Piecewise smooth extreme functions are piecewise linear

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Abstract

The infinite relaxations in Integer Programming were introduced by Gomory and Johnson to provide a general framework for the theory of cutting planes: the so-called valid functions, and in particular the minimal and extreme functions, can be seen as automatic rules for the generation of cuts. However, while many extreme functions are piecewise linear and therefore easy to describe, the set of extreme functions turns out to have a very complicated mathematical structure, as several extreme functions are known that exhibit a somewhat pathological behavior. In this paper we show that if some smoothness assumption is imposed on an extreme function π, then π is necessarily piecewise linear. More precisely, we show that if a continuous extreme function for the Gomory–Johnson one-dimensional infinite group relaxation is a piecewise $\mathcal{C}^2$ function, then it is a piecewise linear function.

1 Introduction

In the late '60s and early '70s, Gomory [11] and Gomory and Johnson [12, 13] introduced a model called the "infinite group relaxation", which is essentially an infinite dimensional relaxation of any possible integer linear programming problem. Based on this model, they developed the notion of valid function: a valid function can be seen as an automatic rule to generate a cutting plane for any integer linear programming problem, starting from the available data. In this sense, a valid function is a generalization of the well-known Gomory mixed integer cut, which provides, via a closed formula, a valid cut from the optimal tableau of the continuous relaxation of the problem. Gomory and Johnson then defined a hierarchy of valid functions: among all valid functions, the so-called minimal ones are "more interesting", as any cut generated by a valid function is dominated by a cut generated by a minimal valid function; and among all minimal valid functions, the extreme ones are even more interesting, as they generate cuts that are not implied by any pair of (minimal) valid functions.

In this paper we will consider the so-called "one-dimensional pure integer infinite group relaxation", i.e., the model of Gomory and Johnson in which all the variables are constrained to be integer ("pure integer") and the relaxation is obtained from a single constraint of the problem ("one-dimensional"). We will give all necessary definitions for the full understanding of the result of this paper, but we will not explain how this infinite dimensional relaxation can be seen as a model encompassing any possible integer linear programming problem, or how valid functions can be seen as automatic rules for the generation of cuts; for these and other details on this topic, the interested reader can refer to the surveys [2, 6, 7] and to [10, Chapter 6].

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Given $b \in (0,1)$, let I_b denote the one-dimensional pure integer infinite group relaxation, i.e., I_b is the set of finite support functions $s : \mathbb{R} \to \mathbb{Z}_+$ such that $\sum_{x\in\mathbb{R}} xs(x) \in b + \mathbb{Z}$. The function s having finite support means that the set $\{x \in \mathbb{R} : s(x) \neq 0\}$ is finite. (Because of this assumption, the summation $\sum_{x\in\mathbb{R}} xs(x)$ is well defined.) A nonnegative function $\pi : \mathbb{R} \to \mathbb{R}_+$ is *valid* for I_b if $\sum_{x\in\mathbb{R}} \pi(x)s(x) \geq 1$ for every $s \in I_b$. A valid function for I_b is *minimal* if there is no valid function $\tilde{\pi} \neq \pi$ such that $\tilde{\pi}(x) \leq \pi(x)$ for every $x \in \mathbb{R}$.

The following characterization of minimal valid functions for I_b is due to Gomory and Johnson [12]. Before stating this result, we recall that a function $\pi : \mathbb{R} \to \mathbb{R}$ is called *subadditive* if $\pi(x) + \pi(y) \geq \pi(x + y)$ for every $x, y \in \mathbb{R}$.

Theorem 1. *Given $b \in (0,1)$, a function $\pi : \mathbb{R} \to \mathbb{R}_+$ is a minimal valid function for I_b if and only if $\pi(x) = 0$ for all $x \in \mathbb{Z}$, π is subadditive, and $\pi(x) + \pi(b - x) = 1$ for every $x \in \mathbb{R}$. Furthermore, these conditions imply that π is periodic modulo $\mathbb{Z}$ (i.e., $\pi(x + w) = \pi(x)$ for every $x \in \mathbb{R}$ and $w \in \mathbb{Z}$).*

Because of the above periodicity property, we will see minimal valid functions for I_b as functions defined on the interval $[0,1]$. (We could choose the interval $[0,1)$, but working with a compact domain will be convenient.) Then, if for a real number x we denote by $\langle x\rangle$ the value of x modulo 1 (i.e., the fractional part of x), the subadditivity of π reads as follows: $\pi(x) + \pi(y) \geq \pi(\langle x + y\rangle)$ for every $x, y \in [0,1]$. For this reason, we say that a function $\pi : [0,1] \to \mathbb{R}$ is subadditive if this property is satisfied. We can then restate Theorem 1 as follows:

Theorem 2. *Given $b \in (0,1)$, a function $\pi : [0,1] \to \mathbb{R}_+$ is a minimal valid function for I_b if and only if $\pi(0) = \pi(1) = 0$, π is subadditive, and $\pi(x) + \pi(\langle b - x\rangle) = 1$ for every $x \in [0,1]$.*

A valid function for I_b is *extreme* if it cannot be written as the midpoint of two distinct valid functions. Every extreme function is a minimal valid function, but an explicit characterization of the extreme functions is not known. In fact, such a characterization is probably hard to obtain, as the set of extreme functions has a quite complicated mathematical structure, see [5, 8, 4].

Most of the (families of) continuous extreme functions known in the literature are piecewise linear, and therefore easy to describe. (See [7, 15] for recent compendiums of known extreme functions.) Indeed, Gomory and Johnson [14] in 2003 conjectured that every continuous extreme function for I_b is piecewise linear. Nonetheless, some extreme functions that are continuous but not piecewise linear have been discovered in the last years, such as those described in [1, 3]. However, these functions are quite "pathological", as they are defined as the limit of some sequence of functions. It is then reasonable to ask whether some regularity assumption rules out these peculiar extreme functions. Therefore, we investigate the following question: do there exist extreme functions that are continuous and "piecewise smooth" but not piecewise linear? Although nobody ever constructed an extreme function of this type, an answer to this question is not known. We give a negative answer, if by "piecewise smooth" we mean "piecewise $\mathcal{C}^2$". The details are now explained.

Let k be a positive integer. We recall that a function $\pi : [r,s] \to \mathbb{R}$, where $r < s$, is of *class* $\mathcal{C}^k$ if it is differentiable k times on $[r,s]$ and its kth derivative is continuous, where at r (respectively, s) by derivative we mean the right (respectively, left) derivative. We say that a function $\pi : [r,s] \to \mathbb{R}$ is a *piecewise* $\mathcal{C}^k$ *function* if there exist numbers $r = v_0 < v_1 < \cdots < v_m = s$ such that the restriction $\pi|_{[v_{t-1},v_t]}$ is a function of class $\mathcal{C}^k$ for every $t \in \{1, \dots, m\}$. The points $v_0, \dots, v_m$ are named *breakpoints* of π. Note that a piecewise $\mathcal{C}^k$ function π is continuous, but its derivative π' may not exist at the breakpoints of π. In the event that $\pi|_{[v_{t-1},v_t]}$ is affine for every $t \in \{1, \dots, m\}$, π is called *piecewise linear.*

The following is the main result of this paper.

