MIXING SETS LINKED BY BI-DIRECTED PATHS

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Abstract. Recently there has been considerable research on simple mixed-integer sets, called mixing sets, and closely related sets arising in uncapacitated and constant capacity lot-sizing. This in turn has led to study of more general sets, called network-dual sets, for which it is possible to derive extended formulations whose projection gives the convex hull of the network-dual set. Unfortunately this formulation cannot be used (in general) to optimize in polynomial time. Furthermore the inequalities defining the convex hull of a network-dual set in the original space of variables are known only for some special cases.

Here we study two new cases, in which the continuous variables of the network-dual set are linked by a bi-directed path. In the first case, which is motivated by lot-sizing problems with (lost) sales, we provide a description of the convex hull as the intersection of the convex hulls of 2^n mixing sets, where n is the number of continuous variables of the set. However optimization is polynomial as only $n+1$ of the sets are required for any given objective function. In the second case, generalizing single arc flow sets, we describe again the convex hull as the intersection of an exponential number of mixing sets and also give a combinatorial polynomial-time separation algorithm.

Key words. Mixed-integer programming, mixing sets, convex hull descriptions, lot-sizing.

AMS subject classifications. 90C11, 90C57.

1. Introduction.

In the last 10–15 years there has been an increasing interest in the polyhedral study of simple-structured mixed-integer sets, for which several authors have derived convex hull descriptions, cutting planes and separation algorithms. This kind of research is motivated both by the theoretical interest in having as deep an understanding as possible of the polyhedral structure of simple mixed-integer sets, and by the fact that these sets often arise as substructures or strong relaxations of practical problems, such as fixed-charge flow problems and lot-sizing models [20].

One of the most basic mixed-integer sets studied in the recent literature is the *mixing set*

$$MIX^{>} = \{(s, x) \in \mathbb{R}_+ \times \mathbb{Z}^m : s - x_i \geq b_i,\ 1 \leq i \leq m\},$$

which was introduced by Günlük and Pochet [12] as an abstraction of some single-item lot-sizing models. Günlük and Pochet [12] gave a linear-inequality description of the convex hull of this set consisting of an exponential number of facet-defining inequalities, which can be separated in polynomial time [12, 19, 20].

Among the numerous variants of the mixing set that were studied recently [3, 4, 5, 7, 8, 9, 11, 13, 21, 22], there are a number of models (e.g., those appearing in [3, 4, 7, 8, 21] as well as the mixing set itself) that, under a simple change of variables, belong to a family of mixed-integer sets studied by Conforti et al. [2], which we refer to as *network-dual sets*. A network-dual set is a mixed-integer set of the form

$$N = \{(u, v) \in \mathbb{R}^p \times \mathbb{Z}^q : Au + Bv \leq d\}, \tag{1}$$

where $[A \mid B]$ is a network-dual matrix, i.e., the transpose of a network-flow matrix. In other words, each row of $[A \mid B]$ has at most one $+1$ and at most one -1, and all other

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entries are equal to zero. Note that ignoring the rows with a single nonzero entry, $[A \mid B]$ is the arc-node incidence matrix of a directed graph, whose nodes are called *continuous* or *integer* depending on whether the corresponding variable is continuous or integer.

Though Conforti et al. [2] provided a linear-inequality description for the convex hull of any network-dual set by using additional variables (extended formulation), this description is not (in general) of polynomial size, and thus it cannot be used to optimize in polynomial time. Furthermore, a linear-inequality description in the original variables is available only for some special cases [4, 6, 8, 10, 21]. In particular, thanks to the results of [2] and [8], such a description is known whenever no row of A contains two nonzero entries, i.e., no inequality involving two continuous variables appears in the definition of N: in this case the convex hull of N is obtained by intersecting the convex hulls of a small number of mixing sets.

In this paper we explore what happens when inequalities involving two continuous variables are part of the description of N, at least for the special case in which A is the arc-node incidence matrix of a *bi-directed path*, i.e., a digraph consisting of a directed path plus the same path with all the arcs reversed.

The rest of the paper is organized as follows. In §2 we recall some results concerning mixing sets and network-dual sets. In §3 we consider a network-dual set (1) where (i) matrix A defines a bi-directed path and (ii) the arcs linking continuous nodes with integer nodes are either all oriented from the continuous node to the integer node or all oriented the other way round. We show that the convex hull of this set is given by the intersection of the convex hulls of an exponential number of mixing sets, each obtained as a relaxation of the original set. However optimization is polynomial, as only a small number of mixing sets are required for any given objective function. We also point out that this set models a single-item discrete lot-sizing problem with sales.

In §4 we consider a network-dual set (1) in which (i) matrix A defines a bi-directed path P and (ii) the arcs linking a continuous node to an integer node can now be oriented arbitrarily, but they all have the last node of P as one of their endpoints. We describe the convex hull of this set again as the intersection of the convex hulls of an exponential number of mixing sets, and we show that optimization is polynomial also for this set. In this case we also show how the inequalities describing the convex hull can be separated in polynomial time.

Finally, we conclude in §5 by discussing some open questions.

Throughout the paper we use the following notation. Given a nonnegative integer n, we define $[n] = \{1, \dots, n\}$, with $[n] = \varnothing$ if $n = 0$. Given a vector a with indices in $[n]$ and a subset $T \subseteq [n]$, we define $a(T) = \sum_{k \in T} a_t$. When $T = \{i, i+1, \dots, j\}$, we sometimes write $a_{i,j}$ instead of $a(T)$. In other words, $a_{i,j} = \sum_{k=i}^{j} a_k$.

2. Mixing sets and network-dual sets. In this section we recall some results concerning mixing sets and network-dual sets.

The *mixing set* $MIX^>$ is defined as the following mixed-integer set:

$$MIX^> : \quad \begin{aligned} s - x_i &\geq b_i,\ i \in [m], && (2)\\ s &\geq 0, && (3)\\ x_i &\in \mathbb{Z}, \quad i \in [m], && (4) \end{aligned}$$

for some rational numbers $b_1, \dots, b_m$. This set was introduced formally by Günlük and Pochet [12]. (We note that in the standard mixing set, inequality (2) is usually written in the form $s + x_i \geq b_i$; however this is just a change of the sign of the integer variables.) The following result gives a linear-inequality description for the convex hull of $MIX^>$, denoted $\operatorname{conv}(MIX^>)$.

PROPOSITION 2.1. [12] *Define* $f_i = b_i - (\lceil b_i \rceil - 1)$*. The polyhedron* $\operatorname{conv}(MIX^>)$ *is described by* $s \geq 0$ *and the two families of* mixing inequalities

$$s - \sum_{r=1}^{q} (f_{i_r} - f_{i_{r-1}})(x_{i_r} + \lceil b_{i_r} \rceil) \geq 0, \tag{5}$$

$$s - \sum_{r=1}^{q} (f_{i_r} - f_{i_{r-1}})(x_{i_r} + \lceil b_{i_r} \rceil) - (1 - f_{i_q})(x_{i_1} + \lceil b_{i_1} \rceil - 1) \geq 0, \tag{6}$$

for all sequences of indices $i_1, \ldots, i_q$ *such that* $f_{i_1} \leq \cdots \leq f_{i_q}$*, with* $f_{i_0} = 0$*.*

When inequality $s \geq 0$ is omitted in the definition of $MIX^>$, the convex hull is given only by (6).

By Proposition 2.1, the polyhedron $\operatorname{conv}(MIX^>)$ is described by an exponential number of inequalities. However, inequalities (5)–(6) can be separated in time $\mathcal{O}(m \log m)$ (this was first observed in [19] for the case of binary x-variables and later extended to the case of general integer x-variables [12, 20]). Furthermore, Miller and Wolsey [17] gave a tight extended formulation for $\operatorname{conv}(MIX^>)$ with $\mathcal{O}(m)$ variables and constraints.

If one defines the *reversed mixing set* $MIX^<$ by the constraints

$$MIX^< : \quad \begin{aligned} & s - y_j \leq c_j, \ j \in [n], \\ & s \leq u, \\ & y_j \in \mathbb{Z}, \quad j \in [n], \end{aligned}$$

for rational numbers $c_1, \ldots, c_n, u$, then it is clear that, under a simple change of variables, this set is essentially equivalent to a mixing set (2)–(4). It follows that the convex hull of the above set is also described by mixing inequalities.

We call *generalized mixing set* a combination of the two sets given above, namely a set $GMIX$ of the form

$$GMIX : \quad \begin{aligned} & s - x_i \geq b_i, \ i \in [m], \\ & s - y_j \leq c_j, \ j \in [n], \\ & l \leq s \leq u, \\ & x_i, y_j \in \mathbb{Z}, \ \ i \in [m], \ j \in [n]. \end{aligned}$$

As shown in [8], the convex hull of the above set is given by the intersection of the convex hulls of the sets $MIX^>$ and $MIX^<$, plus some simple linear constraints on the integer variables.

PROPOSITION 2.2. [8]

$$\operatorname{conv}(GMIX) = \operatorname{conv}(MIX^>) \cap \operatorname{conv}(MIX^<) \cap Q,$$

where Q *is the polyhedron defined by the following inequalities:*

$$-x_i \geq \lceil b_i - u \rceil, \quad i \in [m], \tag{7}$$

$$-y_j \leq \lfloor c_j - l \rfloor, \quad j \in [n], \tag{8}$$

$$y_j - x_i \geq \lceil b_i - c_j \rceil, \ i \in [m], \ j \in [n]. \tag{9}$$

A very similar result holds if one or both bounds on s are omitted in $GMIX$: if no lower (resp., upper) bound on s is given, then (8) (resp., (7)) disappears. We also remark that inequalities (7)–(9) describe the projection of $\text{conv}(GMIX)$ onto the (x, y)-space.

