_m\}.
\end{aligned}$$

The unique solution of this system is:

$$\begin{aligned}
\sigma_{i_1} &= 0 \\
\sigma_{i_t} &= \sum_{\ell=1}^{t-1} b_{i_{\ell+1} j_\ell} - \sum_{\ell=1}^{t-1} b_{i_\ell j_\ell}, && 2 \le t \le m \\
r_{j_t} &= \sum_{\ell=1}^{t} b_{i_\ell j_\ell} - \sum_{\ell=1}^{t-1} b_{i_{\ell+1} j_\ell}, && 1 \le t \le m-1 \\
\sigma_i &= b_{i j_{p(i)}} - r_{j_{p(i)}}, && i \notin \{i_1, \dots, i_m\} \\
r_j &= b_{i_{q(j)+1} j} - \sigma_{i_{q(j)+1}}, && j \notin \{j_1, \dots, j_{m-1}\}.
\end{aligned}$$

As each of these variables σ_i, r_j takes a value of the form $\Phi(\alpha)/3^{n^2+1}$, by Lemma 10 (i) we have that $\sigma_{i_t} > \frac{1}{2} b_{i_t j_{t-1}} > 0$ for $2 \le t \le m$, $r_{j_t} > \frac{1}{2} b_{i_t j_t} > 0$ for $1 \le t \le m-1$,

$\sigma_i > \frac{1}{2}b_{ij_{p(i)}} > 0$ for $i \notin \{i_1, \ldots, i_m\}$, and $r_j > \frac{1}{2}b_{i_{q(j)+1}j} > 0$ for $j \notin \{j_1, \ldots, j_{m-1}\}$. Therefore the nonnegativity constraints are satisfied.

Now we show that the other constraints are satisfied. Consider the i, j constraint with $j \notin \{j_1, \ldots, j_{m-1}\}$. We distinguish some cases.

1. $p(i) \le q(j)$. Then $\sigma_i + r_j \ge r_j > \frac{1}{2}b_{i_{q(j)+1}j} \ge \frac{1}{2}b_{i_{p(i)+1}j} \ge \frac{3}{2}b_{ij} > b_{ij}$.

2. $p(i) > q(j)$ and $i \notin \{i_1, \ldots, i_m\}$. Then $\sigma_i + r_j \ge \sigma_i > \frac{1}{2}b_{ij_{p(i)}} \ge \frac{1}{2}b_{ij_{q(j)+1}} \ge \frac{3}{2}b_{ij} > b_{ij}$.

3. $p(i) = q(j) + 1$ and $i = i_t$ for some $1 \le t \le m$ (thus $p(i) = t = q(j) + 1$). In this case the i, j constraint is satisfied at equality by construction.

4. $p(i) > q(j) + 1$ and $i = i_t$ for some $1 \le t \le m$ (thus $p(i) = t > q(j) + 1$). Then $\sigma_i + r_j \ge \sigma_i > \frac{1}{2}b_{ij_{t-1}} \ge \frac{1}{2}b_{ij_{q(j)+1}} \ge \frac{3}{2}b_{ij} > b_{ij}$.

The argument with $i \notin \{i_1, \ldots, i_m\}$ is similar.

Finally suppose that $i = i_t$ and $j = j_u$ with $u \notin \{t-1, t\}$. If $u > t$, $\sigma_i + r_j \ge r_j > \frac{1}{2}b_{i_uj_u} \ge \frac{3}{2}b_{i_tj_u} > b_{ij}$. If $u < t-1$, $\sigma_i + r_j \ge \sigma_i > \frac{1}{2}b_{i_tj_{t-1}} \ge \frac{3}{2}b_{i_tj_u} > b_{ij}$.

This shows that the solution is feasible and as it is unique, it defines a vertex of the above polyhedron.

Now let $a_{ij} = (j-1)n + i$, so that $b_{ij} = 3^{a_{ij}}/3^{n^2+1}$ and take

$$\alpha = (a_{i_1j_1}, a_{i_2j_1}, a_{i_2j_2}, a_{i_3j_2}, \ldots, a_{i_mj_{m-1}}).$$

As $\sigma_n = \Phi(\alpha)/3^{n^2+1}$, it follows from Lemma 10 (ii) that in any two vertices constructed as above by different sequences $(i_1, \ldots, i_m)$, $(j_1, \ldots, j_{m-1})$ and $(i'_1, \ldots, i'_{m'})$, $(j'_1, \ldots, j'_{m'-1})$, the values of σ_n are distinct numbers in the interval $(0, 1)$. As the number of such sequences is exponential in n, this proves Theorem 9.