Theorem 3. *Given $b \in (0, 1)$, let $\pi : [0, 1] \to \mathbb{R}_+$ be an extreme function for I_b. If π is a piecewise $\mathcal{C}^2$ function, then it is piecewise linear.*

Section 2 is devoted to proving Theorem 3, while Section 3 contains some final remarks.

2 Proof of the theorem

From now on we fix $b \in (0, 1)$ and let $\pi : [0, 1] \to \mathbb{R}_+$ be a minimal valid function for I_b, where π is a piecewise $\mathcal{C}^2$ function that is not piecewise linear. We will prove that π is a convex combination of two distinct valid functions, thus showing that π is not extreme: this will imply Theorem 3. More precisely, we will show that there exists a nonzero function $\gamma : [0, 1] \to \mathbb{R}$ such that both $\pi + t\gamma$ and $\pi - t\gamma$ are valid functions for some $t > 0$. We will call γ the *perturbation function.*

Let B be the set of breakpoints of π. We assume $b \in B$ without loss of generality. Note that the first derivative π' and the second derivative π'' are well defined and continuous at least over the open set $D := [0, 1] \setminus B$. (In other words, π is of class $\mathcal{C}^2$ over D.) Throughout the paper, we will always see π' and π'' as functions with domain D, while π will be seen as a function with domain $[0, 1]$.

We denote by $\Delta_\pi : [0, 1]^2 \to \mathbb{R}$ the *subadditivity slack* of π, i.e.,

$$\Delta_\pi(x, y) := \pi(x) + \pi(y) - \pi(\langle x + y \rangle) \tag{1}$$

for every $x, y \in [0, 1]$. Note that $\Delta_\pi(x, y) \geq 0$ for all $x, y \in [0, 1]$, as π is subadditive by Theorem 2.

Lemma 4. *Δ_π is a continuous function.*

Proof. Since $\pi(0) = \pi(1)$ by Theorem 2, we can write

$$\Delta_\pi(x, y) = \begin{cases} \pi(x) + \pi(y) - \pi(x + y) & \text{if } x, y \in [0, 1],\ x + y \leq 1, \\ \pi(x) + \pi(y) - \pi(x + y - 1) & \text{if } x, y \in [0, 1],\ x + y \geq 1. \end{cases}$$

Since π is continuous, from this we see that Δ_π is continuous as well. ∎

2.1 Properties of the first derivative

We show that the subadditivity of π implies strong conditions on the first derivative π'.

Lemma 5. *Given $x, y \in [0, 1]$ such that $\Delta_\pi(x, y) = 0$, the following hold:*

(i) *if $x, y \in D$, then $\pi'(x) = \pi'(y)$;*

(ii) *if $x, \langle x + y \rangle \in D$, then $\pi'(x) = \pi'(\langle x + y \rangle)$.*

Proof. We first prove (ii). Take $x, y \in [0,1]$ such that $x, \langle x+y\rangle \in D$ and $\Delta_\pi(x,y) = 0$. By subadditivity of π, for every $\varepsilon \in \mathbb{R}$ we have $\Delta_\pi(\langle x+\varepsilon\rangle, y) \geq 0$. Together with the equality $\Delta_\pi(x,y) = 0$, this implies that

$$\pi(\langle x+\varepsilon\rangle) - \pi(x) \geq \pi(\langle x+y+\varepsilon\rangle) - \pi(\langle x+y\rangle)$$

for every ε. Since $x, \langle x+y\rangle \in D \subseteq (0,1)$, if $|\varepsilon|$ is sufficiently small the above inequality can be rewritten as

$$\pi(x+\varepsilon) - \pi(x) \geq \pi(x+y-h+\varepsilon) - \pi(x+y-h), \tag{2}$$

where $h := (x+y) - \langle x+y\rangle \in \{0,1\}$. Note that h does not depend on ε.

For $\varepsilon > 0$, by dividing inequality (2) by ε and taking the limit for $\varepsilon \to 0^+$, we obtain that the right derivative of π at x is at least as large as that at $x+y-h = \langle x+y\rangle$. Since both x and $\langle x+y\rangle$ are in D, the right derivative at x and $\langle x+y\rangle$ coincides with the derivative at those points. Thus $\pi'(x) \geq \pi'(\langle x+y\rangle)$. Similarly, for $\varepsilon < 0$, by dividing (2) by ε and taking the limit for $\varepsilon \to 0^-$, we obtain that the left derivative of π at x is at most as large as that at $\langle x+y\rangle$, and thus $\pi'(x) \leq \pi'(\langle x+y\rangle)$. This shows that $\pi'(x) = \pi'(\langle x+y\rangle)$.

The proof of (i) is similar, but one starts with the inequality $\Delta_\pi(\langle x+\varepsilon\rangle, \langle y-\varepsilon\rangle) \geq 0$ for every ε such that $|\varepsilon|$ is sufficiently small and then obtains $\pi(x+\varepsilon)-\pi(x) \geq \pi(y)-\pi(y-\varepsilon)$. By dividing by ε (for $\varepsilon \neq 0$) and taking the limits for $\varepsilon \to 0^+$ and $\varepsilon \to 0^-$, the result follows. □

2.2 Finding good intervals

Our proof exploits the existence of intervals contained in D with several useful properties, which are described in the next lemmas. As we will see later, these intervals will contain the support of the perturbation function γ.

Before stating the next results, we recall that a *diffeomorphism of class $\mathcal{C}^1$* between two real intervals is an invertible function of class $\mathcal{C}^1$ whose inverse is also of class $\mathcal{C}^1$; equivalently, a diffeomorphism of class $\mathcal{C}^1$ is an invertible function of class $\mathcal{C}^1$ whose derivative never takes value zero.

Lemma 6. *There exist a nondegenerate closed interval $H \subseteq \pi'(D) \setminus \{0\}$ and a finite family of pairwise disjoint nondegenerate closed intervals $W_1, \dots, W_n \subseteq D$, with $n \geq 1$, such that:*

(i) *$W_1 \cup \cdots \cup W_n = \{x \in D : \pi'(x) \in H\}$;*

(ii) *for every $i \in \{1, \dots, n\}$, the restriction $\pi'|_{W_i} : W_i \to H$ is a diffeomorphism of class $\mathcal{C}^1$.*

Proof. Recall that B is the set of breakpoints of π. We call B' the set of all values taken by the right and left first derivative of π at the points in B.

Claim A. *There exist open intervals $N(a)$, for $a \in B'$, such that $N(a)$ is a neighborhood of a for every $a \in B'$, the set*

$$A := \pi'(D) \setminus \bigcup_{a \in B'} N(a)$$

is compact and has positive Lebesgue measure, and the set $D^* := \{x \in D : \pi'(x) \in A\}$ is compact.

Proof of claim. Since π is a piecewise $\mathcal{C}^2$ function that is not piecewise linear, $\pi''(x) \neq 0$ for some $x \in D$. Then, by the continuity of π'', there is a nondegenerate closed interval $I \subseteq D$ such that $\pi''(x) \neq 0$ for every $x \in I$. Thus π' is not constant over I. Since π' is continuous, this implies that $\pi'(I)$ is a nondegenerate closed interval and therefore $\mu(\pi'(I)) > 0$, where $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$.