The sets $MIX^>$, $MIX^<$ and $GMIX$ are special cases of a larger family of sets studied by Conforti et al. [2], namely the family of all mixed-integer sets of the form

$$N = \{(u, v) \in \mathbb{R}^p \times \mathbb{Z}^q : Au + Bv \leq d\}, \tag{10}$$

where $[A \mid B]$ is a network-dual matrix, i.e., the transpose of a network-flow matrix. In other words, each row of $[A \mid B]$ has at most one $+1$ and at most one -1, and all other entries are equal to zero. We refer to sets of this type as *network-dual sets.* As mentioned in §1, several sets studied in the recent literature [3, 4, 7, 8, 12, 21], most of which have applications in production planning, can be transformed into network-dual sets.

Conforti et al. [2] gave an extended formulation for the convex hull of any network-dual set. The particular form of the extended formulation easily implies the following result.

PROPOSITION 2.3. [2] *Let $N = \{(u, v) \in \mathbb{R}^p \times \mathbb{Z}^q : Au + Bv \leq d\}$ be a network-dual set and let $Dv \leq \beta$ be a linear system involving only the integer variables, where D is a network-dual matrix and β is an integer vector. Then*

$$\text{conv}(N \cap \{(u, v) : Dv \leq \beta\}) = \text{conv}(N) \cap \{(u, v) : Dv \leq \beta\}.$$

Given a network-dual set (10) and assuming that one is looking for a linear-inequality description of $\text{conv}(N)$, Proposition 2.3 implies that one can assume the following without loss of generality.

(i) System $Au + Bv \leq d$ does not contain any inequality involving only integer variables (in other words, A does not have any all-zero row). Otherwise, if some inequalities of this type appear in the system, one can remove them, find the convex hull of the resulting set and then put back the inequalities that have been removed with their right-hand sides rounded up.

(ii) Every integer variable appears with nonzero coefficient in at most one inequality of system $Au + Bv \leq d$. Otherwise, if an integer variable v_t appears in two inequalities, let N' be the set obtained from N by replacing one of the two occurrences of v_t with a new integer variable v'_t. Then N is equivalent to the set $N' \cap \{(u, v, v'_t) : v_t - v'_t = 0\}$. Since, by Proposition 2.3, $\text{conv}(N' \cap \{(u, v, v'_t) : v_t - v'_t = 0\}) = \text{conv}(N') \cap \{(u, v, v'_t) : v_t - v'_t = 0\}$, it is sufficient to find a linear-inequality description for $\text{conv}(N')$ and then identify variables v_t and v'_t.

(iii) No inequality of system $Au + Bv \leq d$ involves only one variable. Otherwise, it is easy to introduce a dummy integer variable v_0 in such a way that all the inequalities involve two variables. If the resulting set is called N', then N is equivalent to the set $N' \cap \{(u, v, v_0) : v_0 = 0\}$. Since, by Proposition 2.3, $\text{conv}(N' \cap \{(u, v, v_0) : v_0 = 0\}) = \text{conv}(N') \cap \{(u, v, v_0) : v_0 = 0\}$, it is sufficient to find a linear-inequality description for $\text{conv}(N')$ and then remove variable v_0.

Altogether, the above observations show that one can always assume that $[A \mid B]$ is the arc-node incidence matrix of a digraph in which there is no arc linking two integer nodes, and all the integer nodes have degree one.

In the particular case in which there is in addition no arc linking two continuous nodes, a network-dual set can be written as follows:

$$s_t - x_i^t \geq b_i^t,\ t \in [\ell],\ i \in [m_t], \tag{11}$$
$$s_t - y_j^t \leq c_j^t,\ t \in [\ell],\ j \in [n_t], \tag{12}$$
$$x_i^t, y_j^t \in \mathbb{Z},\quad t \in [\ell],\ i \in [m_t],\ j \in [n_t]. \tag{13}$$

For each fixed $t \in [\ell]$, the above is a generalized mixing set without bounds on the continuous variables. Therefore (11)–(13) is the intersection of ℓ generalized mixing sets defined on disjoint sets of variables, and thus its convex hull is simply given by the intersection of the convex hulls of these ℓ generalized mixing sets. Then a linear-inequality description for the convex hull of (11)–(13) follows immediately.

To study a totally general network-dual set, one has to consider the intersection of generalized mixing sets (11)–(13) plus network-dual inequalities linking the continuous variables. In this paper we address this study by focusing on some special cases. In particular, we assume that the continuous variables are linked by a bi-directed path. In other words, we consider a network-dual set of the type

$$s_t - x_i^t \geq b_i^t, \qquad t \in [\ell],\ i \in [m_t], \tag{14}$$
$$s_t - y_j^t \leq c_j^t, \qquad t \in [\ell],\ j \in [n_t], \tag{15}$$
$$l_t \leq s_t - s_{t-1} \leq u_t,\ t \in [\ell], \tag{16}$$
$$x_i^t, y_j^t \in \mathbb{Z}, \qquad t \in [\ell],\ i \in [m_t],\ j \in [n_t], \tag{17}$$

with $s_0 = 0$.

Since, as illustrated later in §5, finding a linear-inequality description for (14)–(17) seems to be hard in general, we will consider two special cases in §§3–4.

3. Mixing sets linked by a bi-directed path.

3.1. The convex hull. Here we consider the case of a network-dual set obtained as the intersection of mixing sets of the type $MIX^>$, with the continuous variables linked by a bi-directed path. In other words, we study a set of the form (14)–(17) where there is no inequality (15):

$$s_t - x_i^t \geq b_i^t, \qquad t \in [\ell],\ i \in [m_t], \tag{18}$$
$$l_t \leq s_t - s_{t-1} \leq u_t,\ t \in [\ell], \tag{19}$$
$$x_i^t \in \mathbb{Z}, \qquad t \in [\ell],\ i \in [m_t]. \tag{20}$$

We initially assume that all of the constraints (19) are part of the system, and we will discuss later how the formulation changes when only some of them are enforced, i.e $u_t = +\infty$ and/or $l_t = -\infty$ for one or several t. We assume that $l_t \leq u_t$ for $t \in [\ell]$, as otherwise there is no feasible solution.

Under the change of variables $\sigma_t = s_t - s_{t-1}$ for $t \in [\ell]$, (18)–(20) takes the form

$$X: \quad \begin{aligned} &\sigma_{1,t} - x_i^t \geq b_i^t,\ t \in [\ell],\ i \in [m_t], && (21)\\ &l_t \leq \sigma_t \leq u_t,\ \ t \in [\ell], && (22)\\ &x_i^t \in \mathbb{Z}, \quad t \in [\ell],\ i \in [m_t]. && (23) \end{aligned}$$

Let X denote the set defined by (21)–(23). For each $\varnothing \neq T \subseteq [\ell]$ the following set X_T is a valid relaxation for X:

$$X_T: \quad \begin{aligned} &\sigma(T) - x_i^t \geq b_i^t + l(T \setminus [t]) - u([t] \setminus T),\ t \in [\ell],\ i \in [m_t], &(24)\\ &\sigma(T) \geq l(T), &(25)\\ &x_i^t \in \mathbb{Z}, \qquad t \in [\ell],\ i \in [m_t]. &(26)\end{aligned}$$

Constraint (24) is valid for X because it is obtained by summing (21) with inequalities $\sigma_k \geq l_k$ for $k \in T \setminus [t]$ and $-\sigma_k \geq -u_k$ for $k \in [t] \setminus T$.

Since $\sigma(T)$ can be treated as a single continuous variable in (24)–(26), each relaxation X_T is essentially a mixing set, and thus a linear-inequality description for its convex hull is known (see Proposition 2.1).

When $T = \varnothing$, a similar relaxation can be constructed:

$$\begin{aligned} -x_i^t &\geq b_i^t - u_{1,t},\ t \in [\ell],\ i \in [m_t],\\ x_i^t &\in \mathbb{Z}, \qquad t \in [\ell],\ i \in [m_t]. \end{aligned}$$

This is not a mixing set, as there is no continuous variable. The convex hull of the above set is obviously described by the inequalities

$$Q: \qquad -x_i^t \geq \lceil b_i^t - u_{1,t} \rceil,\ t \in [\ell],\ i \in [m_t]. \tag{27}$$

We denote by Q the polyhedron defined by (27). It is immediate to see that Q is the projection of $\mathrm{conv}(X)$ onto the x-space.

The next proposition shows that by taking the convex hulls of all the relaxations X_T, along with inequalities (27) and the original upper bounds on the continuous variables, one finds the convex hull of (21)–(23).

Proposition 3.1.

$$\mathrm{conv}(X) = \bigcap_{\varnothing \neq T \subseteq [\ell]} \mathrm{conv}(X_T) \cap Q \cap \{(\sigma, x) : \sigma_t \leq u_t,\ t \in [\ell]\}. \tag{28}$$

Proof. Let P be the polyhedron on the right-hand side of equality (28). It is clear that $\mathrm{conv}(X) \subseteq P$. Since $\mathrm{conv}(X)$ and P have the same rays, to prove that $P \subseteq \mathrm{conv}(X)$ we proceed as follows: we take any linear objective function $p\sigma + qx$ such that the optimization problem $\min\{p\sigma + qx : (\sigma, x) \in X\}$ has finite optimum, and show that then the problem

$$\min\{p\sigma + qx : (\sigma, x) \in P\} \tag{29}$$

has an optimal solution that belongs to X.

