Maximal S-free convex sets and the Helly number

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Abstract

Given a subset S of $\mathbb{R}^d$, the Helly number $h(S)$ is the largest size of an inclusionwise minimal family of convex sets whose intersection is disjoint from S.

A convex set is S-free if its interior contains no point of S. The parameter $f(S)$ is the largest number of maximal faces in an inclusionwise maximal S-free convex set.

We study the relation between the parameters $h(S)$ and $f(S)$. Our main result is that $h(S) \leq (d+1)f(S)$ for every nonempty proper closed subset S of $\mathbb{R}^d$. We also study the Helly number of the Cartesian product of two discrete sets.

1 Introduction

Given a nonempty subset S of $\mathbb{R}^d$, a family $\{C_1, \dots, C_m\}$ of convex subsets of $\mathbb{R}^d$ is a *critical family for S* (of size m) if

$$\bigcap_{j\in[m]} C_j \cap S = \emptyset \quad \text{and} \quad \bigcap_{j\in[m]\setminus\{i\}} C_j \cap S \neq \emptyset \ \forall i \in [m],$$

where $[m] = \{1, \dots, m\}$. The *Helly number* of S is

$$h(S) = \sup\{m : m \text{ is the size of a critical family for } S\}.$$

A famous result of Helly states that $h(\mathbb{R}^d) = d+1$ (see, e.g., [5]). Doignon [14], Bell [10] and Scarf [18] prove $h(\mathbb{Z}^d) = 2^d$. Hence in $\mathbb{R}^d$ infeasibility of a linear program can be certified by selecting at most $d+1$ inequalities and infeasibility of an integer linear program can be certified by selecting at most 2^d inequalities.

Given a nonempty subset S of $\mathbb{R}^d$, a subset C of $\mathbb{R}^d$ is an *S-free convex set* if C is closed, convex and the interior of C is disjoint from S. An S-free convex set C is *maximal* if $C = C'$ for every S-free convex set C' such that $C \subseteq C'$. It is known that every S-free convex set is contained in a maximal one (see, e.g., [9, Theorem 1.1]).

Averkov [2] defines the *facet number* $f(S)$ as follows. If every maximal S-free convex set is a polyhedron, we let

$$f(S) = \sup\{m : m \text{ is the number of facets of a maximal } S\text{-free convex set}\}.$$

If there exists a maximal S-free convex set which is not a polyhedron, we let $f(S) = \infty$.

Notice that every hyperplane is S-free, as its interior is empty. Furthermore if C is a maximal S-free convex set with empty interior, then C is a hyperplane. (This follows readily

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from maximality.) Therefore every maximal S-free convex set is a hyperplane if and only if S is dense in $\mathbb{R}^d$. As a hyperplane is a polyhedron with no facets, the above argument shows that $f(S) = 0$ if and only if S is dense in $\mathbb{R}^d$. (This is the only discrepancy with the definition given in [2], as Averkov sets $f(S) = -\infty$ when S is dense in $\mathbb{R}^d$.)

When $S = \mathbb{Z}^d$, Lovász [16] shows $f(S) = 2^d$. Basu et al. [6] show $f(S) \le 2^d$ when $S = \mathbb{Z}^d \cap P$, where P is a rational polyhedron. The same bound is proven by Morán and Dey [17] when $S = \mathbb{Z}^d \cap K$, where K is a convex set. Averkov [2] shows that $f(S) = h(S)$ when S is a discrete subset of $\mathbb{R}^d$. (S is *discrete* if $|S \cap B|$ is finite for every bounded set B in $\mathbb{R}^d$.)

When $S = \mathbb{Z}^p \times \mathbb{R}^q$, $f(S)$ and $h(S)$ differ: Lovász [16] proves $f(S) = 2^p$, while Hoffman [15] and Averkov and Weismantel [4] show $h(S) = 2^p(q+1)$. When $S = (\mathbb{Z}^p \times \mathbb{R}^q) \cap K$, where K is a convex set, Averkov [2] proves that $f(S) \le 2^p$ and $h(S) \le 2^p(q+1)$.

Remark also that if a polyhedron P is a maximal S-free convex set, and $H_1^{\le} \cap \cdots \cap H_m^{\le}$ is an irredundant representation of P as the intersection of closed half-spaces, then the family of corresponding open half-spaces $\{H_1^{<}, \dots, H_m^{<}\}$ is a critical family for S. (The maximality of P and the irredundancy of the representation imply that $\bigcap_{j \in [m] \setminus \{i\}} H_j^{<} \cap S \neq \emptyset$ for all $i \in [m]$). Therefore this shows that $f(S) \le h(S)$ when $f(S)$ is finite. When $f(S) = \infty$, the same inequality can be proven with a limit argument, see e.g. [2].

Our motivation for the study of $f(S)$ comes from Integer Programming. We consider the model

$$X = X(R, S) := \{x \in \mathbb{R}^n_+ : Rx \in S\}$$

where $R = (r_1, \dots, r_n)$ is a $d \times n$ real matrix and $S \subseteq \mathbb{R}^d$ is a nonempty closed set with $0 \notin S$. This model has been the focus of current research which is surveyed, e.g., in [6, 7, 8] (see also [12, Chapter 6]).

Since $0 \notin S$ and S is closed, one can show that 0 does not lie in the closed convex hull of X. We are interested in *separating* 0 from X: that is, we want to generate linear inequalities of the form $cx \ge 1$ that are valid for X.