For every $a \in B'$, define the interval $N(a) := (a - \varepsilon, a + \varepsilon)$, where $\varepsilon := \frac{\mu(\pi'(I))}{2(|B'|+1)}$. Then, since $I \subseteq D$,

$$\begin{aligned}\mu(A) &\geq \mu\left(\pi'(I) \setminus \bigcup_{a \in B'} N(a)\right)\\ &\geq \mu(\pi'(I)) - \sum_{a \in B'} \mu(N(a))\\ &\geq \mu(\pi'(I)) - \frac{|B'|}{|B'|+1}\,\mu(\pi'(I)) > 0,\end{aligned}$$

where in the third inequality we used the fact that $\mu(N(a)) = 2\varepsilon = \frac{\mu(\pi'(I))}{|B'|+1}$ for every $a \in B'$.

It remains to show that A and D^* are compact sets. Assume that $B = \{v_0, \dots, v_m\}$, where $0 = v_0 < v_1 < \cdots < v_m = 1$. For $t \in \{1, \dots, m\}$, define $g_t := \pi'|_{[v_{t-1}, v_t]}$, where $g_t(v_{t-1})$ is equal to the right derivative of π at v_{t-1}, and $g_t(v_t)$ is equal to the left derivative of π at v_t. Then

$$\begin{aligned}D^* &= \left\{x \in D : \pi'(x) \notin \bigcup_{a \in B'} N(a)\right\}\\ &= \bigcup_{t=1}^{m} \left\{x \in [v_{t-1}, v_t] : g_t(x) \notin \bigcup_{a \in B'} N(a)\right\},\end{aligned} \tag{3}$$

where the last equality follows from the fact that $g_t(v_{t-1}), g_t(v_t) \in B'$ for every $t \in \{1, \dots, m\}$. Since $\bigcup_{a \in B'} N(a)$ is an open set and g_t is continuous for every $t \in \{1, \dots, m\}$, (3) shows that D^* is a finite union of compact sets, and therefore it is compact. Furthermore $A = \pi'(D^*)$ by definition of D^*, and thus A is compact by the continuity of π'. $\diamond$

Claim B. *There exists a nondegenerate closed interval $H \subseteq A \setminus \{0\}$ such that no point $x \in D$ satisfies simultaneously $\pi'(x) \in H$ and $\pi''(x) = 0$.*

Proof of claim. Assume that for every nondegenerate interval $I \subseteq A$ there exists a point $x \in D^*$ such that $\pi'(x) \in I$ and $\pi''(x) = 0$. Then the set $K := \pi'(\{x \in D^* : \pi''(x) = 0\})$ is a dense subset of A. Since K is closed (as π', π'' are continuous and D^* is compact by Claim A), this implies that $K = A$, and thus K has positive Lebesgue measure by Claim A. In particular, the set $\pi'(\{x \in D : \pi''(x) = 0\})$ has also positive Lebesgue measure. This is a contradiction to the fact that for every $\mathcal{C}^1$ function $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$ is an open set, the set of critical values $f(\{x \in E : f'(x) = 0\})$ has Lebesgue measure zero [16, Theorem 4.3]. (Here we take $E = D$ and $f = \pi'$.)

The above argument shows that there is a nondegenerate interval $I \subseteq A$ such that no point $x \in D^*$ satisfies simultaneously $\pi'(x) \in I$ and $\pi''(x) = 0$. Since $D^* = \{x \in D : \pi'(x) \in A\}$, we can say that no point $x \in D$ satisfies simultaneously $\pi'(x) \in I$ and $\pi''(x) = 0$. The desired result now follows by choosing H to be any nondegenerate closed interval contained in $I \setminus \{0\}$. $\diamond$

For the sequel of the proof of this lemma, we recall that a connected component of a topological space $\mathcal{T}$ is an inclusionwise maximal connected subset of $\mathcal{T}$. It is well known that every connected component of $\mathcal{T}$ is a closed (and of course connected) set, and the connected components of $\mathcal{T}$ form a partition of $\mathcal{T}$ (see, e.g., [17]).

Define $W := \{x \in D : \pi'(x) \in H\} = \{x \in D^* : \pi'(x) \in H\}$. Note that W is closed, because D^* and H are closed (Claims A and B) and π' is continuous. Thus, if we see W as a topological subspace of $\mathbb{R}$ (endowed with its usual topology), the closed sets in W are precisely the closed sets in $\mathbb{R}$ that are contained in W. It follows that the connected components of W are closed connected subsets of $\mathbb{R}$, i.e., (possibly degenerate) closed intervals.

Claim C. *If C is a connected component of W, then $\pi'|_C : C \to H$ is a diffeomorphism of class $\mathcal{C}^1$.*

Proof of claim. Let C be a connected component of W. Since C is a nonempty closed interval, we can write it in the form $C = [r, s]$, where $r \le s$. By Claim B, $\pi''(x) \ne 0$ for every $x \in [r, s]$, and therefore, by continuity of π'', the sign of π'' over $[r, s]$ is always the same, say it is always positive (the other case is similar). Then π' is strictly increasing over $[r, s]$, and therefore it is one-to-one over $[r, s]$. This shows that π' induces a diffeomorphism of class $\mathcal{C}^1$ between C and $\pi'(C)$. It remains to show that $\pi'(C) = H$.

Assume that $\pi'(s)$ does not coincide with the right endpoint of H. Then, as $s \in D$ and π' is strictly increasing over $[r, s]$ and continuous on D, there exists $\varepsilon > 0$ such that $\pi'([r, s + \varepsilon]) \subseteq H$, a contradiction to the fact that $[r, s]$ is a connected component of W. Therefore $\pi'(s)$ is the right endpoint of H. Similarly, $\pi'(r)$ is the left endpoint of H. Since π' is continuous and strictly increasing, this shows that $\pi'(C) = H$. $\diamond$

Claim D. *The number of connected components of W is finite.*

Proof of claim. Let $\mathcal{C}$ be the collection of all connected components of W. We claim that for every $C \in \mathcal{C}$ there is an open interval Z_C such that $W \cap Z_C = C$. To see this, first note that, as argued in the proof of Claim C, the sign of π'' over C is always the same, say it is always positive (the other case is similar). Then, by the continuity of π'', there exists an open interval Z_C such that $C \subseteq Z_C \subseteq D$ and $\pi''(x) > 0$ for every $x \in Z_C$. This implies that π' is strictly increasing over Z_C. Then, as $\pi'(C) = H$ by Claim C, $\pi'(Z_C \setminus C) \cap H = \emptyset$. This implies that $(Z_C \setminus C) \cap W = \emptyset$, and thus $W \cap Z_C = C$, as claimed.