6.2 Sufficient conditions for compactness of the formulation

We now describe conditions that ensure the existence of a compact formulation for a mixed-integer set X of the type MIX^{2TU}. Since X is described by a linear system $Ax \ge b$ where A is a TU matrix with at most two nonzero entries per row, the constraints defining X are of the following type:

$$
\begin{aligned}
x_i + x_j &\ge l_{ij}^{++}, &\quad (i,j) &\in N^{++} && (25)\\
x_i - x_j &\ge l_{ij}^{+-}, &\quad (i,j) &\in N^{+-} && (26)\\
-x_i - x_j &\ge l_{ij}^{--}, &\quad (i,j) &\in N^{--} && (27)\\
x_i &\ge l_i, & i &\in N^l &&\\
x_i &\le u_i, & i &\in N^u &&\\
x_i &\text{ integer}, & i &\in I, &&
\end{aligned}
$$

where $N^{++}, N^{+-}, N^{--} \subseteq N \times N$ and $N^l, N^u, I \subseteq N$. Wlog we assume that if $(i,j) \in N^{++}$ then $(j,i) \notin N^{++}$ and if $(i,j) \in N^{--}$ then $(j,i) \notin N^{--}$.

Let $G_X = (V, E)$ be the undirected graph with node set $V = L = N \setminus I$ corresponding to the continuous variables of X. E contains an edge ij for each inequality of the type (25)–(27) with $i, j \in L$ appearing in the linear system that defines X. Notice that since A is a TU matrix, then, for fixed i, j, the system $Ax \geq b$ can contain either inequalities of type (26) or inequalities of type (25),(27), but not both. Therefore, for each pair of nodes i, j in V, E contains at most two parallel edges connecting i and j.

We impose a *bi-orientation* ω on G_X: to each edge $e \in E$ (corresponding to an inequality $a_i x_i + a_j x_j \geq l_{ij}$) and each endnode i of e, we associate the value $\omega(e, i) = tail$ if $a_i = 1$, the value $\omega(e, i) = head$ otherwise. Thus each edge of G_X could have one head and one tail (if corresponding to an inequality (26)), two tails (if corresponding to an inequality (25)) or two heads (if corresponding to an inequality (27)).

Given a path $P = (v_0, e_1, v_1, e_1, \ldots, v_t)$ in G_X, where $v_0, \ldots, v_t \in V$ and $e_1, \ldots, e_t \in E$, we want to define the *ω-length* of P, $l_\omega(P)$. To do this, we first define the *reverse* of an edge $e \in E$ as the edge obtained by turning each head (resp. tail) of e into a tail (resp. head).

We construct a path $P' = (v_0, e'_1, v_1, e'_1, \ldots, v_t)$ from P by reversing some edges, so that v_0 is a tail of e_1, and every node $v_j, 1 \leq j < t$ is a head of one edge of P' and a tail of the other. Note that given P, the path P' is unique.

Now we define $l_\omega(P) = \sum_{j=1}^{t} \varepsilon(P, e_j) l_{e_j}$, where l_e is the right-hand-side of the inequality corresponding to edge e and

$$\varepsilon(P, e_j) = \begin{cases} -1 & \text{if } e_j \text{ has been reversed in } P' \\ +1 & \text{otherwise.} \end{cases}$$

We also define the list $\mathcal{G} = \{g_1, \ldots, g_\ell\}$ as the set of values $f(l_\omega(P))$ for all paths P in G_X.

Theorem 11 *Let X be a mixed-integer set of the type MIX^{2TU} and define $\mathcal{G}$ as above. Then X admits a list which is complete whose length is $\mathcal{O}(m\ell)$, where m is the number of inequalities in the description of X and $\ell = |\mathcal{G}|$.*

Proof: Let $= (\bar{x}_L, \bar{x}_I)$ be a vertex of conv(X). Then $\bar{x}_L$ is a vertex of the polyhedron defined by the inequalities:

$$\begin{aligned}
a_i x_i + a_j x_j &\geq l^{**}_{ij} & (i,j) &\in N^{**},\ i, j \in L & (28)\\
a_i x_i &\geq l^{**}_{ij} - a_j \bar{x}_j & (i,j) &\in N^{**},\ i \in L,\ j \in I & (29)\\
a_j x_j &\geq l^{**}_{ij} - a_i \bar{x}_i & (i,j) &\in N^{**},\ i \in I,\ j \in L & (30)\\
x_i &\geq l_i & i &\in L \cap N^\ell & (31)\\
x_i &\leq u_i & i &\in L \cap N^u, & (32)
\end{aligned}$$

where if the original inequality is of type (25), then $a_i = a_j = 1$ and $**$ stands for $++$, and the other cases are defined accordingly.