The theory of cut-generating functions studies the above separation problem for fixed S, independently of the matrix R. A function $\psi : \mathbb{R}^d \to \mathbb{R}$ is a *cut-generating function* if for every $n \in \mathbb{N}$ and every $d \times n$ matrix $R = (r_1, \dots, r_n)$, the inequality

$$\sum_{i=1}^{n} \psi(r_i) x_i \ge 1$$

is valid for $X(R, S)$. A cut-generating function ψ is *minimal* if $\psi = \psi'$ for every cut-generating function ψ' such that $\psi' \le \psi$. Since $X \subseteq \mathbb{R}^n_+$, we can restrict to minimal cut-generating functions.

Minimal cut-generating functions are related to maximal S-free convex sets that contain 0 in their interior, see e.g. [11]. In particular, if a maximal S-free convex set containing 0 in its interior is a polyhedron $P = \{x \in \mathbb{R}^d : a_i x \le 1, i \in I\}$, then the function ψ defined as

$$\psi(r) = \max_{i \in I} a_i r \tag{1}$$

is a minimal cut-generating function. Furthermore, if $f(S) < \infty$, then (1) can be computed as the maximum of at most $f(S)$ linear functions.

Since $f(S) \le h(S)$ for every nonempty subset $S \subseteq \mathbb{R}^d$ [2], $f(S)$ is finite whenever $h(S)$ is finite. However it was not known whether the finiteness of $f(S)$ implies the finiteness of $h(S)$.

More generally, the question of characterizing the sets S for which $f(S)$ can be bounded from below in terms of $h(S)$ was largely open. Here we study this issue.

We now summarize our results. In Section 2 we prove properties of the Helly number and introduce the int-Helly number, a concept that is crucial for our results. In Section 3 we prove that $h(S) \leq (d+1)f(S)$ when S is a closed subset of $\mathbb{R}^d$. This bound is tight: if $S \subsetneq \mathbb{R}^d$ is a d-dimensional closed convex set, then $f(S) = 1$ while $h(S) = d+1$.

In Section 4 we study the case when S can be expressed as the Cartesian product of two sets. When $S = S_1 \times \mathbb{R}^q$ and S_1 is discrete, we show $h(S) = h(S_1)(q+1)$. This proves the tightness of the inequality $h(S) \leq h(S_1)(q+1)$ of Averkov and Weismantel [4], and extends the result $h(\mathbb{Z}^p \times \mathbb{R}^q) = 2^p(q+1)$ of Hoffman [15] and Averkov and Weismantel [4]. Finally, when $S = S_1 \times S_2$ and S_1, S_2 are both discrete, we show $h(S_1 \times S_2) \geq h(S_1)h(S_2)$. This inequality is tight, e.g., for $S_1 = \mathbb{Z}^p$ and $S_2 = \mathbb{Z}^q$.

The Helly number has been generalized in several ways. For instance, while the traditional setting for the study of the Helly number is for subsets of $\mathbb{R}^d$, Hoffman [15] develops a purely set-theoretic framework. More recently, Aliev et al. [1] study the largest size of an inclusionwise minimal family of closed half-spaces whose intersection contains exactly k integer points.

2 Helly number and int-Helly number

Given a convex subset C of $\mathbb{R}^d$, we denote with $\dim(C)$, $\mathrm{int}(C)$, $\mathrm{relint}(C)$, $\mathrm{cl}(C)$ and $\mathrm{aff}(C)$ the dimension, interior, relative interior, closure and affine hull of C. Given $V \subseteq \mathbb{R}^d$, we indicate with $\mathrm{conv}(V)$ its convex hull and with $\langle V \rangle$ the linear space generated by V. Finally, if L is an affine subspace of $\mathbb{R}^d$, we define $L^* = \{y \in \mathbb{R}^d : y(x - x_0) = 0 \; \forall x \in L\}$, where x_0 is any point in L; i.e., L^* is the orthogonal complement of the unique linear space that can be obtained by translating L.

Theorems 1 and 3 were proven by Hoffman [15], see also [13]. We give short proofs for the sake of completeness.

Theorem 1 *Given a nonempty subset S of $\mathbb{R}^d$,*

$$h(S) = \sup\left\{|T| : T \subseteq S \textit{ finite}, \textstyle\bigcap_{t \in T} \mathrm{conv}(T \setminus \{t\}) \cap S = \emptyset\right\}. \tag{2}$$

Proof. Let $T \subseteq S$ be finite and satisfy $\bigcap_{t \in T} \mathrm{conv}(T \setminus \{t\}) \cap S = \emptyset$. Then the family $\{\mathrm{conv}(T \setminus \{t\}),\, t \in T\}$ is a critical family of size $|T|$.

Conversely, let $\{C_1, \dots, C_m\}$ be a critical family. Then for every $i \in [m]$ there exists an element $t_i \in \bigcap_{j \in [m] \setminus \{i\}} C_j \cap S$. Let $T = \{t_1, \dots, t_m\}$. As $\mathrm{conv}(T \setminus \{t_j\}) \subseteq C_j$ for $j \in [m]$ and $\bigcap_{j \in [m]} C_j \cap S = \emptyset$, we have that $\bigcap_{j \in [m]} \mathrm{conv}(T \setminus \{t_j\}) \cap S = \emptyset$ as well. $\square$

The next corollary appears e.g. in [4].

Corollary 2 *Given a nonempty subset S of $\mathbb{R}^d$,*

$$h(S) = \sup\{m : m \textit{ is the size of a critical family for } S \textit{ consisting of closed half-spaces}\}.$$

Proof. By Theorem 1, it suffices to show the following:

Let T be a finite subset of S satisfying $\bigcap_{t\in T}\operatorname{conv}(T\setminus\{t\})\cap S=\emptyset$. Then there exists a critical family of closed half-spaces of size at least $|T|$.