Therefore for every $C \in \mathcal{C}$ there is an open interval Z_C such that $W \cap Z_C = C$. The collection of intervals $\mathcal{I} := \{Z_C : C \in \mathcal{C}\}$ forms an open covering of W, which is a compact set (as it is closed and bounded). Then there must be a finite subcollection $\mathcal{I}' \subseteq \mathcal{I}$ that still covers W. However, each C in $\mathcal{C}$ intersects only one of the elements of $\mathcal{I}$, and therefore $\mathcal{I}' = \mathcal{I}$. It follows that $\mathcal{I}$ is finite, and therefore so is $\mathcal{C}$. $\diamond$

We denote by $W_1, \dots, W_n$ the connected components of W, where $n \ge 1$ because $W \ne \emptyset$. Since $W = W_1 \cup \cdots \cup W_n$, these closed intervals satisfy condition (i) of the lemma. Moreover, by Claim C, condition (ii) is also satisfied. Finally, the intervals $W_1, \dots, W_n$ are nondegenerate because they are diffeomorphic to the nondegenerate interval H. $\square$

From now on we assume that intervals $H, W_1, \dots, W_n$ satisfying the conditions of Lemma 6 are given. We define $W := W_1 \cup \cdots \cup W_n$.

For every $i, j \in \{1, \ldots, n\}$, we define the following function ϕ_{ij}, which will play a crucial role throughout the proof:

$$\phi_{ij} := \left(\pi'|_{W_j}\right)^{-1} \circ \pi'|_{W_i}. \tag{4}$$

Note that this function is well defined thanks to Lemma 6.

Lemma 7. *For every $i, j, k \in \{1, \ldots, n\}$, the following properties hold:*

(i) *$\phi_{ij} : W_i \to W_j$ is a diffeomorphism of class $\mathcal{C}^1$;*

(ii) *$\pi'(\phi_{ij}(x)) = \pi'(x)$ for every $x \in W_i$;*

(iii) *ϕ_{ii} is the identity function, $\phi_{ij} = \phi_{ji}^{-1}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$.*

Proof. Property (i) follows from the fact that $\pi'|_{W_i}$ and $\pi'|_{W_j}$ are both diffeomorphisms of class $\mathcal{C}^1$ by Lemma 6 (ii). To prove (ii), note that if $x \in W_i$ then

$$\pi'(\phi_{ij}(x)) = \pi'\left((\pi'|_{W_j})^{-1}(\pi'(x))\right) = \pi'|_{W_j}\left((\pi'|_{W_j})^{-1}(\pi'(x))\right) = \pi'(x).$$

All the properties in (iii) follow immediately from definition (4). □

Lemma 8. *Let $i, j \in \{1, \ldots, n\}$ and let $x, y \in [0, 1]$ be such that $\Delta_\pi(x, y) = 0$. If $x \in W_i$ and $y \in W_j$, then $y = \phi_{ij}(x)$.*

Proof. Since $W_i, W_j \subseteq D$, Lemma 5 (i) implies that $\pi'(x) = \pi'(y)$. By (4),

$$\phi_{ij}(x) = (\pi'|_{W_j})^{-1}(\pi'(x)) = (\pi'|_{W_j})^{-1}(\pi'(y)) = y,$$

where in the last equality we used the fact that $y \in W_j$. □

The intervals $H, W_1, \ldots, W_n$ already satisfy the useful conditions of Lemma 6, but we will need some additional properties. Therefore in the following lemmas we will show that $H, W_1, \ldots, W_n$ can be chosen to satisfy the conditions of Lemma 6 plus some other requirements. In order to enforce these additional requirements, we will need to "restrict" the intervals $H, W_1 \ldots, W_n$ currently available. This means that we will find a suitable nondegenerate closed interval $I \subseteq W_i$ for some $i \in \{1, \ldots, n\}$, and define the following new intervals: $\widetilde{H} = \pi'(I)$ and $\widetilde{W_j} = \phi_{ij}(I)$ for every $j \in \{1, \ldots, n\}$. In particular, $\widetilde{W_i} = I$, as ϕ_{ii} is the identity function by Lemma 7 (iii). This operation will be referred to as "restricting the intervals by redefining $W_i := I$". The interval I will be chosen in such a way that the desired additional requirements are satisfied. Furthermore, it will be always immediate to see that all the conditions that we will enforce (including those already guaranteed by Lemma 6) will be preserved by any further interval restriction. In particular, note that the operation of restricting intervals is defined in such a way that each ϕ_{ij} remains a diffeomorphism of W_i to W_j.

Lemma 9. *The intervals $H, W_1, \ldots, W_n$ can be chosen to satisfy the following additional property: if $\Delta_\pi(x, y) = 0$ for some $x, y \in [0, 1]$, and two of $x, y, \langle x + y\rangle$ are in B, then the other is not in W.*

Proof. Let S be the set of triplets $(x, y, \langle x+y\rangle)$ that violate the above condition; i.e., $(x, y, \langle x+y\rangle) \in S$ if and only if $x, y \in [0,1]$, $\Delta_\pi(x,y) = 0$, two of $x, y, \langle x+y\rangle$ are in B and the other is in W. Since B is a finite set, so is S. Let S' be the set of real numbers in $[0,1]$ that are an entry of some triplet in S. Since S' is a finite set, we can restrict the intervals $H, W_1, \dots, W_n$ in such a way that none of the restricted intervals $\widetilde{W}_1, \dots, \widetilde{W}_n$ contain an element of S'. Therefore no triplet now violates the desired condition. $\square$

Lemma 10. *The intervals $H, W_1, \dots, W_n$ can be chosen to satisfy the following additional conditions:*

(i) *given $i, j \in \{1, \dots, n\}$, either $\Delta_\pi(x, \phi_{ij}(x)) = 0$ for every $x \in W_i$ or $\Delta_\pi(x, \phi_{ij}(x)) > 0$ for every $x \in W_i$; in the former case, either $\langle x + \phi_{ij}(x)\rangle \in D$ for every $x \in W_i$ or $\langle x + \phi_{ij}(x)\rangle$ is the same element of B for every $x \in W_i$;*

(ii) *given $i \in \{1, \dots, n\}$ and $\bar{y} \in B$, either $\Delta_\pi(x, \bar{y}) = 0$ for every $x \in W_i$ or $\Delta_\pi(x, \bar{y}) > 0$ for every $x \in W_i$.*

Proof. For every pair of indices $i, j \in \{1, \dots, n\}$, if condition (i) is violated we restrict $H, W_1, \dots, W_n$ in such a way that the restricted intervals satisfy (i). Similarly, for every $i \in \{1, \dots, n\}$ and $\bar{y} \in B$, if condition (ii) is violated we restrict $H, W_1, \dots, W_n$ in such a way that the restricted intervals satisfy (ii). Since the number of indices in $\{1, \dots, n\}$ and the number of points in B are finite, after a finite number of interval restrictions we obtain intervals satisfying both (i) and (ii).

We start with (i). Fix indices $i, j \in \{1, \dots, n\}$. There are three cases.

Case 1. Assume that $\Delta_\pi(x, \phi_{ij}(x)) = 0$ and $\langle x + \phi_{ij}(x)\rangle \in B$ for all $x \in W_i$. Since the map $x \mapsto x + \phi_{ij}(x)$ is continuous on W_i (by Lemma 7) and B is finite, $\langle x + \phi_{ij}(x)\rangle$ is the same element of B for all $x \in W_i$. Thus the pair i, j satisfies (i).