We first assume that $p \geq \mathbf{0}$ and then consider the case in which some entries of p are negative.

Case 1: $p \geq 0$.

Let $t_1, \dots, t_\ell$ be a reordering of the elements in $[\ell]$ such that $0 =: p_{t_0} \leq p_{t_1} \leq \cdots \leq p_{t_\ell}$, and for $h \in [\ell]$ define $T_h = \{t_h, t_{h+1}, \dots, t_\ell\}$. In order to show that problem (29) has an optimal solution belonging to X, we prove that the relaxed linear program

$$\min\left\{p\sigma + qx : (\sigma, x) \in \bigcap_{h \in [\ell]} \mathrm{conv}(X_{T_h}) \cap Q\right\} \tag{30}$$

has an optimal solution that belongs to X.

Under the change of variables $\rho_h = \sigma(T_h)$ for $h \in [\ell]$, problem (30) takes the form

$$\min\left\{ \sum_{h\in[\ell]} (p_{t_h} - p_{t_{h-1}})\rho_h + qx : (\rho, x) \in \bigcap_{h\in[\ell]} \mathrm{conv}(Z_{T_h}) \cap Q \right\}, \tag{31}$$

where the sets Z_{T_h} are defined as follows:

$$Z_{T_h}: \quad \begin{aligned} \rho_h - x_i^t &\geq b_i^t + l(T_h \setminus [t]) - u([t] \setminus T_h), \ t \in [\ell],\ i \in [m_t], && (32)\\ \rho_h &\geq l(T_h), && (33)\\ x_i^t &\in \mathbb{Z}, \qquad t \in [\ell],\ i \in [m_t]. && (34) \end{aligned}$$

All the extreme points of the feasible region of problem (31) have integer x-components, as this polyhedron is the intersection of mixing sets defined on disjoint sets of variables, plus some bounds on the integer variables (see Proposition 2.3). It follows that problem (31) has an optimal solution $(\bar\rho, \bar x)$ with $\bar x$ integer. Since the coefficients of variables $\rho_1, \dots, \rho_\ell$ in the objective function are all nonnegative, we can assume that $\bar\rho_1, \dots, \bar\rho_\ell$ are minimal.

In the three claims below, we prove that the point $(\bar\sigma, \bar x)$ that corresponds to $(\bar\rho, \bar x)$ under the change of variables satisfies (21)–(23). For this purpose, define $\beta_i^t = \bar x_i^t + b_i^t$ for $t \in [\ell]$ and $i \in [m_t]$. In order to reduce the number of cases that need to be analyzed, we would like to be able to treat constraints (32)–(33) as a single family of inequalities. To do so, it is convenient to define $\beta_1^0 = 0$ and $m_0 = 1$. Then (32)–(33) evaluated at $(\bar\rho, \bar x)$ give the following single family of inequalities:

$$\bar\rho_h \geq \beta_i^t + l(T_h \setminus [t]) - u([t] \setminus T_h),\ h \in [\ell],\ t \in \{0\} \cup [\ell],\ i \in [m_t]. \tag{35}$$

Also note that (27) implies that

$$\beta_i^t \leq u_{1,t},\ t \in \{0\} \cup [\ell],\ i \in [m_t]. \tag{36}$$

Claim 1: $\bar\sigma_{t_h} \geq l_{t_h}$ for $h \in [\ell]$.
Proof of claim. If $h = \ell$, the inequality to be verified is $\bar\rho_\ell \geq l_{t_\ell}$. However this condition is clearly satisfied, as it is included in (35) (with $h = \ell$ and $t = 0$). So we assume $h < \ell$. Then the inequality to be verified is $\bar\rho_h - \bar\rho_{h+1} \geq l_{t_h}$. By the minimality of $\bar\rho_{h+1}$, we have $\bar\rho_{h+1} = \beta_i^t + l(T_{h+1} \setminus [t]) - u([t] \setminus T_{h+1})$ for some indices t and i. Together with inequality $\bar\rho_h \geq \beta_i^t + l(T_h \setminus [t]) - u([t] \setminus T_h)$, this implies that

$$\bar\rho_h - \bar\rho_{h+1} \geq \begin{cases} l_{t_h} & \text{if } t_h > t,\\ u_{t_h} & \text{otherwise.} \end{cases}$$

Thus $\bar\rho_h - \bar\rho_{h+1} \geq l_{t_h}$ in all cases.

Claim 2: $\bar\sigma_{t_h} \leq u_{t_h}$ for $h \in [\ell]$.
Proof of claim. If $h = \ell$, the inequality is $\bar\rho_\ell \leq u_{t_\ell}$. By the minimality of $\bar\rho_\ell$, we have $\bar\rho_\ell = \beta_i^t + l(T_\ell \setminus [t]) - u([t] \setminus T_\ell)$ for some indices t and i, i.e.,

$$\bar\rho_\ell = \begin{cases} \beta_i^t + l_{t_\ell} - u_{1,t} & \text{if } t_\ell > t,\\ \beta_i^t + u_{t_\ell} - u_{1,t} & \text{otherwise.} \end{cases}$$

Inequality (36) then implies that $\bar\rho_\ell \leq u_{t_\ell}$. So we assume $h < \ell$. Then the inequality to be checked is $\bar\rho_h - \bar\rho_{h+1} \leq u_{t_h}$. By the minimality of $\bar\rho_h$, we have $\bar\rho_h = \beta_i^t +$

$l(T_h \setminus [t]) - u([t] \setminus T_h)$ for some indices t and i. Together with inequality $\bar\rho_{h+1} \geq \beta_i^t + l(T_{h+1} \setminus [t]) - u([t] \setminus T_{h+1})$, this implies that

$$\bar\rho_h - \bar\rho_{h+1} \leq \begin{cases} l_{t_h} & \text{if } t_h > t, \\ u_{t_h} & \text{otherwise.} \end{cases}$$

Thus $\bar\rho_h - \bar\rho_{h+1} \leq u_{t_h}$ in all cases.

CLAIM 3: $\bar\sigma_{1,t} - \bar x_i^t \geq b_i^t$ for $t \in [\ell]$ and $i \in [m_t]$.

Proof of claim. Given $k \in [\ell]$ we define h_k as the unique index $h \in [\ell]$ such that $t_h = k$. In other words, the two mappings $h \mapsto t_h$ and $k \mapsto h_k$ are inverse of each other. Then the inequality that we want to check can be written as

$$\sum_{k \in [t]} (\bar\rho_{h_k} - \bar\rho_{h_k+1}) \geq \beta_i^t. \tag{37}$$

We prove (37) by induction on t.

Let $t \in [\ell]$ and $i \in [m_t]$ be fixed. If $\bar\rho_{h_k} - \bar\rho_{h_k+1} = u_k$ for all $k \in [t]$, then $\sum_{k\in[t]}(\bar\rho_{h_k} - \bar\rho_{h_k+1}) = u_{1,t} \geq \beta_i^t$ by (36), and inequality (37) is satisfied. Therefore we assume that $\bar\rho_{h_k} - \bar\rho_{h_k+1} < u_k$ for at least one index $k \in [t]$, and we define π as the index such that $h_\pi = \min_{k\in[t]}\{h_k : \bar\rho_{h_k} - \bar\rho_{h_k+1} < u_k\}$, where $\bar\rho_{\ell+1} = 0$.

If $h_\pi < \ell$, by the minimality of $\bar\rho_{h_\pi+1}$ we have

$$\bar\rho_{h_\pi+1} = \beta_j^\tau + l(T_{h_\pi+1} \setminus [\tau]) - u([\tau] \setminus T_{h_\pi+1}) \tag{38}$$

for some indices τ and j. We claim that $\pi > \tau$. To see this, observe that since $\bar\rho_{h_\pi} \geq \beta_j^\tau + l(T_{h_\pi} \setminus [\tau]) - u([\tau] \setminus T_{h_\pi})$ and since $T_{h_\pi} = T_{h_\pi+1} \cup \{t_{h_\pi}\} = T_{h_\pi+1} \cup \{\pi\}$, condition $\pi \leq \tau$ would imply $\bar\rho_{h_\pi} - \bar\rho_{h_\pi+1} \geq u_\pi$, contradicting the definition of π. Thus $\pi > \tau$. If $h_\pi = \ell$ instead, we define $\tau = 0$, so that (38) still holds.