For $t\in T$, let $\mathcal{H}_t$ be a finite family of closed half-spaces whose intersection is $\operatorname{conv}(T\setminus\{t\})$ and let $\mathcal{H}=\bigcup_{t\in T}\mathcal{H}_t$. We have that $\bigcap_{H\in\mathcal{H}}H\cap S=\bigcap_{t\in T}\operatorname{conv}(T\setminus\{t\})\cap S=\emptyset$. As every half-space in $\mathcal{H}$ contains at least $|T|-1$ points of T, the intersection of $|T|-1$ of them contains a point of T (and thus of S). Therefore the family $\mathcal{H}$ of half-spaces contains a critical family of size at least $|T|$. □

Given $S\subseteq\mathbb{R}^d$, a subset T of S is in *S-convex position* if T is finite and $\operatorname{conv}(T)\cap S=\operatorname{vert}(\operatorname{conv}(T))$ (where $\operatorname{vert}(P)$ stands for the set of vertices of a polytope P). Note that when this happens we have $\operatorname{conv}(T)\cap S=\operatorname{vert}(\operatorname{conv}(T))=T$.

Theorem 3 *Given a nonempty subset S of $\mathbb{R}^d$ which is discrete,*

$$h(S)=\sup\{|T|:T\subseteq S\text{ in }S\text{-convex position}\}.\tag{3}$$

Proof. We show that (2) and (3) are equivalent.

If $T\subseteq S$ is in S-convex position, then $\bigcap_{t\in T}\operatorname{conv}(T\setminus\{t\})\cap S=\emptyset$.

Conversely, let $T\subseteq S$ be a finite set such that $\bigcap_{t\in T}\operatorname{conv}(T\setminus\{t\})\cap S=\emptyset$. Since S is discrete, we may assume that there is no $T'\neq T$ with $|T'|=|T|$ and $T'\subset\operatorname{conv}(T)\cap S$ that satisfies $\bigcap_{t\in T'}\operatorname{conv}(T'\setminus\{t\})\cap S=\emptyset$.

Since $\operatorname{vert}(\operatorname{conv}(T))\subseteq\operatorname{conv}(T)\cap S$, if T is not in S-convex position there exists $s^*\in(\operatorname{conv}(T)\cap S)\setminus\operatorname{vert}(\operatorname{conv}(T))$. If $s^*\in T$, then $s^*\in\bigcap_{t\in T}\operatorname{conv}(T\setminus\{t\})\cap S$, a contradiction. Therefore $s^*\in S\setminus T$. Since $\bigcap_{t\in T}\operatorname{conv}(T\setminus\{t\})\cap S=\emptyset$, then $s^*\notin\operatorname{conv}(T\setminus\{\hat t\})$ for some $\hat t\in T$. Let $T'=T\cup\{s^*\}\setminus\{\hat t\}$. As $\operatorname{conv}(T')\subset\operatorname{conv}(T)$, we have that $\bigcap_{t\in T'}\operatorname{conv}(T'\setminus\{t\})\cap S=\emptyset$. Since $\hat t\in T\setminus T'$, this contradicts the choice of T. □

Given a subset $T\subset\mathbb{R}^d$ in S-convex position such that $\operatorname{aff}(T)=\mathbb{R}^d$, a polytope P is a *T-facet polytope* if P has $|T|$ facets and each facet F of P contains exactly one point of T, which is in $\operatorname{relint}(F)$. Hence $\operatorname{conv}(T)\subset P$ and $\dim(P)=d$. Assuming that $0\in\operatorname{int}(\operatorname{conv}(T))$, a T-facet polytope P can be constructed as follows. As T is in S-convex position and $0\in\operatorname{int}(\operatorname{conv}(T))$, given $v\in T$ there exists a_v such that

$$a_vv=1\text{ and }a_vv'<1\text{ for every }v'\in T\setminus\{v\}.\tag{4}$$

Let $P=\{x\in\mathbb{R}^d:a_vx\le 1,\ v\in T\}$. As $\operatorname{aff}(T)=\mathbb{R}^d$, P is a polytope. Since a_v satisfies (4), we have that $F_v=P\cap\{x\in\mathbb{R}^d:a_vx=1\}$ is a facet of P, $v\in\operatorname{relint}(F_v)$ and $F_v\cap T=\{v\}$. Hence P is a T-facet polytope.

Theorem 4 *Let S be a discrete subset of $\mathbb{R}^d$ containing more than one point. If $h(S)<\infty$, then there exists an S-free convex set P such that:*

a) P is a full-dimensional polyhedron of the form $P=P_0+\operatorname{aff}(S)^*$, where $P_0\subset\operatorname{aff}(S)$ is a T-facet polytope (with respect to $\operatorname{aff}(S)$) for some $T\subseteq S$ in S-convex position;

b) P has $h(S)$ facets and every facet of P contains exactly one point of S, which is in its relative interior.

Proof. We first assume that $\text{aff}(S) = \mathbb{R}^d$. Since $h(S)$ is finite, by Theorem 3 there exists a subset $T \subseteq S$ in S-convex position, where $|T| = h(S)$. Note that $\text{aff}(T) = \mathbb{R}^d$: if not, since S is discrete and $\text{conv}(T)$ is compact, we could add a point of S to T and obtain a larger set in S-convex position, a contradiction to Theorem 3.

Since we may assume w.l.o.g. that $0 \in \text{int}(\text{conv}(T))$, by the above construction there exists a T-facet polytope P_0 with $h(S) = |T|$ facets. Since $\text{aff}(S)^* = \{0\}$, $P = P_0$ satisfies a).