Let $S_{\bar{x}}$ be a set of $|L|$ independent inequalities among (28)–(32) that define $\bar{x}_L$. Then it is well known (and easy to see) that the edges corresponding to inequalities of type (28) in $S_{\bar{x}}$ define a forest $F_{\bar{x}}$ in G_X. Let $C_{\bar{x}} = (V(C_{\bar{x}}), E(C_{\bar{x}}))$ be a connected component of such a forest. Since $|V(C_{\bar{x}})| = |E(C_{\bar{x}})| + 1$, $C_{\bar{x}}$ contains a unique "root" node r whose value is determined by one of the bounds (29)–(32) and therefore the fractional part of $\bar{x}_r$ takes $\mathcal{O}(m)$ possible values, where m is the number of inequalities in the description of X.

If v is a node of $C_{\bar{x}}$ distinct from r, then the value of $\bar{x}_v$ is determined by the value of $\bar{x}_r$ and the tight inequalities (28) corresponding to the edges in the path P_{vr} in $C_{\bar{x}}$ having v as first vertex and r as last vertex: if e is the edge in P_{vr} incident with r and if P'_{vr} is constructed from P_{vr} as described above, we have

$$\bar{x}_v = \begin{cases} l_\omega(P_{vr}) + \bar{x}_r & \text{if } r \text{ is a head of } e \\ l_\omega(P_{vr}) - \bar{x}_r & \text{otherwise.} \end{cases} \tag{33}$$

Since the list $\mathcal{G}$ has ℓ elements, this shows that the fractional part of each variable $\bar{x}_v$ at a vertex can take at most $\mathcal{O}(m\ell)$ values. □

Corollary 12 *Assume that a mixed-integer set X of the type MIX^{2TU} satisfies at least one of the following conditions:*

(i) *The number of paths connecting two nodes in G_X is bounded by a polynomial function of the size of the description of X;*

(ii) *The number of elements in the sets $\{f(l^{**}_{ij}),\, (i,j) \in N^{**}\}$, where $** \in \{++,+-,--\}$, is bounded by a constant.*

(iii) *G_X is bipartite with vertex classes U, V and the inequalities defining X which contain two continuous variables x_u, x_v ($u \in U, v \in V$) have the form $x_u + x_v \geq b_v - b_u$ for some vector b with indices in $U \cup V$.*

Then X admits a complete list of fractional parts which is compact.

Proof: If (i) holds, the length of the list $\mathcal{G}$ is bounded by a polynomial function. Then Theorem 11 implies that there is a complete list for X which is compact.

Now suppose that (ii) holds and assume

$$\bigcup_{**\in\{++,+-,--\}} \{f(l^{**}_{ij}),\, (i,j) \in N^{**}\} = \{f_1, \ldots, f_t\}.$$

Each value $f(l_\omega(P_{rv}))$ can be expressed as

$$f(l_\omega(P_{rv})) = \sum_{h=1}^{t} \alpha_h f_h,$$

where α_h is an integer for all h. Since G_X has $|L|$ nodes, the maximum length of a path in G_X is $|L|-1$. This implies $|\alpha_h| \leq |L|-1$ for all h. Then the length of the list $\mathcal{G}$ is at most $(2|L|-1)^t$. Thus, by Theorem 11 there is a complete list for X of size $\mathcal{O}(m(2|L|-1)^t) = \mathcal{O}(mn^t)$, as t is a constant.

Finally assume that (iii) holds. In this case it is easy to verify that for $v \in U \cup V$,

$$l_\omega(P_{vr}) = b_r - b_v \tag{34}$$

and thus X admits a complete list which is compact. □

Observation 3 *The example whose complete list has exponential length constructed in Theorem 9 shows that if a mixed-integer set of the type MIX^{2TU} does not satisfy any of the above three conditions, then its complete list may be long.*

Observation 4 *If X is a mixed-integer set of the type MIX^{2TU} such that the size of every connected component of G_X is bounded by a constant, then X satisfies condition* (i) *of the above Corollary.*

7 Examples

We show that several well-studied mixed-integer sets can be transformed into sets of the type MIX^{2TU}, but first we give a precise meaning to the word "transformed".