Now, using $\tau < \pi \leq t$, inequality

$$\bar\rho_{h_\pi} \geq \beta_i^t + l(T_{h_\pi} \setminus [t]) - u([t] \setminus T_{h_\pi})$$

and (38), we find

$$\bar\rho_{h_\pi} - \bar\rho_{h_\pi+1} \geq \beta_i^t - \beta_j^\tau - l\big(T_{h_\pi+1} \cap ([t] \setminus [\tau])\big) - u\big(([t] \setminus [\tau]) \setminus T_{h_\pi}\big). \tag{39}$$

Observe that an index k satisfies $k \notin T_{h_\pi}$ if and only if $k = t_r$ for some $r < h_\pi$, or in other words $h_k = r < h_\pi$. Thus, by the definition of π, we have $\bar\rho_{h_k} - \bar\rho_{h_k+1} = u_k$ for $k \notin T_{h_\pi}$. Now, if we sum (39) with inequalities $\bar\rho_{h_k} - \bar\rho_{h_k+1} \geq u_k$ for $k \in ([t] \setminus [\tau]) \setminus T_{h_\pi}$ and $\bar\rho_{h_k} - \bar\rho_{h_k+1} \geq l_k$ for $k \in T_{h_\pi+1} \cap ([t] \setminus [\tau])$, we obtain

$$\sum_{k \in [t]\setminus[\tau]} (\bar\rho_{h_k} - \bar\rho_{h_k+1}) \geq \beta_i^t - \beta_j^\tau.$$

If $t = 1$ (base step of the induction), as $\tau < \pi \leq t$, we have $\tau = 0$. Then $\beta_j^\tau = 0$ and (37) holds. If $t > 1$ instead, the conclusion follows as by induction we have $\sum_{k\in[\tau]}(\bar\rho_{h_k} - \bar\rho_{h_k+1}) \geq \beta_j^\tau$. This concludes the proof of Claim 3 and the analysis of Case 1.

Case 2: p has some negative components.

Recall that our target is to show that problem (29) has an optimal solution that belongs to X. The proof is by induction on the number of negative entries of p. The base case (i.e., no negative entry in p) has been considered in Case 1 above.

Assume that p has some negative entries and choose one of them, say $p_r < 0$. Then $\sigma_r = u_r$ in any optimal solution of problem (29), and thus problem (29) is equivalent to

$$\min\{p\sigma + qx : (\sigma, x) \in F\}, \tag{40}$$

where F is the face of P induced by inequality $\sigma_r \le u_r$, i.e., $F = \{(\sigma, x) \in P : \sigma_r = u_r\}$.

Let X' be the mixed-integer set obtained by replacing σ_r with u_r in (21)–(23). The set X' has one variable less than X, but it is still a set of the type (21)–(23). So it makes sense to consider the relaxations X'_T for $\varnothing \ne T \subseteq [\ell] \setminus \{r\}$, as well as the polyhedron Q', which is the analogue of Q. Let σ' and p' denote the vectors σ and p respectively, with the r-th component removed. If we define

$$P' = \bigcap_{\varnothing \ne T \subseteq [\ell] \setminus \{r\}} \operatorname{conv}(X'_T) \cap Q' \cap \{(\sigma', x) : \sigma'_t \le u_t,\, t \in [\ell] \setminus \{r\}\},$$

then by induction the optimization problem

$$\min\{p'\sigma' + qx : (\sigma', x) \in P'\} \tag{41}$$

has an optimal solution $(\bar\sigma', \bar x)$ that belongs to X'. If vector $(\bar\sigma', \bar x)$ is extended to $(\bar\sigma, \bar x)$ by setting $\bar\sigma_r = u_r$, we find a vector belonging to $X \cap F$.

To conclude, we show that $(\bar\sigma, \bar x)$ is an optimal solution to problem (29), or, equivalently, to problem (40). To see this, note that for each $\varnothing \ne T \subseteq [\ell] \setminus \{r\}$, the sets X_T and X'_T coincide. Furthermore, Q and Q' are defined by the same inequalities. It follows that $F \subseteq P'$ (or, more formally, for any $(\sigma, x) \in F$, we have $(\sigma', x) \in P'$). Then problem (41) is a relaxation of problem (40). Since $(\bar\sigma, \bar x) \in F$, it follows that $(\bar\sigma, \bar x)$ is an optimal solution to problem (40), and thus also to problem (29). This proves that (29) has an optimal solution that belongs to X when some components of p are negative. □

Note that from the proof of Proposition 4 it follows that linear optimization over X can be carried out in polynomial time as a linear program over the convex hull of n mixing sets (plus some network-dual constraints on the integer variables).

Corollary 3.2. *There is a polynomial-time algorithm to solve linear optimization over the set X.*

The result of Proposition 3.1 can be extended to the case in which only some of the bounds (22) are part of the description of X, as we now illustrate. Let L (respectively, U) be the set of indices t for which a lower (respectively, upper) bound on σ_t is enforced. So the mixed-integer set under consideration is now the following:

$$\begin{aligned} \sigma_{1,t} - x_i^t \ge b_i^t,\ & t \in [\ell],\ i \in [m_t], \\ \sigma_t \ge l_t, \quad & t \in L, \\ \sigma_t \le u_t, \quad & t \in U, \\ x_i^t \in \mathbb{Z}, \quad & t \in [\ell],\ i \in [m_t]. \end{aligned}$$

The relaxations X_T can still be constructed, but now some of the inequalities become meaningless. Specifically, it is possible to write inequality (24) if and only if $T \setminus [t] \subseteq L$ and $[t] \setminus T \subseteq U$; similarly, it is possible to write inequality (25) if and only if $T \subseteq L$. However, the relaxations that one obtains are still mixing sets (with or without a lower bound on the continuous variable), thus their convex hulls are given by mixing inequalities. Analogously, Q is now defined by (27) only for the indices t such that $[t] \subseteq U$. With these modifications in mind, one can see that the same result as that of Proposition 3.1 holds.

3.2. An application: discrete lot-sizing with sales. We show here that the single-item discrete lot-sizing problem with sales can be modeled as a mixed-integer set of the type (18)–(20).

The single-item discrete lot-sizing problem with sales is as follows. Given a horizon of n periods and lower and upper bounds l_t and u_t respectively on the amount that can be sold in period t, one has to decide in which periods to produce in order to maximize the total profit, i.e., the difference between the revenue from sales and the costs of production and storage. In each period the production is either 0 or at full capacity C, say $C = 1$ without loss of generality. The per-unit production and holding costs are denoted p_t and h_t respectively, while the sales price of the item is r_t. This problem can be formulated as the following mixed-integer program:

$$\max \sum_{t=1}^{n} (r_t v_t - p_t x_t - h_t s_t) - h_0 s_0 \tag{42}$$

$$\begin{aligned} \text{subject to} \quad & s_{t-1} + x_t = v_t + s_t, && t \in [n], && (43)\\ & s_t \geq 0,\ l_t \leq v_t \leq u_t, && t \in [n], && (44)\\ & x_t \in \{0,1\}, && t \in [n], && (45) \end{aligned}$$

where for each period t, x_t is the amount produced, v_t is the amount sold, and s_t is the stock at the end of the period (with s_0 being the initial stock variable). After using (43) to eliminate variable s_t for $t \in [n]$, the feasible region of the above problem becomes

$$\begin{aligned} & s_0 + x_{1,t} \geq v_{1,t}, && t \in [n], && (46)\\ & s_0 \geq 0,\ l_t \leq v_t \leq u_t, && t \in [n], && (47)\\ & x_t \in \{0,1\}, && t \in [n]. && (48) \end{aligned}$$

Defining $\sigma_t = v_{1,t} - s_0$ for $t \in \{0\} \cup [n]$ and $y_t = x_{1,t}$, (46)–(48) can be rewritten as

$$\begin{aligned} & \sigma_t - y_t \leq 0, && t \in [n], && (49)\\ & l_t \leq \sigma_t - \sigma_{t-1} \leq u_t, && t \in [n], && (50)\\ & 0 \leq y_t - y_{t-1} \leq 1, && t \in [n], && (51)\\ & y_t \in \mathbb{Z}, && t \in [n], && (52) \end{aligned}$$

with $\sigma_0 \leq 0$, $y_0 = 0$.

After changing the sign of the inequalities (49) and ignoring for the moment constraints (51), the above is a mixed-integer set of the type (18)–(20). Thus Proposition 3.1 gives the convex hull of the above set when inequalities (51) are omitted. However, by Proposition 2.3 we know that it is sufficient to intersect this convex hull with constraints (51) to obtain the convex hull of (49)–(52). Thus the result of this section yields a linear-inequality description for the convex hull of the feasible region of the single-item discrete lot-sizing problem with sales. Furthermore, Corollary 3.2 implies that the single-item discrete lot-sizing problem with sales can be solved in polynomial time.

In earlier work Loparic [14] showed the polynomiality of the constant-capacity lot-sizing problem with sales using a dynamic programming algorithm based on regeneration intervals. For the uncapacitated version, Loparic et al. [15] gave a valid inequality description of the convex hull of solutions. The corresponding inequalities can, not surprisingly, be viewed as uncapacitated versions of the mixing inequalities that can be obtained for the discrete lot-sizing set with sales described above.