In order to conclude that P is an S-free convex set and satisfies b), it suffices to show that $P \cap S = T$. Assume the contrary, i.e., $(P \cap S) \setminus T \neq \emptyset$. Since S is discrete and $\text{conv}(T) \cap S = T$, there exists a point $s^* \in (P \cap S) \setminus T$ such that $T \cup \{s^*\}$ is in S-convex position. This contradicts the fact that $h(S)$ satisfies (3) and $|T| = h(S)$.

We now assume that $\text{aff}(S) \subsetneq \mathbb{R}^d$. Since S contains more than one point, the dimension of $\text{aff}(S)$ is at least one. Then, by considering $\text{aff}(S)$ as the ambient space, the above argument shows that there exists an S-free convex set which is a T-facet polytope $P_0 \subset \text{aff}(S)$, where $T \subseteq S$ is in S-convex position and $|T| = h(S)$. Therefore, in the ambient space $\mathbb{R}^d$, the polyhedron $P = P_0 + \text{aff}(S)^*$ is an S-free convex set and satisfies a) and b). $\square$

We remark that if a discrete set S contains more than one point and $h(S) < \infty$, Theorem 4 implies that $f(S) = h(S)$, a result of Averkov [2]. To see this, consider a polyhedron P satisfying the conditions of Theorem 4. If C is a closed convex set that properly contains P, then $\text{relint}(F) \subseteq \text{int}(C)$ for some facet F of P. Since the polyhedron P is S-free and contains one point of S in the relative interior of each facet, C is not S-free. This shows that P is a maximal S-free convex set. As P has $h(S)$ facets, we have $f(S) \geq h(S)$, and since $f(S) \leq h(S)$, this proves that $f(S) = h(S)$.

2.1 int-Helly number

In order to bound $f(S)$ from below in terms of $h(S)$, we introduce the int-Helly number. Given a nonempty subset S of $\mathbb{R}^d$, convex sets $C_1, \ldots, C_m$ form an *int-critical family* (of size m) if

$$\bigcap_{j\in[m]} \text{int}(C_j) \cap S = \emptyset \quad \text{and} \quad \bigcap_{j\in[m]\setminus\{i\}} \text{int}(C_j) \cap S \neq \emptyset \ \forall i \in [m].$$

The *int-Helly number* of S is

$$h^\circ(S) = \sup\{m : m \text{ is the size of an int-critical family for } S\}.$$

Since a family $\{C_1, \ldots, C_m\}$ is int-critical for S if and only if $\{\text{int}(C_1), \ldots, \text{int}(C_m)\}$ is a critical family for S, we have that $h^\circ(S) \leq h(S)$, and this inequality can be strict, as shown in an example in Section 3. If S is a nonempty subset of $\mathbb{R}^d$, the inequality $f(S) \leq h^\circ(S)$ can be proven using Averkov's arguments [2] for the proof of the inequality $f(S) \leq h(S)$.

The following theorem and corollary are analagous to Theorem 1 and Corollary 2, respectively.

Theorem 5 *Given a nonempty subset S of $\mathbb{R}^d$,*

$$h^\circ(S) = \sup\left\{|T| : T \subseteq S \textit{ finite}, \textstyle\bigcap_{t\in T} \text{conv}(T \setminus \{t\}) \cap \text{cl}(S) = \emptyset\right\}.$$

Proof. Let $T \subseteq S$ be finite and satisfy $\bigcap_{t\in T} \operatorname{conv}(T \setminus \{t\}) \cap \operatorname{cl}(S) = \emptyset$. We construct an int-critical family of size $|T|$.

For $\varepsilon > 0$, let $B(\varepsilon)$ be the closed ball of radius ε centered at the origin. We will use the fact that if a compact set and a closed set are disjoint then their distance is strictly positive. Note that the sets $\operatorname{conv}(T \setminus \{t\})$ for $t \in T$ are compact, the set $\operatorname{cl}(S)$ is closed, and the intersection of these sets is empty. Then the minimum distance between these $|T| + 1$ sets is some $\varepsilon > 0$. It follows that $\bigcap_{t\in T} \big(\operatorname{conv}(T \setminus \{t\}) + B(\varepsilon/2)\big) \cap \operatorname{cl}(S) = \emptyset$. Then the family $\{\operatorname{conv}(T \setminus \{t\}) + B(\varepsilon/2),\ t \in T\}$ is an int-critical family of size $|T|$.

Conversely, let $\{C_1, \dots, C_m\}$ be an int-critical family. Then $\bigcap_{j\in[m]} \operatorname{int}(C_j) \cap S = \emptyset$, which implies $\bigcap_{j\in[m]} \operatorname{int}(C_j)\cap\operatorname{cl}(S) = \emptyset$. Since $\{C_1, \dots, C_m\}$ is an int-critical family, for every $i \in [m]$ there exists an element $t_i \in \bigcap_{j\in[m]\setminus\{i\}} \operatorname{int}(C_j) \cap S$. Let $T = \{t_1, \dots, t_m\}$. As $\operatorname{conv}(T \setminus \{t_j\}) \subseteq \operatorname{int}(C_j)$ for $j \in [m]$ and $\bigcap_{j\in[m]} \operatorname{int}(C_j)\cap\operatorname{cl}(S) = \emptyset$, we have that $\bigcap_{j\in[m]} \operatorname{conv}(T\setminus\{t_j\})\cap\operatorname{cl}(S) = \emptyset$ as well. $\square$

Corollary 6 *Given a nonempty subset S of $\mathbb{R}^d$,*

$$h^\circ(S) = \sup\{m : m \textit{ is the size of an int-critical family for } S \textit{ consisting of closed half-spaces}\}.$$

Proof. By Theorem 5, it suffices to show the following:

Let T be a finite subset of S satisfying $\bigcap_{t\in T} \operatorname{conv}(T \setminus \{t\}) \cap \operatorname{cl}(S) = \emptyset$. Then there exists an int-critical family of closed half-spaces of size at least $|T|$.