7.1 Mixed-integer linear mappings

Theorem 13 *Consider the transformation defined by* $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, *where* $(x, y) \in \mathbb{R}^{m+n}$, $(x', y') \in \mathbb{R}^{m'+n'}$, $m + n = m' + n'$ *and* $A \in \mathbb{R}^{(m+n)\times(m'+n')}$ *is nonsingular. The following are equivalent:*

(i) *For each* $(x, y) \in \mathbb{R}^{m+n}$, y *is integral if and only if* y' *is integral.*

(ii) $m = m'$, $n = n'$ *and* $A = \begin{bmatrix} A_1 & A_2 \\ \mathbf{0} & U \end{bmatrix}$, *where* $A_1 \in \mathbb{R}^{m\times m}$ *is nonsingular,* $A_2 \in \mathbb{R}^{m\times n}$ *and* $U \in \mathbb{R}^{n\times n}$ *is unimodular.*

Proof: (i)⇒(ii) Suppose $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_1 \in \mathbb{R}^{m'\times m}$, $A_2 \in \mathbb{R}^{m'\times n}$, $A_3 \in \mathbb{R}^{n'\times m}$ and $A_4 \in \mathbb{R}^{n'\times n}$. If $A_3 \neq \mathbf{0}$, one of its entries is a nonzero number a. Wlog we assume that this entry is in the first row and first column of A_3. Then $A \begin{pmatrix} e_1/2a \\ \mathbf{0} \end{pmatrix}$ contains a component equal to $1/2$ in the entry corresponding to y'_1, contradicting (i). Thus $A_3 = \mathbf{0}$.
If $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ is the inverse of A ($B_1 \in \mathbb{R}^{m\times m'}$, $B_2 \in \mathbb{R}^{m\times n'}$, $B_3 \in \mathbb{R}^{n\times m'}$ and $B_4 \in \mathbb{R}^{n\times n'}$), a similar argument shows that $B_3 = \mathbf{0}$.
Thus we obtain $y' = A_4 y$, $y = B_4 y'$ for each y. We now prove that this implies $n = n'$. Equation $y = B_4 A_4 y$ for all y yields $B_4 A_4 = I_n$, thus $\mathrm{rk} A_4 \geq n$. Since A_4 is $n' \times n$, this implies $n' \geq n$. Similarly, starting from $y' = A_4 B_4 y'$ for all y', one obtains $n \geq n'$. Thus $n = n'$ and consequently $m = m'$. (i) then implies that A_4 is unimodular.
(ii)⇒(i) The transformation and its inverse are

$$\begin{cases} x' = A_1 x + A_2 y \\ y' = U y \end{cases} \quad \text{and} \quad \begin{cases} x = A_1^{-1}(x' - A_2 U^{-1} y') \\ y = U^{-1} y' \end{cases}.$$

Since U is unimodular, these two transformations preserve the integrality of y and y'. □

Consider an arbitrary mixed-integer set $X = \{x : Ax \geq b;\ x_i \text{ integer}, i \in I\}$ and let $\mathcal{F}$ be a complete list of fractional parts which is compact for X. In general, if we apply a linear mapping of the kind described in Theorem 13 to X, the transformed mixed-integer set X' may not have a complete list which is compact. For instance, let $X = \{x : 0 \leq x_i \leq 2^{-i}, i \in N\}$ (so here $I = \emptyset$; similar examples with $I \neq \emptyset$ can be easily derived from this example). The list $\mathcal{F} = \{0;\ 2^{-i}, i \in N\}$ is complete for X and its size is linear in the size of the description of X. The linear mapping $x'_1 = x_2 + \cdots + x_n$, $x'_i = x_i, i \in N \setminus \{1\}$, transforms X into $X' = \{x' : 0 \leq x'_1 - x'_2 - \cdots - x'_n \leq 2^{-1};\ 0 \leq x'_i \leq 2^{-i}, i \in N \setminus \{1\}\}$. Now, for each subset $S \subseteq N \setminus \{1\}$ the vector

$$x'_i = \begin{cases} 2^{-i} & \text{if } i \in S \\ 0 & \text{if } i \in (N \setminus \{1\}) \setminus S \\ \sum_{j\in S} 2^{-j} & \text{if } i = 1 \end{cases}$$

is a vertex of X'. Since the values of the sum $\sum_{j\in S} 2^{-j}$ are distinct numbers in the interval $[0,1)$ for each S, any complete list for X' contains a number of fractional parts which is exponential in the size of the description of X.