4. General mixing sets linked by a bi-directed path.

4.1. The convex hull. The second special case that we study is a set of the form (14)–(17) in which only the generalized mixing set associated with the last node of the path appears in the system, i.e., the case $m_t = n_t = 0$ for $t < \ell$. Writing m (resp., n) instead of m_ℓ (resp., n_ℓ), and x_i (resp., y_j) instead of x_i^ℓ (resp., y_j^ℓ), the model is

$$\begin{aligned} s_\ell - x_i \geq b_i, &\quad i \in [m],\\ s_\ell - y_j \leq c_j, &\quad j \in [n],\\ l_t \leq s_t - s_{t-1} \leq u_t, &\quad t \in [\ell],\\ x_i, y_j \in \mathbb{Z}, &\quad i \in [m],\ j \in [n], \end{aligned}$$

where $s_0 = 0$ and $l_t \leq u_t$ for $t \in [\ell]$. Using the same change of variables as in §3, i.e., $\sigma_t = s_t - s_{t-1}$ for $t \in [\ell]$, the above set takes the form

$$Y: \quad \begin{aligned} \sigma_{1,\ell} - x_i \geq b_i,\ i \in [m], &\qquad (53)\\ \sigma_{1,\ell} - y_j \leq c_j,\ j \in [n], &\qquad (54)\\ l_t \leq \sigma_t \leq u_t,\ \ t \in [\ell], &\qquad (55)\\ x_i, y_j \in \mathbb{Z},\quad i \in [m],\ j \in [n]. &\qquad (56) \end{aligned}$$

Let Y be the set defined by (53)–(56). For each $\varnothing \neq T \subseteq [\ell]$ the following sets $Y_T^>$ and $Y_T^<$ are valid relaxations for Y:

$$Y_T^>: \quad \begin{aligned} \sigma(T) - x_i \geq b_i - u([\ell] \setminus T),\ i \in [m], &\qquad (57)\\ \sigma(T) \geq l(T), &\qquad (58)\\ x_i \in \mathbb{Z},\qquad i \in [m], &\qquad (59) \end{aligned}$$

and

$$Y_T^<: \quad \begin{aligned} \sigma(T) - y_j \leq c_j - l([\ell] \setminus T),\ j \in [n],\\ \sigma(T) \leq u(T),\\ y_j \in \mathbb{Z},\qquad j \in [n]. \end{aligned}$$

Since the former set is a mixing set and the latter is a reversed mixing set, their convex hulls are known.

It is also easy to see that the following inequalities are valid for Y:

$$Q': \quad \begin{aligned} -x_i \geq \lceil b_i - u_{1,\ell} \rceil,\ i \in [m], &\qquad (60)\\ -y_j \leq \lfloor c_j - l_{1,\ell} \rfloor,\ j \in [n], &\qquad (61)\\ y_j - x_i \geq \lceil b_i - c_j \rceil,\ i \in [m],\ j \in [n]. &\qquad (62) \end{aligned}$$

We denote by Q' the polyhedron defined by (60)–(62).

Much as in §3, we prove that by taking the convex hulls of all the relaxations $Y_T^>$ and $Y_T^<$ along with inequalities (60)–(62), one finds the convex hull of (53)–(56).

PROPOSITION 4.1.

$$\mathrm{conv}(Y) = \bigcap_{\varnothing\neq T\subseteq[\ell]} \mathrm{conv}(Y_T^>) \cap \bigcap_{\varnothing\neq T\subseteq[\ell]} \mathrm{conv}(Y_T^<) \cap Q'. \tag{63}$$

Proof. Let P be the polyhedron on the right-hand side of equality (63). It is clear that $\mathrm{conv}(Y) \subseteq P$. As in the proof of Proposition 3.1, in order to prove that $P \subseteq \mathrm{conv}(Y)$ we show that if $p\sigma + qx + ry$ is a linear objective function such that the optimization problem $\min\{p\sigma + qx + ry : (\sigma, x) \in Y\}$ has finite optimum, then the problem

$$\min\{p\sigma + qx + ry : (\sigma, x, y) \in P\} \tag{64}$$

has an optimal solution that belongs to Y.

Assume that $p_1 \le \cdots \le p_\ell$ and define $\tau = \min\{h : p_h \ge 0\}$, with $\tau = \ell + 1$ if $p_\ell < 0$. For $h \in [\ell]$, let $S_h = \{1, \dots, h\}$ and $T_h = \{h, \dots, \ell\}$. In order to show that problem (64) has an optimal solution belonging to Y, we prove that the relaxed linear program

$$\min\left\{p\sigma + qx + ry : (\sigma, x, y) \in \bigcap_{h=1}^{\tau-1} \mathrm{conv}(Y_{S_h}^<) \cap \bigcap_{h=\tau}^{\ell} \mathrm{conv}(Y_{T_h}^>) \cap Q'\right\} \tag{65}$$

has an optimal solution that belongs to Y.

Under the change of variables

$$\rho_h = \begin{cases} \sigma_{1,h} & \text{if } 1 \le h < \tau, \\ \sigma_{h,\ell} & \text{if } \tau \le h \le \ell, \end{cases}$$

problem (65) takes the form

$$\min\left\{\tilde p\rho + qx + ry : (\rho, x, y) \in \bigcap_{h=1}^{\tau-1} \mathrm{conv}(Z_{S_h}^<) \cap \bigcap_{h=\tau}^{\ell} \mathrm{conv}(Z_{T_h}^>) \cap Q'\right\}, \tag{66}$$

where

$$\tilde p\rho = \sum_{h=1}^{\tau-2}(p_h - p_{h+1})\rho_h + p_{\tau-1}\rho_{\tau-1} + p_\tau\rho_\tau + \sum_{h=\tau+1}^{\ell}(p_h - p_{h-1})\rho_h \tag{67}$$

(with $p_0 = p_{\ell+1} = 0$), and the sets $Z_{T_h}^>$ and $Z_{S_h}^<$ are defined as follows:

$$Z_{T_h}^> : \quad \begin{aligned} \rho_h - x_i &\ge b_i - u_{1,h-1}, \; i \in [m], && (68)\\ \rho_h &\ge l_{h,\ell}, && (69)\\ x_i &\in \mathbb{Z}, \qquad i \in [m], && (70) \end{aligned}$$

and

$$Z_{S_h}^< : \quad \begin{aligned} \rho_h - y_j &\le c_j - l_{h+1,\ell}, \; j \in [n], && (71)\\ \rho_h &\le u_{1,h}, && (72)\\ y_j &\in \mathbb{Z}, \qquad j \in [n]. && (73) \end{aligned}$$

All the extreme points of the feasible region of problem (66) have integral x- and y-components, as this polyhedron is the intersection of mixing sets and reversed mixing sets defined on disjoint sets of variables, plus some bounds on the integer variables (see Proposition 2.3). Then problem (66) has an optimal solution $(\bar\rho, \bar x, \bar y)$ with $\bar x$ and $\bar y$ integer. As the coefficients of variables $\rho_1, \dots, \rho_{\tau-1}$ in the objective function are negative, while those of variables $\rho_\tau, \dots, \rho_\ell$ are nonnegative, we can assume that $\bar\rho_1, \dots, \bar\rho_{\tau-1}$ are maximal and $\bar\rho_\tau, \dots, \bar\rho_\ell$ are minimal.

We now prove that the point $(\bar\sigma, \bar x, \bar y)$ that corresponds to $(\bar\rho, \bar x, \bar y)$ under the change of variables, satisfies (53)–(56). For this purpose, define $\beta_i = \bar x_i + b_i$ for $i \in [m]$, and $\gamma_j = \bar y_j + c_j$ for $j \in [n]$. Note that inequalities (60)–(62) imply that

$$\beta_i \le u_{1,\ell},\ i \in [m], \tag{74}$$

$$\gamma_j \ge l_{1,\ell},\ j \in [n], \tag{75}$$

$$\beta_i \le \gamma_j,\ \ i \in [m],\ j \in [n]. \tag{76}$$

First we prove that $\bar\sigma_h \ge l_h$ for $h \in [\ell]$.

1. Assume first that $h < \tau$. If $h = 1$, the inequality to be verified is $\bar\rho_1 \ge l_1$. By the maximality of $\bar\rho_1$, we have either $\bar\rho_1 = \gamma_j - l_{2,\ell}$ for some j or $\bar\rho_1 = u_1$. In the former case inequality (75) implies that $\bar\rho_1 \ge l_1$, while in the latter case we have $\bar\rho_1 = u_1 \ge l_1$. So we assume $1 < h < \tau$. Then the inequality can be written as $\bar\rho_h - \bar\rho_{h-1} \ge l_h$. By the maximality of $\bar\rho_h$, we have either $\bar\rho_h = \gamma_j - l_{h+1,\ell}$ for some j or $\bar\rho_h = u_{1,h}$. In the former case inequality $\bar\rho_{h-1} \le \gamma_j - l_{h,\ell}$ implies that $\bar\rho_h - \bar\rho_{h-1} \ge l_h$, while in the latter case inequality $\bar\rho_{h-1} \le u_{1,h-1}$ implies that $\bar\rho_h - \bar\rho_{h-1} \ge u_h \ge l_h$.

2. Now assume that $h \ge \tau$. If $h = \ell$, the inequality to be verified is $\bar\rho_\ell \ge l_\ell$. However this inequality is part of conditions (69). So we assume $\tau \le h < \ell$. Then the inequality is $\bar\rho_h - \bar\rho_{h+1} \ge l_h$. By the minimality of $\bar\rho_{h+1}$, we have either $\bar\rho_{h+1} = \beta_i - u_{1,h}$ for some i or $\bar\rho_{h+1} = l_{h+1,\ell}$. In the former case inequality $\bar\rho_h \ge \beta_i - u_{1,h-1}$ implies that $\bar\rho_h - \bar\rho_{h+1} \ge u_h \ge l_h$, while in the latter case inequality $\bar\rho_h \ge l_{h,\ell}$ implies that $\bar\rho_h - \bar\rho_{h+1} \ge l_h$.

With a symmetric argument one proves that $\bar\sigma_h \le u_h$ for $h \in [\ell]$.