For $t \in T$, let $\mathcal{H}_t$ be a finite family of closed half-spaces whose intersection is $\operatorname{conv}(T \setminus \{t\})$ and let $\mathcal{H} = \bigcup_{t\in T} \mathcal{H}_t$. We have that $\bigcap_{H\in\mathcal{H}} H \cap \operatorname{cl}(S) = \bigcap_{t\in T} \operatorname{conv}(T \setminus \{t\}) \cap \operatorname{cl}(S) = \emptyset$. Since $\bigcap_{H\in\mathcal{H}} H$ is a compact set and $\operatorname{cl}(S)$ is closed, for every $H \in \mathcal{H}$ there is a closed half-space H' strictly containing H such that $\bigcap_{H'\in\mathcal{H}'} \operatorname{int}(H') \cap S = \emptyset$, where $\mathcal{H}' = \{H' : H \in \mathcal{H}\}$. As every half-space in $\mathcal{H}'$ contains at least $|T| - 1$ points of T in its interior, the intersection of $|T| - 1$ of them contains a point of T (and thus of S) in its interior. Therefore the family $\mathcal{H}'$ of half-spaces contains an int-critical family of size at least $|T|$. $\square$

Since $\{C_1, \dots, C_m\}$ is an int-critical family if and only if $\{\operatorname{int}(C_1), \dots, \operatorname{int}(C_m)\}$ is a critical family, the above corollary implies that

$$h^\circ(S) = \sup\{m : m \text{ is the size of a critical family for } S \text{ consisting of open half-spaces}\}.$$

Furthermore, the following result follows immediately by comparing Theorems 1 and 5.

Corollary 7 *Given a nonempty subset S of $\mathbb{R}^d$ which is closed, we have that $h^\circ(S) = h(S)$.*

3 Bounding $f(S)$ in terms of $h^\circ(S)$

We first construct a set $S \subseteq \mathbb{R}^3$ which is not closed such that $f(S) = 1$ and $h(S) = \infty$. Let $S = \{x \in \mathbb{R}^3 : x_3 \le 0\} \setminus \{x \in \mathbb{R}^3 : x_3 = 0, x_1^2 + x_2^2 \le 1\}$. Since the half-space $H = \{x \in \mathbb{R}^3 : x_3 \ge 0\}$ is the only maximal S-free convex set, $f(S) = 1$. For every $n \ge 3$ we construct a critical family of size $n + 1$, thus showing $h(S) = \infty$. Let P be an n-gon whose vertices satisfy $x_3 = 0$, $x_1^2 + x_2^2 = 1$, and let $l_1, \dots, l_n$ be its edges. For $i = 1, \dots, n$ let C_i be the closed half-space containing P and whose boundary contains the set $l_i + \langle(0, 0, 1)\rangle$. Then $\{H, C_1, \dots, C_n\}$ is a critical family.

This example shows that when S is not closed, $f(S)$ cannot be bounded from below in terms of $h(S)$. However the following theorem bounds $h^\circ(S)$ in terms of $f(S)$.

Theorem 8 *Let S be a nonempty subset of $\mathbb{R}^d$ which is not dense in $\mathbb{R}^d$. If $f(S)$ is finite, then*

$$h^\circ(S) \leq (d+1)f(S).$$

Proof. By Corollary 6, it suffices to show the following:

Let $\{H_1,\ldots,H_m\}$ be an int-critical family of closed half-spaces. Then there exists a maximal S-free convex set that is a polyhedron with at least $\frac{m}{d+1}$ facets.

Assume first that $\bigcap_{i\in[m]} \operatorname{int}(H_i) = \emptyset$. Since $\bigcap_{i\in[m]\setminus\{j\}} \operatorname{int}(H_i) \neq \emptyset$ for all $j \in [m]$, $\{H_1,\ldots,H_m\}$ is also an int-critical family for $\mathbb{R}^d$. Since $h^\circ(\mathbb{R}^d) = h(\mathbb{R}^d) = d+1$, we have $m \leq d+1$. Since S is nonempty and not dense in $\mathbb{R}^d$, we have that $f(S) \geq 1$ and therefore there exists a maximal S-free polyhedron whose number of facets is at least $f(S) \geq 1 \geq \frac{m}{d+1}$, thus proving the result.

Assume now $\bigcap_{i\in[m]} \operatorname{int}(H_i) \neq \emptyset$, hence $P = \bigcap_{i\in[m]} H_i$ is a full-dimensional S-free polyhedron. For $i \in [m]$, let $a^i x \leq b_i$ be an inequality that defines H_i. Let P' be the polyhedron defined by the inequalities $a^i x \leq b'_i$, $i \in [m]$, where $b'_1,\ldots,b'_m$ are defined recursively as follows (see also Bell [10]): for $j = 1,\ldots,m$, let b'_j be the supremum of the values β such that the polyhedron defined by the system

$$\begin{aligned}
&a^i x \leq b'_i, \quad i = 1,\ldots,j-1,\\
&a^j x \leq \beta\\
&a^i x \leq b_i, \quad i = j+1,\ldots,m,
\end{aligned}$$

is S-free.