Case 2. Assume that $\Delta_\pi(x, \phi_{ij}(x)) = 0$ for all $x \in W_i$ and there exists $\bar{x} \in W_i$ such that $\langle \bar{x} + \phi_{ij}(\bar{x})\rangle \in D$. Then, since D is an open set contained in $(0,1)$ and ϕ_{ij} is continuous, there exists a nondegenerate closed interval $I \subseteq W_i$ such that $\langle x + \phi_{ij}(x)\rangle \in D$ for all $x \in I$. If we restrict the intervals by redefining $W_i := I$, then the pair i, j satisfies (i).

Case 3. Assume that there exists $\bar{x} \in W_i$ such that $\Delta_\pi(\bar{x}, \phi_{ij}(\bar{x})) > 0$. Then, by the continuity of Δ_π and ϕ_{ij} (Lemma 4 and Lemma 7), there exists a nondegenerate closed interval $I \subseteq W_i$ such that $\Delta_\pi(x, \phi_{ij}(x)) > 0$ for all $x \in I$. If we restrict the intervals by redefining $W_i := I$, then the pair i, j satisfies (i).

We now consider condition (ii). Fix $i \in \{1, \dots, n\}$ and $\bar{y} \in B$. If $\Delta_\pi(x, \bar{y}) = 0$ for every $x \in W_i$, condition (ii) is already satisfied. So we assume that there exists $\bar{x} \in W_i$ such that $\Delta_\pi(\bar{x}, \bar{y}) > 0$. Then, as Δ_π is continuous, there exists a nondegenerate closed interval $I \subseteq W_i$ such that $\Delta_\pi(x, \bar{y}) > 0$ for every $x \in W_i$. If we restrict the intervals by redefining $W_i := I$, then the pair i, j satisfies (ii). $\square$

We introduce some terminology describing some of the possible situations appearing in Lemma 10. Given $i, j \in \{1, \dots, n\}$, if $\Delta_\pi(x, \phi_{ij}(x)) = 0$ for every $x \in W_i$, we say that i, j form an *additive pair*. If, in addition, $\langle x + \phi_{ij}(x)\rangle \in D$ for every $x \in W_i$, the pair is called *nondegenerate*, while if $\langle x + \phi_{ij}(x)\rangle$ is the same element of B for every $x \in W_i$, the pair is *degenerate*. Furthermore, if $i \in \{1, \dots, n\}$ and $\bar{y} \in B$ are such that $\Delta_\pi(x, \bar{y}) = 0$ for every $x \in W_i$, we say that $i, \bar{y}$ form a *(degenerate) additive pair*. (Note that we speak of "additive pairs" both for pairs of the type i, j with $i, j \in \{1, \dots, n\}$, and for pairs of the type $i, \bar{y}$ with $i \in \{1, \dots, n\}$ and $\bar{y} \in B$; this will never generate confusion, as the type of pair will be always specified.)

Lemma 11. *The following properties hold:*

(i) *if $i, j \in \{1, \dots, n\}$ form a nondegenerate additive pair, then there exists $k \in \{1, \dots, n\}$ such that $\langle x + \phi_{ij}(x) \rangle = \phi_{ik}(x)$ for all $x \in W_i$;*

(ii) *if $i \in \{1, \dots, n\}$ and $\bar{y} \in B$ form an additive pair, then there exists $k \in \{1, \dots, n\}$ such that $\langle x + \bar{y} \rangle = \phi_{ik}(x)$ for all $x \in W_i$.*

Proof. We first prove (i). Assume that $i, j \in \{1, \dots, n\}$ form a nondegenerate additive pair, i.e., $\Delta_\pi(x, \phi_{ij}(x)) = 0$ and $\langle x + \phi_{ij}(x) \rangle \in D$ for every $x \in W_i$. By Lemma 5 (ii), $\pi'(x) = \pi'(\langle x + \phi_{ij}(x) \rangle)$ for every $x \in W_i$. Lemma 6 (i) then implies that for every $x \in W_i$ there is an index $k(x) \in \{1, \dots, n\}$ such that $\langle x + \phi_{ij}(x) \rangle \in W_{k(x)}$. However, since the map $x \mapsto x + \phi_{ij}(x)$ is continuous and the intervals $W_1, \dots, W_n$ are pairwise disjoint subsets of the open interval $(0, 1)$, $k(x)$ has to be the same index for all $x \in W_i$, call it k. Thus $\langle x + \phi_{ij}(x) \rangle \in W_k$ for every $x \in W_i$. Since $\pi'(x) = \pi'(\langle x + \phi_{ij}(x) \rangle)$, by (4) we obtain $\phi_{ik}(x) = (\pi'|_{W_k})^{-1}(\pi'(x)) = \langle x + \phi_{ij}(x) \rangle$ for every $x \in W_i$.

We now prove (ii). Assume that $i \in \{1, \dots, n\}$ and $\bar{y} \in B$ form an additive pair, i.e., $\Delta_\pi(x, \bar{y}) = 0$ for every $x \in W_i$. By Lemma 9, $\langle x + \bar{y} \rangle \in D$ for every $x \in W_i$. The proof now continues exactly as in (i), except that $\phi_{ij}(x)$ has to be replaced with $\bar{y}$. ☐

2.3 Properties of the second derivative

The next two lemmas give some useful information about the second derivative π''.

Lemma 12. *If $i, j \in \{1, \dots, n\}$ form an additive pair and $x \in W_i$, then $\pi''(x) = \pi''(\phi_{ij}(x))\phi'_{ij}(x)$.*

Proof. Since i, j form an additive pair, we have $\Delta_\pi(x, \phi_{ij}(x)) = 0$ for every $x \in W_i$. As $x, \phi_{ij}(x) \in D$ for every $x \in W_i$, by Lemma 5 (i) we have $\pi'(x) = \pi'(\phi_{ij}(x))$ for every $x \in W_i$. Since both π' and ϕ_{ij} are differentiable, we can differentiate the previous equality, thus obtaining $\pi''(x) = \pi''(\phi_{ij}(x))\phi'_{ij}(x)$ for every $x \in W_i$. ☐

Before stating the next lemma, we recall that $\pi'|_{W_i} : W_i \to H$ is a diffeomorphism of class $\mathcal{C}^1$ for every $i \in \{1, \dots, n\}$ (Lemma 6). This implies that π'' is either always positive or always negative on W_i.

Lemma 13. *If $i, j \in \{1, \dots, n\}$ form an additive pair, then π'' is positive on at least one of W_i and W_j.*

Proof. We first consider a degenerate additive pair $i, j \in \{1, \dots, n\}$. In this case, $x + \phi_{ij}(x)$ is constant for $x \in W_i$, and therefore $\phi'_{ij}(x) = -1$ for $x \in W_i$. By Lemma 12, this implies that $\pi''(x) = -\pi''(\phi_{ij}(x))$. Therefore π'' is positive on (exactly) one of W_i and W_j.

We now consider a nondegenerate additive pair $i, j \in \{1, \dots, n\}$. Assume by contradiction that π'' is negative on both W_i and W_j. This implies that π is strictly concave on both W_i and W_j. We will use the following characterization of strict concavity: π is strictly concave on W_i if and only if $\pi(r + \varepsilon) - \pi(r) > \pi(s + \varepsilon) - \pi(s)$ for every $r, s \in W_i$ with $r < s$ and every $\varepsilon > 0$ such that $r + \varepsilon, s + \varepsilon \in W_i$.