Finally, we have to show that $(\bar\sigma, \bar x, \bar y)$ satisfies (53)–(54). If $\tau = 1$, inequality (53) is equivalent to $\rho_1 \ge \beta_i$, which is part of the constraints defining the feasible region of (66) (see the set $Z^{>}_{T_1}$). If $\tau = \ell + 1$, inequality (53) is equivalent to $\rho_\ell \ge \beta_i$. By the maximality of $\bar\rho_\ell$, we have either $\bar\rho_\ell = \gamma_j$ for some j or $\bar\rho_\ell = u_{1,\ell}$. In the former case inequality (76) implies that $\bar\rho_\ell \ge \beta_i$, while in the latter case inequality (74) establishes the claim. So we now assume $1 < \tau \le \ell$. Then inequality (53) is equivalent to $\rho_{\tau-1} + \rho_\tau \ge \beta_i$. By the maximality of $\bar\rho_{\tau-1}$, we have either $\bar\rho_{\tau-1} = \gamma_j - l_{\tau,\ell}$ for some j or $\bar\rho_{\tau-1} = u_{1,\tau-1}$. In the former case inequality $\bar\rho_\tau \ge l_{\tau,\ell}$ and (76) imply that $\bar\rho_{\tau-1} + \bar\rho_\tau \ge \beta_i$, while in the latter case inequality $\bar\rho_\tau \ge \beta_i - u_{1,\tau-1}$ establishes the claim. This proves that $(\bar\sigma, \bar x, \bar y)$ satisfies (53). A perfectly symmetric argument shows that $(\bar\sigma, \bar x, \bar y)$ satisfies (54).

This concludes the proof for the case $p_1 \le \cdots \le p_\ell$. When the p_k's satisfy a different ordering, the proof is the same and one finds the other sets $Y^{>}_T$ and $Y^{<}_T$. ∎

As for the set X discussed in §3, the above proof shows the following.

COROLLARY 4.2. *There is a polynomial-time algorithm to solve linear optimization over the set Y.*

The extension of Proposition 4.1 to the case in which only some of the bounds on the continuous variables are enforced in (53)–(56) is similar to that described in the previous section.

4.1.1. Strengthening of the result. In this subsection we present a stronger version of Proposition 4.1, showing that many of the mixing inequalities defining the convex hulls of the mixing sets $Y_T^>$ and $Y_T^<$ are not facet-defining for the polyhedron $\mathrm{conv}(Y)$. We show this for the mixing sets of the type $Y_T^>$, as the argument for the reversed mixing sets $Y_T^<$ is perfectly symmetric.

To describe the mixing inequalities that define the convex hull of the sets $Y_T^>$, we need some notation, which will also be used in the next subsection. First of all, by translating the σ-variables, we can assume without loss of generality that $l_t = 0$ for $t \in [\ell]$ (this will simplify notation). Let $(\bar\sigma, \bar x, \bar y)$ be a point satisfying the initial linear system (53)–(55). Given a subset $T \subseteq [\ell]$ and an index $i \in [m]$, we denote by b_i^T the right-hand side of (57), i.e., $b_i^T = b_i - u([\ell] \setminus T)$. We also define $f_i^T = b_i^T - (\lceil b_i^T \rceil - 1)$ and $B_i^T = \bar x_i + \lceil b_i^T \rceil$. Finally, given a subset of indices $\varnothing \neq I \subseteq [m]$, we define $M_1^{T,I}$ and $M_2^{T,I}$ as the left-hand sides of the mixing inequalities for $Y_T^>$ associated with subset I, evaluated at $(\bar\sigma, \bar x, \bar y)$:

$$M_1^{T,I} = \bar\sigma(T) - \sum_{r=1}^{q} (f_{i_r}^T - f_{i_{r-1}}^T) B_{i_r}^T, \tag{77}$$

$$M_2^{T,I} = M_1^{T,I} - (1 - f_{i_q}^T)(B_{i_1}^T - 1), \tag{78}$$

where $i_1, \dots, i_q$ is an ordering of the elements in I such that $f_{i_1}^T \le \cdots \le f_{i_q}^T$, with $f_{i_0}^T = 0$. We will refer to inequalities of the form (77) (resp., (78)) as mixing inequalities of the first (resp., second) type.

LEMMA 4.3. *$M_2^{T,I} \le M_2^{V,I}$ for any $\varnothing \neq I \subseteq [m]$ and any two subsets $\varnothing \neq V \subseteq T \subseteq [\ell]$.*

Proof. It is sufficient to consider the case $|T| = |V| + 1$. Let τ be the unique element in $T \setminus V$ and define $\varphi = u_\tau - \lfloor u_\tau \rfloor$. Let $i_1, \dots, i_q$ be an ordering of the elements in I such that $f_{i_1}^T \le \cdots \le f_{i_q}^T$ and assume that $f_{i_{\pi-1}}^T \le \varphi < f_{i_\pi}^T$ for some index $\pi \in [q]$, where $f_{i_0}^T = 0$ (the case $\varphi \ge f_{i_q}^T$ can be treated similarly). Since $b_i^V = b_i^T - u_\tau$ for $i \in [m]$, we have

$$f_{i_r}^V = \begin{cases} f_{i_r}^T - \varphi & \text{if } r \ge \pi, \\ f_{i_r}^T - \varphi + 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad B_{i_r}^V = \begin{cases} B_{i_r}^T - \lfloor u_\tau \rfloor & \text{if } r \ge \pi, \\ B_{i_r}^T - \lfloor u_\tau \rfloor - 1 & \text{otherwise.} \end{cases}$$

It follows that $f_{i_\pi}^V \le \cdots \le f_{i_q}^V \le f_{i_1}^V \le \cdots \le f_{i_{\pi-1}}^V$. Then

$$\begin{aligned} M_2^{V,I} &= \bar\sigma(V) - (f_{i_\pi}^T - \varphi)(B_{i_\pi}^T - \lfloor u_\tau \rfloor) - \sum_{r=\pi+1}^{q} (f_{i_r}^T - f_{i_{r-1}}^T)(B_{i_r}^T - \lfloor u_\tau \rfloor) \\ &\quad - (f_{i_1}^T - f_{i_q}^T + 1)(B_{i_1}^T - \lfloor u_\tau \rfloor - 1) - \sum_{r=2}^{\pi-1} (f_{i_r}^T - f_{i_{r-1}}^T)(B_{i_r}^T - \lfloor u_\tau \rfloor - 1) \\ &\quad - (1 - (f_{i_{\pi-1}}^T - \varphi + 1))(B_{i_\pi}^T - \lfloor u_\tau \rfloor - 1) \\ &= M_2^{T,I} - \bar\sigma_\tau + \lfloor u_\tau \rfloor + \varphi = M_2^{T,I} - \bar\sigma_\tau + u_\tau \ge M_2^{T,I}, \end{aligned}$$

where the second equality follows from tedious but straightforward calculation, and the inequality holds because $(\bar\sigma, \bar x, \bar y)$ satisfies (55). ∎

The above lemma implies that out of all the mixing inequalities of the type $M_2^{T,I} \ge 0$, only those having $T = [\ell]$ are needed in the description of $\mathrm{conv}(Y)$. Keeping in mind that a similar result can be proven for the mixing inequalities defining the polyhedra $\mathrm{conv}(Y_T^<)$, we have the following strengthening of Proposition 4.1.

COROLLARY 4.4. *In the description of* $\mathrm{conv}(Y)$ *given in* (63)*, the mixing inequalities of the second type for the sets* $Y_T^>$ *and* $Y_T^<$ *with* $T \neq [\ell]$ *can be dropped.*

4.2. Separation of the inequalities. Both sets X (defined by (21)–(23)) and Y (defined by (53)–(56)) are generalizations of the *splittable flow arc set* studied by Magnanti et al. [16] and Atamtürk and Rajan [1] as a relaxation of some multi-commodity flow capacitated network design problems. The splittable flow arc set is defined by the constraints

$$\sigma_{1,\ell} - x \geq b, \tag{79}$$
$$l_t \leq \sigma_t \leq u_t,\ t \in [\ell], \tag{80}$$
$$x \in \mathbb{Z}. \tag{81}$$

This set is the special case of (21)–(23) in which $m_\ell = 1$ and $m_t = 0$ for all $t < \ell$, and also the special case of (53)–(56) in which $m = 1$ and $n = 0$.

Magnanti et al. [16] proved that the convex hull of (79)–(81) is described by an exponential family of inequalities, called *residual capacity inequalities*, which can be viewed as simple MIR-inequalities (see [18]) derived from suitable relaxations of (79)–(81). Their result is a special case of both Propositions 3.1 and 4.1. Atamtürk and Rajan [1] gave a separation algorithm for these inequalities, whose running time is $\mathcal{O}(\ell)$.

Since simple MIR-inequalities are a special case of mixing inequalities and since, for a given mixing set, the mixing inequalities can be separated in polynomial time [20], it is natural to wonder whether the separation algorithm of Atamtürk and Rajan [1] can be extended to the more general sets studied in this paper. As for the residual capacity inequalities, the main difficulty is due to the fact that though separation is easy for a fixed mixing set, here we have polyhedra described by an exponential number of mixing sets, and the problem of selecting the right mixing set is nontrivial. However, we show below that for the set Y it is possible to determine *a priori* which mixing sets can provide a most violated inequality. Then it is sufficient to apply the mixing-inequalities separation algorithm to those particular mixing sets.

Let $(\bar{\sigma}, \bar{x}, \bar{y})$ be a point satisfying the initial linear system (53)–(55). We show how to check in polynomial time whether $(\bar{\sigma}, \bar{x}, \bar{y})$ belongs to the convex hull of (53)–(56). Recall that by Proposition 4.1 the convex hull is $\bigcap_T \mathrm{conv}(Y_T^>) \cap \bigcap_T \mathrm{conv}(Y_T^<) \cap Q'$. Here we consider only the inequalities defining $\bigcap_T \mathrm{conv}(Y_T^>)$. Indeed, the sets $\bigcap_T \mathrm{conv}(Y_T^<)$ can be treated similarly thanks to symmetry arguments, and it is trivial to check in polynomial time whether $(\bar{\sigma}, \bar{x}, \bar{y})$ satisfies the inequalities defining Q'.