Claim. *P' is a full-dimensional S-free polyhedron with m facets. Furthermore, given any $j \in [m]$ and $\varepsilon > 0$, there is a point $s \in S$ such that $a^j s < b'_j + \varepsilon$ and $a^i s < b'_i$ for every $i \in [m] \setminus \{j\}$.*

Proof of claim. Since $\{H_1,\ldots,H_m\}$ is an int-critical family, $b'_1,\ldots,b'_m$ are all finite. Therefore P' is a full-dimensional S-free polyhedron with m facets. The last part of the claim follows from the definition of $b'_1,\ldots,b'_m$. ◇

Let Q be a maximal S-free convex set containing P'. Since $f(S)$ is finite, Q is a polyhedron. Let $\{c^k x \leq d_k,\ k \in [q]\}$ be the set of inequalities that are facet defining for Q and are supporting for P'.

For $k \in [q]$, let $F_k = P' \cap \{x \in \mathbb{R}^d : c^k x = d_k\}$. Since F_k is a nonempty face of P', by the theorem of Carathéodory for cones the system

$$\sum_{i\in[m]} u_i a^i = c^k, \quad \sum_{i\in[m]} u_i b'_i = d_k, \quad u \in \mathbb{R}^m_+ \tag{5}$$

admits a solution $u^k \in \mathbb{R}^m_+$ whose support has cardinality at most $d - \dim(F_k)$. Note that $u^k \neq 0$. We show below that for every $j \in [m]$ there exists $k \in [q]$ such that $u^k_j > 0$. By this, $dq \geq m$, which implies that the number of facets of Q is at least $\frac{m}{d} > \frac{m}{d+1}$.

Let $j \in [m]$ be fixed. As stated above, we prove that there exists $k \in [q]$ such that $u^k_j > 0$. By the claim, for every integer $t \geq 1$ there is a point $s^t \in S$ such that

$$\begin{aligned}
&a^j s^t < b'_j + 1/t\\
&a^i s^t < b'_i, \quad i \in [m] \setminus \{j\}.
\end{aligned}$$

Since Q is S-free, for every t there is a facet-defining inequality $\gamma^t x \le \delta_t$ for Q such that $\gamma^t s^t \ge \delta_t$. Then, as the number of facets of Q is finite, there is a facet-defining inequality $\gamma^* x \le \delta^*$ for Q such that $\gamma^* s^t \ge \delta^*$ for infinitely-many indices t. W.l.o.g., we assume that $\gamma^* s^t \ge \delta^*$ for every index $t \ge 1$.

We claim that the inequality $\gamma^* x \le \delta^*$ is supporting for P'. Assume by contradiction that this is not the case. Then the distance between P' and the hyperplane $\gamma^* x = \delta^*$ would be some $\varepsilon > 0$. Since $\gamma^* s^t \ge \delta^*$ for all t, the distance between every s^t and P' would be at least ε, a contradiction to the choice of the sequence $(s_t)_{t\in\mathbb{N}}$.

Therefore $\gamma^* x \le \delta^*$ is a facet-defining inequality for Q and a supporting inequality for P'. This means that $\gamma^* = c^k$ and $\delta^* = d_k$ for some $k \in [q]$ (up to scaling by a positive factor). To conclude, we show that $u^k_j > 0$.

Assume by contradiction that $u^k_j = 0$. Recall that $a^i s^t < b'_i$ for every $i \in [m] \setminus \{j\}$ and for every t. Since u^k solves system (5) and $u^k \ne 0$, we obtain $c^k s^t < d_k$ for every t. This is a contradiction, as $\gamma^* s^t \ge \delta^*$ for all t, and $(\gamma^*, \delta^*) = (c^k, d_k)$. □

Notice that the above proof shows that $h^\circ(S) = (d+1)f(S)$ if and only if there exists an int-critical family of closed half-spaces whose intersection has empty interior, and in this case $f(S) = 1$ and $h^\circ(S) = d + 1$.

Theorem 8 implies that $h^\circ(S) < h(S)$ for the example given at the beginning of Section 3 (indeed one can show that $h^\circ(S) = 4$ for that example).

Theorem 8 and Corollary 7 also imply the following:

Corollary 9 *Let S be a nonempty proper subset of $\mathbb{R}^d$ which is closed. If $f(S)$ is finite, then*

$$h(S) \le (d+1)f(S).$$

4 Free sums of polytopes and Cartesian products

Given polytopes $P_1 \subset \mathbb{R}^p$ and $P_2 \subset \mathbb{R}^q$, the *free sum* of P_1 and P_2 is the polytope in $\mathbb{R}^p \times \mathbb{R}^q$ defined as follows:

$$P_1 \oplus P_2 = \operatorname{conv}((P_1 \times \{0\}) \cup (\{0\} \times P_2)).$$

The following result can be found, e.g., in [3].

Remark 10 *Assume both P_1 and P_2 contain the origin in their interior and let $a^i_1 x_1 \le 1$, $i \in [m_1]$, $a^j_2 x_2 \le 1$, $j \in [m_2]$ be irredundant descriptions of P_1, P_2 respectively. Then*

a) The system $a^i_1 x_1 + a^j_2 x_2 \le 1$, $i \in [m_1]$, $j \in [m_2]$ provides an irredundant description of $P_1 \oplus P_2$.

b) If $F_1 \subsetneq P_1$ is a face of P_1 and $F_2 \subsetneq P_2$ is a face of P_2, then $F_1 \oplus F_2$ is a face of $P_1 \oplus P_2$ of dimension $\dim(F_1) + \dim(F_2) + 1$. Furthermore all faces of $P_1 \oplus P_2$ except $P_1 \oplus P_2$ itself arise this way.

The next theorem generalizes the equality $h(\mathbb{Z}^p \times \mathbb{R}^q) = 2^p(q+1)$ of Hoffman [15] and Averkov and Weismantel [4].