However, for the mixed-integer sets we study below (except the sets INT in Section 7.3 and $BIP(I)$ in Section 7.5), we will consider linear mappings of the kind of Theorem 13 which give rise to mixed-integer sets of the type MIX^{2TU} satisfying at least one of the conditions of Corollary 12. Thus, in these cases the existence of a complete list which is compact is guaranteed. Furthermore, for these sets a complete list which is compact is explicitly given.

7.2 The continuous mixing set with flows

The continuous mixing set with flows $CFLOWMIX$, studied in Conforti et al. [3], is

$$\begin{aligned} s + r_j + x_j &\geq b_j, && j \in N \\ x_j &\leq y_j, && j \in N \\ s \geq 0,\ r_j \geq 0,\ x_j &\geq 0,\ y_j \geq 0 \text{ integer}, && j \in N. \end{aligned}$$

As explained in [3], this set provides both a relaxation of the single item constant capacity lot-sizing problem with backlogging and an exact formulation of the two stage stochastic lot-sizing problem with constant capacities and backlogging.

The following observation shows that the above mixed-integer set can be transformed into a set of the type MIX^{2TU}. Let $FLOW$ be the following set:

$$\begin{aligned} \sigma_j + x_j &\geq b_j, && j \in N \\ x_j &\leq y_j, && j \in N \\ s \geq 0,\ \sigma_j - s \geq 0,\ x_j &\geq 0,\ y_j \geq 0 \text{ integer}, && j \in N. \end{aligned}$$

Since the constraint matrix of the above system is a TU matrix with at most two nonzero entries per row, $FLOW$ is a mixed-integer set of the type MIX^{2TU}.

Observation 5 *The linear transformation:*

$$s = s,\ \ \sigma_j = s + r_j,\ \ x_j = x_j,\ \ y_j = y_j,\ \ j \in N$$

maps $CFLOWMIX$ into $FLOW$.

Remark that if X is a mixed-integer set of the type $FLOW$, then the graph G_X (as defined in Section 6.2) is a tree, with leaves corresponding to variables x_j. Therefore G_X satisfies condition (i) of Corollary 12. Below we explicitly give a complete list for $FLOW$ which is compact.

Lemma 14 *The list $\mathcal{F} = \{0;\ f(b_j),\ j \in N;\ f(b_i - b_j),\ i,j \in N\}$ is complete for $FLOW$.*

Proof: We use the same notation as in the proof of Theorem 11. For a connected component $C_{\bar{x}}$ of $F_{\bar{x}}$, the root r corresponds to a variable which assumes an integer value. Then, by equation (33) we only need to compute the values $f(l_\omega(P))$ for all P in G_X. It is easy to check that the list $\mathcal{F} = \{0;\ f(b_j),\ j \in N;\ f(b_i - b_j),\ i,j \in N\}$ includes all these values. $\square$

Therefore the result of Section 5 provides an extended formulation of the set conv$(FLOW)$ which is compact. Applying the inverse of the above linear transformation gives an extended formulation of conv$(CFLOWMIX)$ which is compact.

We now introduce several faces of the polyhedron conv$(CFLOWMIX)$ that have been studied.

7.2.1 The continuous mixing set

The continuous mixing set is the mixed-integer set $CMIX$ defined as follows:

$$\begin{aligned} s + r_j + y_j \geq b_j, & \quad j \in N \\ s \geq 0,\ r_j \geq 0,\ y_j \geq 0 \text{ integer}, & \quad j \in N. \end{aligned}$$

Clearly the polyhedron $\text{conv}(CMIX)$ is the face of $\text{conv}(CFLOWMIX)$ defined by the equations $x_j = y_j, j \in N$. An extended formulation for $\text{conv}(CMIX)$ which is compact was given by Miller and Wolsey [13]. Later Van Vyve [16] gave a more compact extended formulation and a linear inequality description of $\text{conv}(CMIX)$ in the original space.