Since π'' is negative on both W_i and W_j, we deduce from Lemma 12 that ϕ'_{ij} is always positive and thus ϕ_{ij} is a strictly increasing function. Since $\phi_{ji} = \phi_{ij}^{-1}$ by Lemma 7 (iii), this implies that ϕ_{ji} is also a strictly increasing function.

Let x be any point in the interior of W_i. Note that $\phi_{ij}(x)$ is in the interior of W_j. By the continuity of ϕ_{ij} and ϕ_{ji}, there exists $\varepsilon > 0$ such that $x + \varepsilon \in W_i$, $\phi_{ij}(x+\varepsilon) + \varepsilon \in W_j$, $\phi_{ij}(x) + \varepsilon \in W_j$, and $\phi_{ji}(\phi_{ij}(x) + \varepsilon) + \varepsilon \in W_i$. We have

$$\pi(x) + \pi(\phi_{ij}(x+\varepsilon) + \varepsilon) \geq \pi(\langle x + \phi_{ij}(x+\varepsilon) + \varepsilon \rangle) = \pi(x+\varepsilon) + \pi(\phi_{ij}(x+\varepsilon)), \tag{5}$$

where the inequality is due to the subadditivity of π, and the equality holds because i, j form an additive pair. Similarly,

$$\begin{aligned} \pi(\phi_{ij}(x)) + \pi(\phi_{ji}(\phi_{ij}(x) + \varepsilon) + \varepsilon) &\geq \pi(\langle \phi_{ij}(x) + \phi_{ji}(\phi_{ij}(x) + \varepsilon) + \varepsilon \rangle) \\ &= \pi(\phi_{ij}(x) + \varepsilon) + \pi(\phi_{ji}(\phi_{ij}(x) + \varepsilon)). \end{aligned} \tag{6}$$

We then have

$$\begin{aligned} \pi(x+\varepsilon) - \pi(x) &\leq \pi(\phi_{ij}(x+\varepsilon) + \varepsilon) - \pi(\phi_{ij}(x+\varepsilon)) \\ &< \pi(\phi_{ij}(x) + \varepsilon) - \pi(\phi_{ij}(x)) \\ &\leq \pi(\phi_{ji}(\phi_{ij}(x) + \varepsilon) + \varepsilon) - \pi(\phi_{ji}(\phi_{ij}(x) + \varepsilon)) \\ &< \pi(x+\varepsilon) - \pi(x), \end{aligned}$$

where the first inequality follows from (5), the second inequality holds because $\phi_{ij}(x) < \phi_{ij}(x+\varepsilon)$ (as ϕ_{ij} is strictly increasing) and π is strictly concave over W_j, the third inequality follows from (6), and the last inequality holds because $x < \phi_{ji}(\phi_{ij}(x) + \varepsilon)$ (as ϕ_{ji} is strictly increasing) and π is strictly concave over W_i. We have obtained a contradiction. $\square$

2.4 Lower bounds on Δ_π

We have so far obtained several technical results that will be useful for the proof of Theorem 3. We recall that we are given a minimal valid function $\pi : [0,1] \to \mathbb{R}_+$ which is a piecewise $\mathcal{C}^2$ function but not piecewise linear, and we want to show that π is not extreme by finding a nonzero perturbation function $\gamma : [0,1] \to \mathbb{R}$ such that both $\pi + t\gamma$ and $\pi - t\gamma$ are valid functions for some $t > 0$. In particular, when constructing γ we have to ensure that $\pi + t\gamma$ and $\pi - t\gamma$ are subadditive for some $t > 0$. In order to have this condition satisfied, it is important to study the behavior of Δ_π close to its zeros, because, informally speaking, for these points it is harder to maintain subadditivity when π is perturbed. The results of this subsection show that if we start from a point (x, y) in which the additivity slack $\Delta_\pi(x, y)$ is zero and we move "a little bit away from (x, y)", then Δ_π increases sufficiently fast.

For every $i, j \in \{1, \ldots, n\}$, define

$$\Gamma_{ij} := \{(x, \phi_{ij}(x)) : x \in W_i\}.$$

Since ϕ_{ij} is a $\mathcal{C}^1$ function by Lemma 7, Γ_{ij} is a $\mathcal{C}^1$ curve in the plane. Thus, given $x \in W_i$, it makes sense to define the unit normal vector $\mathbf{n}_{ij}(x)$ to the curve Γ_{ij} at the point $(x, \phi_{ij}(x))$. Furthermore, the function $x \mapsto \mathbf{n}_{ij}(x)$ is a continuous map of W_i to $\mathbb{R}^2$. Recall that the unit normal vector $\mathbf{n}_{ij}(x)$ is

$$\mathbf{n}_{ij}(x) = \left(\frac{-\phi'_{ij}(x)}{\sqrt{1 + (\phi'_{ij}(x))^2}}, \frac{1}{\sqrt{1 + (\phi'_{ij}(x))^2}} \right). \tag{7}$$

Let V_1 be any nondegenerate closed interval contained in the interior of W_1, and define $V_i := \phi_{1i}(V_1)$ for every $i \in \{2, \dots, n\}$. Note that $V_1, \dots, V_n$ are nondegenerate closed intervals satisfying the following properties:

- V_i is contained in the interior of W_i for every $i \in \{1, \dots, n\}$;
- for every $i, j \in \{1, \dots, n\}$, $\phi_{ij}|_{V_i}$ is a diffeomorphism of V_i to V_j.

For every $i, j \in \{1, \dots, n\}$, we define the curve

$$\overline{\Gamma}_{ij} := \Gamma_{ij} \cap (V_i \times V_j) = \{(x, \phi_{ij}(x)) : x \in V_i\}.$$

In the sequel, $\bar{\varepsilon}$ will denote a fixed positive number satisfying the following properties:

$$[x - 2\bar{\varepsilon}, x + 2\bar{\varepsilon}] \subseteq W_i \text{ for every } x \in V_i \text{ and } i \in \{1, \dots, n\}. \tag{8}$$

$$[x - \bar{\varepsilon}, x + \bar{\varepsilon}] \subseteq [0, 1] \text{ for every } x \in B \setminus \{0, 1\}. \tag{9}$$

The existence of such a number $\bar{\varepsilon}$ can be easily verified.

The next three lemmas give lower bounds on Δ_π. Here and in the following, we denote by $\|\mathbf{v}\|$ the Euclidean norm of a vector $\mathbf{v} \in \mathbb{R}^2$.

Lemma 14. *There exist $\varepsilon_1 \in (0, \bar{\varepsilon}]$ and $c_1 > 0$ such that the following holds: if $i, j \in \{1, \dots, n\}$ form an additive pair, then $\Delta_\pi((x, y) + \lambda \mathbf{n}_{ij}(x)) \geq c_1 \lambda^2$ for every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\varepsilon_1, \varepsilon_1]$.*

Proof. Define $\eta := \inf\{|\pi''(x)| : x \in W\}$. As π'' is continuous on W and W is compact, this infimum is indeed a minimum. Since, by Lemma 6, $\pi'|_{W_i}$ is a diffeomorphism for every $i \in \{1, \dots, n\}$, we have $\pi''(x) \neq 0$ for every $x \in W$, and therefore $\eta > 0$. We will show that the claimed result holds with $c_1 := \eta/8$ and some ε_1 to be determined later. Since the number of indices in $\{1, \dots, n\}$ is finite, it will be sufficient to show the existence of ε_1 for a fixed additive pair. Therefore in the following we fix $i, j \in \{1, \dots, n\}$ forming an additive pair.