Theorem 11 *Let S be a nonempty subset of $\mathbb{R}^p$ which is discrete. Then*

$$h(S \times \mathbb{R}^q) = h(S)(q+1).$$

Proof. Averkov and Weismantel [4] show that $h(S \times \mathbb{R}^q) \leq h(S)(q+1)$. Therefore it suffices to prove $h(S \times \mathbb{R}^q) \geq h(S)(q+1)$.

Remark that if $\{C_1, \ldots, C_m\}$ is a critical family for S, the family $\{C_1 \times \mathbb{R}^q, \ldots, C_m \times \mathbb{R}^q\}$ is critical for $S \times \mathbb{R}^q$. Therefore if $h(S) = \infty$, then $h(S \times \mathbb{R}^q) = \infty$ as well. So we assume $h(S) < \infty$.

If S contains only one point, then $h(S) = 1$ and $S \times \mathbb{R}^q$ is equivalent to $\mathbb{R}^q$. Therefore $h(S \times \mathbb{R}^q) = h(S)(q+1) = q+1$ in this case.

We now assume that S contains more than one point. Given a critical family for S of size $h(S)$, we construct a critical family for $S \times \mathbb{R}^q$ of size $h(S)(q+1)$. Thus $h(S \times \mathbb{R}^q) \geq h(S)(q+1)$ and the theorem follows.

Since S is a discrete set containing more than one point and $h(S) < \infty$, there exists a maximal S-free convex set which is a polyhedron P satisfying the properties of Theorem 4.

We first assume $\operatorname{aff}(S) = \mathbb{R}^p$. Hence P is a full-dimensional polytope. By possibly translating S, we may assume that the origin is in the interior of P. Let Δ be a full-dimensional simplex in $\mathbb{R}^q$ containing the origin in its interior and let $Q = P \oplus \Delta \subseteq \mathbb{R}^p \times \mathbb{R}^q$.

Since $f(S) = h(S)$ as S is discrete [2], by Remark 10 a) Q has $h(S)(q+1)$ facets. We show that the half-spaces containing Q and defining its facets form an int-critical family for $S \times \mathbb{R}^q$.

Since $Q \subseteq P \times \mathbb{R}^q$ and P is S-free, Q is an $(S \times \mathbb{R}^q)$-free polytope. Therefore it suffices to show that for every facet F of Q the polyhedron Q_F, defined as the intersection of the half-spaces associated with the facets of Q distinct from F, contains a point of $S \times \mathbb{R}^q$ in its interior.

By Remark 10 b), $F = F_P \oplus F_\Delta$, where F_P and F_Δ are facets of P and Δ, respectively. Since Δ has $q+1$ facets, by Remark 10 a) $F_P \times \{0\}$ is contained in $q+1$ facets of Q. Let $(a,0)x \leq b$ be an inequality that supports $F_P \times \{0\}$ and let $c^i x \leq d_i$, $i \in [q+1]$ be inequalities that define the facets of Q that contain $F_P \times \{0\}$, where $c^{q+1}x \leq d_{q+1}$ is the inequality that defines F.

Since $F_P \times \{0\}$ is contained in $q+1$ facets of Q, the system $(a,0) = \sum_{i=1}^{q+1} \lambda_i c^i$ admits a unique solution, say $\bar{\lambda}$, and $\bar{\lambda}_i > 0$ for $i \in [q+1]$. Hence the system $(a,0) = \sum_{i=1}^{q} \lambda_i c^i$ is infeasible. This shows that the inequality $(a,0)x \leq b$ does not define a face of Q_F. Hence the set $\{x \in \operatorname{int}(Q_F) : (a,0)x = b\}$ has dimension $p+q-1$.

As P satisfies the properties of Theorem 4, $\operatorname{relint}(F_P) \cap S$ contains a point $\bar{s}$. Since $(a,0)\begin{pmatrix} \bar{s} \\ y \end{pmatrix} = b$ for every $y \in \mathbb{R}^q$ and the set $\{x \in \operatorname{int}(Q_F) : (a,0)x = b\}$ has dimension $p+q-1$, we have that

$$\operatorname{int}(Q_F) \cap \left\{ \begin{pmatrix} \bar{s} \\ y \end{pmatrix} : (a,0)\begin{pmatrix} \bar{s} \\ y \end{pmatrix} = b \right\} \neq \emptyset.$$

This concludes the proof in the case $\operatorname{aff}(S) = \mathbb{R}^p$.

Assume now that $\operatorname{aff}(S) \subsetneq \mathbb{R}^p$ and let P_0 be a polytope satisfying the properties of Theorem 4. By considering $\operatorname{aff}(S) \times \mathbb{R}^q$ as the ambient space, the argument for the case $\operatorname{aff}(S) = \mathbb{R}^p$ shows that $P_0 \oplus \Delta$ is a full-dimensional polytope and the half-spaces supporting its facets form an int-critical family. Hence, in the ambient space $\mathbb{R}^p \times \mathbb{R}^q$, the half-spaces supporting the facets of $(P_0 \oplus \Delta) + (\mathrm{aff}(S)^* \times \{0\})$, where 0 is the origin in $\mathbb{R}^q$, form an int-critical family. $\square$

We now consider the Helly number of the Cartesian product of two discrete sets.