7.2.2 The mixing set with flows

The mixing set with flows $FLOWMIX$ is defined as follows:

$$\begin{aligned} s + x_j \geq b_j, & \quad j \in N \\ x_j \leq y_j, & \quad j \in N \\ s \geq 0,\ x_j \geq 0,\ y_j \geq 0 \text{ integer}, & \quad j \in N. \end{aligned}$$

The polyhedron $\text{conv}(FLOWMIX)$ is the face of $\text{conv}(CFLOWMIX)$ defined by the equations $r_j = 0, j \in N$. Conforti et al. [2] described $\text{conv}(FLOWMIX)$ both with an extended formulation and in the original (s, x, y)-space.

7.2.3 The $\geq$-mixing set

The $\geq$-mixing set $MIX^{\geq}$ is defined as follows:

$$\begin{aligned} s + y_j \geq b_j, & \quad j \in N \\ s \geq 0,\ y_j \geq 0 \text{ integer}, & \quad j \in N. \end{aligned}$$

The polyhedron $\text{conv}(MIX^{\geq})$ is the face of $\text{conv}(FLOWMIX)$ defined by the equations $x_j = y_j, j \in N$. By dropping the nonnegativity of y one finds the mixing set MIX defined in the introduction, which was first studied explicitly by Günlük and Pochet [9].

The following observation shows that the $\geq$-mixing set admits a complete list that is shorter than that of the set described in Lemma 14.

Observation 6 *If $(\bar{s}, \bar{y})$ is a vertex of $\text{conv}(MIX^{\geq})$, then $\bar{s} = 0$ or $f(\bar{s}) = f(b_j)$ for some $j \in N$. Therefore $\{0;\ f(b_j),\ j \in N\}$ is a complete list for $MIX^{\geq}$.*

An identical result holds for the set MIX.

7.3 The intersection set

The intersection set INT, discussed in Conforti et al. [3], is defined as follows:

$$\begin{aligned} \sigma_i + r_j + y_j \geq b_{ij}, & \quad i, j \in N \\ \sigma_i \geq 0,\ r_j \geq 0,\ y_j \geq 0 \text{ integer}, & \quad i, j \in N. \end{aligned}$$

Observation 7 *The linear transformation:*

$$y_j = y_j,\ \ \sigma_i = \sigma_i,\ \ \rho_j = r_j + y_j,\ \ i,j \in N$$

maps INT into the following mixed-integer set:

$$\begin{array}{cl} \sigma_i + \rho_j \geq b_{ij}, & i,j \in N \\ \rho_j - y_j \geq 0, & j \in N \\ \sigma_i \geq 0,\ y_j \geq 0 \text{ integer}, & i,j \in N. \end{array}$$

The above mixed-integer set is of the type MIX^{2TU}.

In Section 6.1 it has been shown that in general the set INT does not admit a complete list $\mathcal{F}$ whose size is polynomial in the size of the description of INT (see Observation 1).

7.4 Lot-sizing

Van Vyve [17] showed that the set LOT

$$\begin{array}{cl} s_i + r_j + \sum_{u=i+1}^{j} y_u \geq b_j - b_i, & i,j \in N, j > i \\ s_i \geq 0,\ r_j \geq 0,\ y_j \in \{0,1\}, & i,j \in N \end{array}$$

represents the dominant of the feasible solutions of a lot-sizing problem with constant capacities and backlogging, and provided an extended formulation for conv(LOT) which is compact.

Observation 8 *The linear transformation:*

$$z_0 = 0,\ \ z_j = \sum_{u=1}^{j} y_u,\ \ \sigma_i = s_i - z_i,\ \ \rho_j = r_j + z_j,\ \ i,j \in N \tag{35}$$

maps LOT into the following mixed-integer set:

$$\begin{array}{cl} \sigma_i + \rho_j \geq b_j - b_i, & i,j \in N, j > i \\ \sigma_i + z_i \geq 0, & i \in N \\ \rho_j - z_j \geq 0, & j \in N \\ 0 \leq z_j - z_{j-1} \leq 1, & j \in N \\ z_j \text{ integer}, & j \in N \end{array} \tag{36}$$

The above mixed-integer set is of the type MIX^{2TU}.