Thus we only have to show how one can check in polynomial time whether one of the inequalities defining $\bigcap_T \mathrm{conv}(Y_T^>)$ is violated by $(\bar{\sigma}, \bar{x})$ (the y-variables can be ignored). As in the previous subsection, we assume without loss of generality that $l_t = 0$ for $t \in [\ell]$. Then, by Corollary 4.4, our separation problem concerns all the inequalities $M_1^{T,I} \geq 0$ for $\varnothing \neq T \subseteq [\ell]$ and $\varnothing \neq I \subseteq [m]$, and all the inequalities $M_2^{T,I} \geq 0$ for $T = [\ell]$ and $\varnothing \neq I \subseteq [m]$.

When dealing with mixing inequalities of the second type, we only have to consider the mixing set $Y_{[\ell]}^>$. Thus one can decide in $\mathcal{O}(m \log m)$ time whether there is an inequality of this type violated by $(\bar{\sigma}, \bar{x})$ by applying the separation algorithm for the mixing inequalities of the second type (see [20]) to the set $Y_{[\ell]}^>$.

We now assume that $(\bar\sigma, \bar x)$ violates no mixing inequality of the second type and turn to the mixing inequalities of the first type. We will show that if $(\bar\sigma, \bar x)$ violates an inequality of this type, then there is a most violated inequality with T being one of the sets S_i, $i \in [m]$, where each S_i is a subset whose definition depends only on $(\bar\sigma, \bar x)$:

$$S_i = \{k \in [\ell] : \bar\sigma_k - u_k(\bar x_i - \lceil \bar x_i \rceil + 1) < 0\}.$$

Let $M_1^{T,I}$ be the left-hand side of a most violated mixing inequality of the first type, where the cardinality of I is as small as possible. Let $i_1, \dots, i_q$ be an ordering of the elements in I such that $f_{i_1}^T \le \cdots \le f_{i_q}^T$, with $f_{i_0}^T = 0$. Note that, by the minimality of $|I|$, we have $f_{i_1}^T < \cdots < f_{i_q}^T$. Furthermore, since no mixing inequality of the second type is violated, we have $f_{i_q}^T < 1$.

LEMMA 4.5. *The following chain of inequalities holds:* $0 < B_{i_q}^T < B_{i_{q-1}}^T < \cdots < B_{i_1}^T < 1$.

Proof. With $J = I \setminus \{i_q\}$, inequality $M_1^{T,J} - M_1^{T,I} > 0$ gives $(f_{i_q}^T - f_{i_{q-1}}^T)B_{i_q}^T > 0$, thus $B_{i_q}^T > 0$. For $1 \le r \le q-1$ and $J = I \setminus \{i_r\}$, inequality $M_1^{T,J} - M_1^{T,I} > 0$ gives $(f_{i_r}^T - f_{i_{r-1}}^T)(B_{i_r}^T - B_{i_{r+1}}^T) > 0$, thus $B_{i_r}^T > B_{i_{r+1}}^T$. Finally, summing inequality $-M_1^{T,I} > 0$ with $M_2^{T,I} \ge 0$ gives $-(1 - f_{i_q}^T)(B_{i_1}^T - 1) > 0$, thus $B_{i_1}^T < 1$. ∎

Recalling that $B_i^T = \bar x_i + \lceil b_i^T \rceil$ for $i \in [m]$, Lemma 4.5 implies that $\lceil \bar x_{i_r} \rceil = -\lceil b_{i_r}^T \rceil + 1$ for $r \in [q]$, thus $S_{i_r} = \{k \in [\ell] : \bar\sigma_k - u_k B_{i_r}^T < 0\}$ for $r \in [q]$. To simplify notation, define

$$S = S_{i_1} = \left\{k \in [\ell] : \bar\sigma_k - u_k B_{i_1}^T < 0\right\}. \tag{82}$$

The next two lemmas show that $T = S$.

LEMMA 4.6. $T \subseteq S$.

Proof. Assume first that $f_{i_{\pi-1}}^T \le u(T \setminus S) < f_{i_\pi}^T$ for some $\pi \in [q]$. We show that if $T \not\subseteq S$, then $M_1^{T\cap S,J} < M_1^{T,I}$ for some $J \subseteq I$, contradicting the fact that $M_1^{T,I}$ is the left-hand side of a most violated inequality.

Since $b_i^{T\cap S} = b_i^T - u(T \setminus S)$ for $i \in [m]$, we have

$$f_{i_r}^{T\cap S} = \begin{cases} f_{i_r}^T - u(T\setminus S) & \text{if } r \ge \pi, \\ f_{i_r}^T - u(T\setminus S) + 1 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad B_{i_r}^{T\cap S} = \begin{cases} B_{i_r}^T & \text{if } r \ge \pi, \\ B_{i_r}^T - 1 & \text{otherwise.}\end{cases}$$

It follows that $f_{i_\pi}^{T\cap S} < \cdots < f_{i_q}^{T\cap S} < f_{i_1}^{T\cap S} < \cdots < f_{i_{\pi-1}}^{T\cap S}$. Define $J = \{i_\pi, \dots, i_q\}$. Then

$$M_1^{T\cap S,J} = \bar\sigma(T\cap S) - (f_{i_\pi}^T - u(T\setminus S))B_{i_\pi}^T - \sum_{r=\pi+1}^{q} (f_{i_r}^T - f_{i_{r-1}}^T)B_{i_r}^T$$

and thus

$$\begin{aligned} M_1^{T,I} - M_1^{T\cap S,J} &= \bar\sigma(T\setminus S) - \sum_{r=1}^{\pi-1}(f_{i_r}^T - f_{i_{r-1}}^T)B_{i_r}^T - (u(T\setminus S) - f_{i_{\pi-1}}^T)B_{i_\pi}^T \\ &\ge \bar\sigma(T\setminus S) - \sum_{r=1}^{\pi-1}(f_{i_r}^T - f_{i_{r-1}}^T)B_{i_1}^T - (u(T\setminus S) - f_{i_{\pi-1}}^T)B_{i_1}^T \\ &= \sum_{k\in T\setminus S}(\bar\sigma_k - u_k B_{i_1}^T) > 0, \end{aligned}$$

where the first inequality follows from Lemma 4.5, and the last one holds because of the definition of S and the fact that $T \setminus S \neq \varnothing$.

We now suppose that $u(T \setminus S) \geq f^T_{i_q}$ and show that this contradicts the assumption that $M^{T,I}_1 < 0$. Rearranging (77), we have that

$$\begin{aligned} M^{T,I}_1 &= \bar\sigma(T \cap S) + \sum_{k \in T \setminus S} (\bar\sigma_k - u_k B^T_{i_1}) - (f^T_{i_1} - u(T \setminus S))B^T_{i_1} - \sum_{r=2}^{q} (f^T_{i_r} - f^T_{i_{r-1}})B^T_{i_r} \\ &\geq -(f^T_{i_1} - u(T \setminus S)) - \sum_{r=2}^{q} (f^T_{i_r} - f^T_{i_{r-1}}) = u(T \setminus S) - f^T_{i_q} \geq 0, \end{aligned}$$

where the first inequality holds because of the nonnegativity of $\bar\sigma$, the definition of S and Lemma 4.5. ∎

Lemma 4.7. *$S \subseteq T$.*

Proof. By Lemma 4.6, $T \subseteq S$. Assume that the inclusion is strict. Define $a = \lfloor u(S \setminus T) \rfloor$ and $\varphi = u(S \setminus T) - a$.

Assume first that $1 - f^T_{i_\pi} < \varphi \leq 1 - f^T_{i_{\pi-1}}$ for some $\pi \in [q]$. We show that if $T \subsetneq S$, then $M^{S,I}_2 < 0$, contradicting the initial assumption that no mixing inequality of the second type is violated by $(\bar\sigma, \bar x)$.

Since $b^S_i = b^T_i + u(S \setminus T)$ for $i \in [m]$, we have

$$f^S_{i_r} = \begin{cases} f^T_{i_r} + \varphi & \text{if } r < \pi, \\ f^T_{i_r} + \varphi - 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad B^S_{i_r} = \begin{cases} B^T_{i_r} + a & \text{if } r < \pi, \\ B^T_{i_r} + a + 1 & \text{otherwise.} \end{cases}$$

It follows that $f^S_{i_\pi} < \cdots < f^S_{i_q} < f^S_{i_1} < \cdots < f^S_{i_{\pi-1}}$. Then

$$\begin{aligned} M^{S,I}_2 &= \bar\sigma(S) - (f^T_{i_\pi} + \varphi - 1)(B^T_{i_\pi} + a + 1) - \sum_{r=\pi+1}^{q} (f^T_{i_r} - f^T_{i_{r-1}})(B^T_{i_r} + a + 1) \\ &\quad - (f^T_{i_1} - f^T_{i_q} + 1)(B^T_{i_1} + a) - \sum_{r=2}^{\pi-1} (f^T_{i_r} - f^T_{i_{r-1}})(B^T_{i_r} + a) \\ &\quad - (1 - f^T_{i_{\pi-1}} - \varphi)(B^T_{i_\pi} + a) \\ &= M^{T,I}_1 + \bar\sigma(S \setminus T) - (1 - f^T_{i_q})B^T_{i_1} - (f^T_{i_q} - 1 + \varphi + a) \\ &= M^{T,I}_1 + \bar\sigma(S \setminus T) - (1 - f^T_{i_q})(B^T_{i_1} - 1) - u(S \setminus T) \\ &= M^{T,I}_1 + \sum_{k \in S \setminus T} (\bar\sigma_k - u_k B^T_{i_1}) - (1 - f^T_{i_q} - u(S \setminus T))(B^T_{i_1} - 1) < 0, \end{aligned}$$

where the inequality holds because of the following: (i) $M^{T,I}_1 < 0$ by assumption; (ii) $\bar\sigma_k - u_k B^T_{i_1} < 0$ for all $k \in S$; (iii) $1 - f^T_{i_q} - u(S \setminus T) \leq 1 - f^T_{i_\pi} - a - \varphi < 0$ by the definition of π and because $a \geq 0$; (iv) $B^T_{i_1} - 1 < 0$ by Lemma 4.5.