Theorem 12 *Let $S_1 \subset \mathbb{R}^p$ and $S_2 \subset \mathbb{R}^q$ be nonempty subsets which are both discrete. Then*

$$h(S_1 \times S_2) \geq h(S_1)h(S_2).$$

Proof. Remark that if $\{C_1, \ldots, C_m\}$ is a critical family for S_1, then $\{C_1 \times \mathbb{R}^q, \ldots, C_m \times \mathbb{R}^q\}$ is a critical family for $S_1 \times S_2$. Hence if $h(S_1) = \infty$, then $h(S_1 \times S_2) = \infty$ as well. Similarly, if $h(S_2) = \infty$ then $h(S_1 \times S_2) = \infty$. Therefore we assume that $h(S_1)$ and $h(S_2)$ are both finite.

If S_1 contains only one point, then $h(S_1) = 1$ and $S_1 \times S_2$ is equivalent to S_2. Therefore $h(S_1 \times S_2) = h(S_1)h(S_2)$ in this case. A similar argument can be used if S_2 contains only one point.

We now assume that each of S_1, S_2 contains more than one point. Since $h(S_1)$, $h(S_2)$ are both finite and S_1, S_2 are both discrete, there exist maximal S_1-free and S_2-free convex sets which are polyhedra P_1 and P_2 satisfying the properties of Theorem 4.

We now assume $\mathrm{aff}(S_1) = \mathbb{R}^p$ and $\mathrm{aff}(S_2) = \mathbb{R}^q$. Hence P_1 and P_2 are full-dimensional polytopes. We may assume that both P_1 and P_2 contain the origin in their interior, thus P_1 is defined by an irredundant system $a_1^i x_1 \leq 1,\ i \in [m_1]$, where $m_1 = h(S_1)$, and P_2 is defined by an irredundant system $a_2^j x_2 \leq 1,\ j \in [m_2]$, where $m_2 = h(S_2)$.

Consider the polytope $Q = \{(x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^q : \frac{1}{2}(x_1, x_2) \in P_1 \oplus P_2\}$. By Lemma 10 a), Q has $h(S_1)h(S_2)$ facets. In the following we show that Q is a maximal $(S_1 \times S_2)$-free convex set. This implies that $f(S_1 \times S_2) \geq h(S_1)h(S_2)$, and since $S_1 \times S_2$ is discrete, we conclude that $h(S_1 \times S_2) = f(S_1 \times S_2) \geq h(S_1)h(S_2)$.

We first show that Q is an $(S_1 \times S_2)$-free convex set. By Remark 10 a), the system $a_1^i x_1 + a_2^j x_2 \leq 2,\ i \in [m_1],\ j \in [m_2]$ provides an irredundant description of Q. Let $(s_1, s_2) \in S_1 \times S_2$. Since P_1 is an S_1-free convex set, $a_1^i s_1 \geq 1$ for some $i \in [m_1]$; similarly, $a_2^j s_2 \geq 1$ for some $j \in [m_2]$. Therefore $a_1^i s_1 + a_2^j s_2 \geq 2$. This shows that $(s_1, s_2) \notin \mathrm{int}(Q)$ and therefore Q is an $(S_1 \times S_2)$-free convex set.

In order to show that Q is a maximal S-free convex set, it suffices to show that the relative interior of every facet of Q contains a point of $S_1 \times S_2$. Let F be a facet of Q. By Remark 10 a), there exist $\hat{\imath} \in [m_1]$, $\hat{\jmath} \in [m_2]$ such that $F = \{x \in Q : a_1^{\hat{\imath}} x_1 + a_2^{\hat{\jmath}} x_2 = 2\}$. Since P_1 satisfies the conditions of Theorem 4, S_1 contains a point, say s_1, such that $a_1^{\hat{\imath}} s_1 = 1$ and $a_1^i s_1 < 1$ for all $i \in [m_1] \setminus \{\hat{\imath}\}$. Similarly, S_2 contains a point, say s_2, such that $a_2^{\hat{\jmath}} s_2 = 1$ and $a_2^j s_2 < 1$ for all $j \in [m_2] \setminus \{\hat{\jmath}\}$. Therefore $(s_1, s_2) \in (S_1 \times S_2)$ satisfies $a_1^{\hat{\imath}} s_1 + a_2^{\hat{\jmath}} s_2 = 2$ and $a_1^i s_1 + a_2^j s_2 < 2$ for every pair $(i, j) \in [m_1] \times [m_2] \setminus \{(\hat{\imath}, \hat{\jmath})\}$. This shows that (s_1, s_2) is in the relative interior of F.

This concludes the proof of the theorem in the case $\mathrm{aff}(S_1) = \mathbb{R}^p$ and $\mathrm{aff}(S_2) = \mathbb{R}^q$. We now assume that $\mathrm{aff}(S_1) \subsetneq \mathbb{R}^p$ or $\mathrm{aff}(S_2) \subsetneq \mathbb{R}^q$. By considering $\mathrm{aff}(S_1) \times \mathrm{aff}(S_2)$ as the ambient space, the above argument shows that the polytope Q constructed as above is a maximal $(S_1 \times S_2)$-free convex set in $\mathrm{aff}(S_1) \times \mathrm{aff}(S_2)$. Hence in the ambient space $\mathbb{R}^p \times \mathbb{R}^q$, the polyhedron $Q + (\mathrm{aff}(S_1)^* \times \mathrm{aff}(S_2)^*)$ is a maximal $(S_1 \times S_2)$-free convex set. $\square$

The special case of Theorem 12 when S_1 and S_2 are translated lattices follows from [3, Theorem 5.3]. We also remark that Averkov and Weismantel [4] construct 1-dimensional discrete sets S_1, S_2 such that $h(S_1 \times S_2) > h(S_1)h(S_2)$. Finally, note that the inequality of Theorem 12 is tight, e.g., for $S_1 = \mathbb{Z}^p$ and $S_2 = \mathbb{Z}^q$.

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