Lemma 15 *Let X be the above mixed-integer set. The list $\mathcal{F} = \{0;\ f(b_i - b_j),\ i,j \in N\}$ is complete for X.*

Proof: Again we use the same notation as in the proof of Theorem 11. The graph G_X is bipartite with one vertex class corresponding to variables σ_i and the other corresponding to variables ρ_j. The structure of inequalities (36) shows that condition (iii) of Corollary 12 is satisfied. Since all other constraints have integer right-hand-side, the root r corresponds to a variable which assumes an integer value. Then, by equations (33) and (34), the list given above contains all possible fractional parts taken by the variables at a vertex. □

By the above Lemma and the form of transformation (35), we immediately derive the following result, which was shown by Van Vyve [17]:

Observation 9 *The list $\mathcal{F} = \{0;\ f(b_i - b_j),\ i,j \in N\}$ is complete for LOT.*

The above observation, together with the result of Section 5, provides an extended formulation of conv(LOT) which is compact.

7.5 Bipartite cover inequalities

Given a bipartite graph $G = (U, V; E)$, let (I, L) be a partition of $U \cup V$ with $I \neq \emptyset$ and let $BIP(I)$ be the mixed-integer set:

$$\begin{aligned} x_u + x_v \geq b_{uv}, &\quad uv \in E \\ x_u \geq 0, &\quad u \in L \\ x_u \geq 0 \text{ integer}, &\quad u \in I. \end{aligned}$$

The set $BIP(I)$ is obviously a set of the type MIX^{2TU}. The example of Section 6.1 shows that $BIP(I)$ does not admit in general a complete list which is compact. However, such a list exists in the following two special cases.

The first case is the set $BIP(U)$ (i.e. the integer variables correspond to the nodes of one side of the bipartition of G): Miller and Wolsey [13] show that for the set $BIP(U)$ the list $\{0;\ f(b_{uv}), uv \in E\}$ is complete and they also give a formulation of $BIP(U)$ in the x-space.

The second case is the set $BIP(I)$ with the additional condition that $2b_{uv}$ is integer for all $uv \in E$, that is, $f(b_{uv})$ is either 0 or 1/2 for all $uv \in E$: this set satisfies condition (ii) of Corollary 12. Conforti et al. [4] give a formulation in the x-space of this set.

8 Concluding remarks

One outstanding question that remains concerns the complexity of the optimization problem over the sets MIX^{2TU} when the list of fractional parts has exponential size. More specifically, whether the polyhedron conv(MIX^{2TU}) admits an extended formulation which is compact, even when the list of fractional parts has exponential size. In particular, can Corollary 8 be strengthened such that one has a polynomial algorithm in m, n and $\log D$ or even only in m and n?

Another intriguing challenge is to understand under what conditions the formulation for conv(MIX^{2TU}) in the original x-space can be explicitly described (possibly by projecting the extended formulation introduced in this paper). A fundamental result of this type is the formulation of conv($MIX^{\geq}$) in the original space of Günlük and Pochet [9]. Other results for bipartite cover inequalities (i.e. for conv($BIP(I)$)) can be found in [4] and [13]. Van Vyve [16] gives the formulation for conv($CMIX$). Conforti et al. [2] give the formulation for conv($FLOWMIX$). The formulation of conv(MIX^{2TU}) in the original space when there is a single integer variable was given by Di Summa [6]. To the best of our knowledge, this is what is known so far.

Another aspect is the fact that a set MIX^{2TU} is equivalent to a set MIX^{DN} and that the extended formulation introduced in this paper involves a system of inequalities $A(x, \mu) \geq b$ where A is a dual network matrix and b is an integral vector. The associated optimization problem can therefore be solved in the extended space as a dual of a network flow problem. Can this be used to develop new algorithms for optimization and/or separation? Computationally, what is the most effective use of the formulation for MIX^{2TU}, when the description of a set MIX^{2TU} is a relaxation of a more complicated mixed-integer set? Should one use the dual network formulation (11)–(15), the same formulation but with the δ variables as in (3)–(8) rather than the μ variables, cutting planes and separation, or other?

A last question concerns the extension of our model. Recently it has been shown that several problems that involve the optimization of a linear function over a generalization of the mixing set $MIX^{\geq}$, but whose description does not involve a TU matrix, are solvable in polynomial time. Two such sets are the mixing set with divisible capacities in Conforti and Wolsey [5] and the mixing-MIR set with divisible capacities (Van Vyve [15], de Farias and Zhao [18]). To what extent, if any, can the results here be extended to these problems?

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