Now assume that $0 \leq \varphi \leq 1 - f^T_{i_q}$. A calculation similar to that carried out above gives again

$$M^{S,I}_2 = M^{T,I}_1 + \sum_{k \in S \setminus T} (\bar\sigma_k - u_k B^T_{i_1}) - (1 - f^T_{i_q} - u(S \setminus T))(B^T_{i_1} - 1).$$

However, in this case we can conclude that $M_2^{S,I} < 0$ only if $u(S \setminus T) \geq 1 - f_{i_q}^T$. Therefore it remains to consider the case when $0 \leq u(S \setminus T) \leq 1 - f_{i_q}^T$. In this case we have

$$
\begin{aligned}
M_1^{S,I} &= \bar{\sigma}(S) - (f_{i_1}^T + u(S \setminus T))B_{i_1} - \sum_{r=2}^{q}(f_{i_r}^T - f_{i_{r-1}}^T)B_{i_r}^T \\
&= M_1^{T,I} + \bar{\sigma}(S \setminus T) - u(S \setminus T)B_{i_1}^T \\
&= M_1^{T,I} + \sum_{k \in S \setminus T} (\bar{\sigma}_k - u_k B_{i_1}^T) < M_1^{T,I},
\end{aligned}
$$

where the inequality follows from the definition of S and the fact that $S \setminus T$ is nonempty. However, this contradicts the fact that $M_1^{T,I}$ is the left-hand side of a most violated inequality. $\square$

Therefore $T = S_{i_1}$. Since i_1 is unknown but certainly lies in $[m]$, it suffices to consider all the sets S_i for $i \in [m]$.

We then have the following algorithm for checking whether $(\bar{\sigma}, \bar{x})$ violates one of the inequalities defining $\bigcap_T \operatorname{conv}(Y_T^{>})$:

1. Apply the separation algorithm for mixing inequalities of the second type [20] to the set $Y_{[\ell]}^{>}$; if there is a violated inequality, return it and stop.
2. For $i \in [m]$, apply the separation algorithm for mixing inequalities of the first type [20] to the set $Y_T^{>}$ with $T = S_i = \{k \in [\ell] : \bar{\sigma}_k - u_k(\bar{x}_i - \lceil \bar{x}_i \rceil + 1) < 0\}$. If there is a violated inequality, return it and stop.
3. If no violated inequality has been found during the above steps, there is no violated inequality.

If Step 2 is executed for all $i \in [m]$ rather than stopping when a violated inequality is found, this algorithm finds a most violated inequality (if a violated inequality exists).

Step 1 can be carried out in time $\mathcal{O}(m \log m)$. In Step 2, before applying the separation algorithm for the mixing inequalities, one has to determine the set S_i and the right-hand sides of the mixing set $Y_{S_i}^{>}$ for $i \in [m]$. For this purpose, it is convenient to have on ordering $i_1, \ldots, i_m$ of the elements of $[m]$ such that $\bar{x}_{i_1} - \lceil \bar{x}_{i_1} \rceil \leq \cdots \leq \bar{x}_{i_m} - \lceil \bar{x}_{i_m} \rceil$, and an ordering $k_1, \ldots, k_\ell$ of the elements of $[\ell]$ such that $\bar{\sigma}_{k_1}/u_{k_1} \leq \cdots \leq \bar{\sigma}_{k_\ell}/u_{k_\ell}$. These orderings can be obtained with $\mathcal{O}(m \log m + \ell \log \ell)$ operations. Then $S_{i_1} \subseteq \cdots \subseteq S_{i_m}$, and with another $\mathcal{O}(m + \ell)$ operations one can obtain all the sets and the right-hand sides needed. Finally, for each $i \in [m]$ the execution of the separation algorithm for the set $Y_{S_i}^{>}$ requires $\mathcal{O}(m \log m)$ operations. Thus the overall running time of the above algorithm is $\mathcal{O}(\ell \log \ell + m^2 \log m)$.

With a similar algorithm the inequalities defining $\bigcap_T \operatorname{conv}(Y_T^{<})$ be separated in time $\mathcal{O}(\ell \log \ell + n^2 \log n)$. The inequalities defining Q', i.e., (60)–(62), can be separated in time $\mathcal{O}(mn)$. Thus the overall running time of the separation algorithm is $\mathcal{O}(\ell \log \ell + m^2 \log m + n^2 \log n)$.

PROPOSITION 4.8. *The inequalities defining the polyhedron* $\operatorname{conv}(Y)$ *can be separated in time* $\mathcal{O}(\ell \log \ell + m^2 \log m + n^2 \log n)$.

5. Concluding remarks and open questions. For the two sets studied in §§3–4, the convex hull turns out to be essentially the intersection of the convex hulls of (generalized) mixing sets. A natural question is whether a similar result holds for the more general set (14)–(17). However, this seems to be false even for very small instances. For example, it can be checked that one of the facet-inducing inequalities for the convex hull of the set

$$\begin{gathered} s_1 - x_1 \geq 4.8,\ s_1 - x_2 \geq 5.4, \\ s_2 - y_1 \leq 2.6,\ s_2 - y_2 \leq 2.8, \\ s_1 - s_2 \geq 0,\ x_1, x_2, y_1, y_2 \in \mathbb{Z} \end{gathered}$$

is the inequality $s_1 - s_2 - 0.2x_1 - 0.6x_2 + 0.2y_1 + 0.6y_2 \geq 2.4$, which does not seem to be a mixing inequality for any (reasonable) relaxation of the set. This indicates that mixing sets are not enough to describe the convex hull of a general set of the type (14)–(17). (It is interesting to note that if inequality $s_2 - s_1 \geq 0$ is replaced by the equation $s_2 - s_1 = 0$ in the above constraints, then the resulting set is just a generalized mixing set.)

Even though the convex hull of (14)–(17) cannot be described in terms of mixing sets, still it would be interesting to prove some result showing that the convex hull of (14)–(17) is equal to the intersection of simpler sets. However, our efforts in this direction have been vain so far.

Furthermore, it is not clear whether the separation algorithm described in §4.2 can be extended to the set X. The results presented in §4.2 rely upon the fact that the "cyclic ordering" of the fractional parts of the right-hand sides of inequalities (24) is the same for all relaxations $Y_T^{>}$. Since this is not the case for the relaxations X_T considered in §3, it appears hard to extend the result. However, since linear optimization over X can be carried out in polynomial time (see Corollary 3.2), it is reasonable to hope that a polynomial-time combinatorial algorithm for solving the separation problem exists.

A final open question concerns the lot-sizing model with sales of §3.2. When the amount produced in each period can take any fractional value between 0 and 1, one obtains the constant-capacity single-item lot-sizing model with sales:

$$\max \sum_{t=1}^{n} (r_t v_t - p_t x_t - q_t y_t - h_t s_t) - h_0 s_0 \tag{83}$$

$$\text{subject to} \qquad s_{t-1} + x_t = v_t + s_t, \qquad t \in [n], \tag{84}$$

$$s_t \geq 0,\ l_t \leq v_t \leq u_t, \qquad t \in [n], \tag{85}$$

$$0 \leq x_t \leq y_t,\ y_t \in \{0,1\}, \qquad t \in [n], \tag{86}$$

where y_t is a set-up variable indicating whether production takes place in period t, and q_t is the associated set-up cost (the meaning of the other variables and parameters is as in §3.2). For each fixed $k \in [n]$, the following mixed-integer set is a relaxation of (84)–(86):

$$\begin{aligned} s_{k-1} + y_{k,t} &\geq v_{k,t}, && k \leq t \leq n, \\ s_{k-1} \geq 0,\ l_t \leq v_t &\leq u_t, && k \leq t \leq n, \\ y_t &\in \{0,1\}, && k \leq t \leq n. \end{aligned}$$

Note that this set is the feasible region of a discrete lot-sizing problem with sales of the form (46)–(48). If we denote it by $X_k^{DLSI-CC-SL}$, then the set $\bigcap_{k=1}^{n} X_k^{DLSI-CC-SL}$ is a relaxation of (84)–(86), called the Wagner-Whitin relaxation of the problem and denoted $X^{WW-CC-SL}$. Based on analogous results valid for other lot-sizing models (see, e.g., [20]), it is reasonable to conjecture that

$$\operatorname{conv}\left(X^{WW-CC-SL}\right) = \bigcap_{k=1}^{n} \operatorname{conv}\left(X_k^{DLSI-CC-SL}\right).$$

Currently we do not have any counterexample to this conjecture.

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