By Lemma 13, π'' is positive on W_i or W_j. We assume that π'' is positive on W_j; if this is not the case, it is sufficient to swap the role of i and j in this proof. (Indeed, $\Delta_\pi(x, y) = \Delta_\pi(y, x)$ for every $x, y \in [0, 1]$.)

For every $(x, y) \in \overline{\Gamma}_{ij}$ (i.e., $x \in V_i$ and $y = \phi_{ij}(x) \in V_j$), we write $\mathbf{n}_{ij}(x) = (\alpha_{ij}(x), \beta_{ij}(x))$. Note that since $|\alpha_{ij}(x)| \leq 1$ and $|\beta_{ij}(x)| \leq 1$, by (8) we have $x + \lambda\alpha_{ij}(x) \in W_i$, $y + \lambda\beta_{ij}(x) \in W_j$, and $x + \lambda(\alpha_{ij}(x) + \beta_{ij}(x)) \in W_i$ for every $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$. In particular, these three values are all in the interval $(0, 1)$.

By the subadditivity of π, for every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$ we have

$$\begin{aligned} \Delta_\pi((x, y) + \lambda \mathbf{n}_{ij}(x)) &= \Delta_\pi(x + \lambda\alpha_{ij}(x), y + \lambda\beta_{ij}(x)) \\ &= \pi(x + \lambda\alpha_{ij}(x)) + \pi(y + \lambda\beta_{ij}(x)) - \pi(\langle x + y + \lambda(\alpha_{ij}(x) + \beta_{ij}(x))\rangle) \\ &\geq \pi(x + \lambda\alpha_{ij}(x)) + \pi(y + \lambda\beta_{ij}(x)) - \pi(x + \lambda(\alpha_{ij}(x) + \beta_{ij}(x))) - \pi(y). \end{aligned}$$

We denote the last expression obtained above by $\psi_x(\lambda)$:

$$\psi_x(\lambda) := \pi(x + \lambda\alpha_{ij}(x)) + \pi(y + \lambda\beta_{ij}(x)) - \pi(x + \lambda(\alpha_{ij}(x) + \beta_{ij}(x))) - \pi(y).$$

(We remark that this expression depends solely on x and λ, as $y = \phi_{ij}(x)$. However, we do not replace y with $\phi_{ij}(x)$ to keep the notation slightly simpler.) Then it is sufficient to show that there exists $\varepsilon_1 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \geq c_1\lambda^2$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_1, \varepsilon_1]$.

Note that $\psi_x(0) = 0$. Furthermore, as observed above, $x+\lambda\alpha_{ij}(x) \in W_i \subseteq D$, $y+\lambda\beta_{ij}(x) \in W_j \subseteq D$, and $x + \lambda(\alpha_{ij}(x) + \beta_{ij}(x)) \in W_i \subseteq D$ for every $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, and therefore we can differentiate $\psi_x(\lambda)$ with respect to λ:

$$\begin{aligned}\psi'_x(\lambda) &= \alpha_{ij}(x)\pi'(x + \lambda\alpha_{ij}(x)) + \beta_{ij}(x)\pi'(y + \lambda\beta_{ij}(x)) \\ &\quad - (\alpha_{ij}(x) + \beta_{ij}(x))\pi'(x + \lambda(\alpha_{ij}(x) + \beta_{ij}(x))),\end{aligned}$$

hence

$$\begin{aligned}\psi'_x(0) &= \alpha_{ij}(x)\pi'(x) + \beta_{ij}(x)\pi'(y) - (\alpha_{ij}(x) + \beta_{ij}(x))\pi'(x) \\ &= \beta_{ij}(x)(\pi'(y) - \pi'(x)) = 0,\end{aligned}$$

where the last equality follows from Lemma 5 (i) and the fact that i, j form an additive pair.

Moreover, since

$$\begin{aligned}\psi''_x(\lambda) &= (\alpha_{ij}(x))^2\pi''(x + \lambda\alpha_{ij}(x)) + (\beta_{ij}(x))^2\pi''(y + \lambda\beta_{ij}(x)) \\ &\quad - (\alpha_{ij}(x) + \beta_{ij}(x))^2\pi''(x + \lambda(\alpha_{ij}(x) + \beta_{ij}(x))),\end{aligned} \tag{10}$$

we have

$$\begin{aligned}\psi''_x(0) &= (\alpha_{ij}(x))^2\pi''(x) + (\beta_{ij}(x))^2\pi''(y) - (\alpha_{ij}(x) + \beta_{ij}(x))^2\pi''(x) \\ &= \frac{(\phi'_{ij}(x))^2\pi''(x) + \pi''(y) - (-\phi'_{ij}(x) + 1)^2\pi''(x)}{1 + (\phi'_{ij}(x))^2} \\ &= \frac{\pi''(y) + (2\phi'_{ij}(x) - 1)\pi''(x)}{1 + (\phi'_{ij}(x))^2} \\ &= \frac{(1 + 2(\phi'_{ij}(x))^2 - \phi'_{ij}(x))\pi''(y)}{1 + (\phi'_{ij}(x))^2},\end{aligned}$$

where in the second equality we used the fact that $\alpha_{ij}(x)$ and $\beta_{ij}(x)$ are the entries of $\mathbf{n}_{ij}(x)$ given in (7), and the last equality follows from Lemma 12. Since the function $t \mapsto \frac{1+2t^2-t}{1+t^2}$ can be easily verified (via standard calculus) to have minimum value larger than $1/2$, and $\pi''(y) \geq \eta$ by the definition of η and the fact that π'' is positive over W_j, it follows that $\psi''_x(0) \geq \eta/2$ for every $x \in V_i$.

Define a function $g : V_i \times [-\bar{\varepsilon}, \bar{\varepsilon}] \to \mathbb{R}$ by setting $g(x, \lambda) := \psi''_x(\lambda)$ for every $x \in V_i$ and $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$. Equation (10) shows that g is a continuous function. Since $V_i \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ is a compact set, by the Heine–Cantor theorem g is uniformly continuous. Then there exists $\varepsilon_1 \in (0, \bar{\varepsilon}]$ such that $|g(\tilde{x}, \tilde{\lambda}) - g(x, \lambda)| \leq \eta/4$ for every $(x, \lambda), (\tilde{x}, \tilde{\lambda}) \in V_i \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ with $\|(\tilde{x}, \tilde{\lambda}) - (x, \lambda)\| \leq \varepsilon_1$. In particular, if we take $\tilde{x} = x$ and $\tilde{\lambda} = 0$, we have that $|\psi''_x(\lambda) - \psi''_x(0)| \leq \eta/4$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_1, \varepsilon_1]$. Since $\psi''_x(0) \geq \eta/2$ for every $x \in V_i$, this implies that $\psi''_x(\lambda) \geq \eta/4$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_1, \varepsilon_1]$.

Since ψ_x is a function of class $\mathcal{C}^2$ over $[-\varepsilon_1, \varepsilon_1]$, by Taylor's theorem with the remainder in mean-value form, we have $\psi_x(\lambda) = \psi_x(0) + \psi'_x(0)\lambda + \frac{\psi''_x(\nu)}{2}\lambda^2$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_1, \varepsilon_1]$, where ν lies between 0 and λ. Since $\psi_x(0) = \psi'_x(0) = 0$ and $\psi''_x(\nu) \geq \eta/4$, we obtain $\psi_x(\lambda) \geq c_1\lambda^2$. This concludes the proof of the lemma. □

For degenerate additive pairs $i, j \in \{1, \ldots, n\}$, the lower bound of Lemma 14 can be improved from quadratic to linear, as shown in the next lemma. Note that for a degenerate additive pair, since $x + \phi_{ij}(x)$ is constant for $x \in W_i$, we have $\phi'_{ij}(x) = -1$ and thus the normal vector $\mathbf{n}_{ij}(x)$ is a scalar multiple of the vector $(1, 1)$. This is why in the next lemma we write $(x + \lambda, y + \lambda)$ instead of $(x, y) + \lambda\mathbf{n}_{ij}(x)$.

Lemma 15. *There exist $\varepsilon_2 \in (0, \bar{\varepsilon}]$ and $c_2 > 0$ such that the following holds: if $i, j \in \{1, \dots, n\}$ form a degenerate additive pair, then $\Delta_\pi(x+\lambda, y+\lambda) \geq c_2|\lambda|$ for every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\varepsilon_2, \varepsilon_2]$.*

Proof. Define

$$\rho := \inf \left\{ |\pi'(x) - \pi'(y)| : x \in V_i,\ y \text{ is an endpoint of } W_i,\ i \in \{1, \dots, n\} \right\}. \tag{11}$$

Since, for every $i \in \{1, \dots, n\}$, the closed interval V_i is contained in the interior of W_i and π' is strictly monotonic on W_i (by Lemma 6), we have $\rho > 0$. We will show that the claimed result holds with $c_2 := \rho$ and some ε_2 to be determined later. Since the number of indices in $\{1, \dots, n\}$ is finite, it will be sufficient to show the existence of ε_2 for a fixed degenerate additive pair. Therefore in the following we fix $i, j \in \{1, \dots, n\}$ forming a degenerate additive pair. Moreover, by Lemma 13, we can assume that π'' is positive on W_i.

For every $(x, y) \in \overline{\Gamma}_{ij}$ (i.e., $x \in V_i$ and $y = \phi_{ij}(x) \in V_j$), by (8) we have $x + \lambda \in W_i$ and $y + \lambda \in W_j$ for every $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$. In particular, these two values are in the interval $(0, 1)$.

Since i, j form a degenerate additive pair, there exists $\bar{z} \in B$ such that $\langle x + y \rangle = \bar{z}$ for every $(x, y) \in \Gamma_{ij}$. In other words, $\phi_{ij}(x) = \langle \bar{z} - x \rangle$ for every $x \in W_i$. In particular, if we write $W_i = [\alpha_i, \beta_i]$ and $W_j = [\alpha_j, \beta_j]$, we have $\langle \alpha_i + \beta_j \rangle = \langle \beta_i + \alpha_j \rangle = \bar{z}$, $\phi_{ij}(\alpha_i) = \beta_j$, and $\phi_{ij}(\beta_i) = \alpha_j$.

Case 1. We first consider the case $\lambda \geq 0$. For every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [0, \bar{\varepsilon}]$ we have

$$\begin{aligned}
\Delta_\pi(x + \lambda, y + \lambda) &= \pi(x + \lambda) + \pi(y + \lambda) - \pi(\langle x + y + 2\lambda \rangle) \\
&= \pi(x + \lambda) + \pi(y + \lambda) - \pi(\langle \alpha_i + \beta_j + 2\lambda \rangle) \\
&\geq \pi(x + \lambda) + \pi(y + \lambda) - \pi(\alpha_i + 2\lambda) - \pi(\beta_j),
\end{aligned}$$

where the second equality holds because $\langle x + y \rangle = \bar{z} = \langle \alpha_i + \beta_j \rangle$, and the inequality is due to the subadditivity of π. Then, if we define

$$\psi_x(\lambda) := \pi(x + \lambda) + \pi(y + \lambda) - \pi(\alpha_i + 2\lambda) - \pi(\beta_j),$$

it is sufficient to show that there exists $\varepsilon_2 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \geq c_2\lambda$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_2]$.

Since i, j form an additive pair and because of the properties $\langle x+y \rangle = \langle \alpha_i+\beta_j \rangle$, $y = \phi_{ij}(x)$, and $\beta_j = \phi_{ij}(\alpha_i)$, we have $\psi_x(0) = (\pi(x)+\pi(y)) - (\pi(\alpha_i)+\pi(\beta_j)) = \Delta_\pi(x, y) - \Delta_\pi(\alpha_i, \beta_j) = 0$. Furthermore, thanks to the fact that $x + \lambda \in W_i \subseteq D$, $\alpha_i + 2\lambda \in W_i \subseteq D$ (as $\lambda \geq 0$) and $y + \lambda \in W_j \subseteq D$, we can differentiate with respect to λ:

$$\psi'_x(\lambda) = \pi'(x + \lambda) + \pi'(y + \lambda) - 2\pi'(\alpha_i + 2\lambda),$$

hence the right derivative at 0 is $\psi'_x(0) = \pi'(x) + \pi'(y) - 2\pi'(\alpha_i) = 2(\pi'(x) - \pi'(\alpha_i)) = 2|\pi'(x) - \pi'(\alpha_i)| \geq 2\rho$, where the second equation is due to Lemma 5 (i), the third equation holds because π' is increasing over W_i (as π'' is positive over W_i), and the inequality follows from the definition of ρ.

Define a function $g : V_i \times [0, \bar{\varepsilon}] \to \mathbb{R}$ by setting $g(x, \lambda) := \psi'_x(\lambda)$ for every $x \in V_i$ and $\lambda \in [0, \bar{\varepsilon}]$. Since g is continuous and $V_i \times [0, \bar{\varepsilon}]$ is compact, g is uniformly continuous. Then there exists $\varepsilon_2 \in (0, \bar{\varepsilon}]$ such that $|g(\tilde{x}, \tilde{\lambda}) - g(x, \lambda)| \leq \rho$ for every $(x, \lambda), (\tilde{x}, \tilde{\lambda}) \in V_i \times [0, \bar{\varepsilon}]$ with $\|(\tilde{x}, \tilde{\lambda}) - (x, \lambda)\| \leq \varepsilon_2$. In particular, if we take $\tilde{x} = x$ and $\tilde{\lambda} = 0$, we have that $|\psi'_x(\lambda) - \psi'_x(0)| \leq \rho$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_2]$. Since $\psi'_x(0) \geq 2\rho$ for every $x \in V_i$, this implies that $\psi'_x(\lambda) \geq \rho$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_2]$.

Since $\psi_x(0) = 0$, we have $\psi_x(\lambda) = \int_0^\lambda \psi'_x(\nu) d\nu \geq \rho\lambda = c_2\lambda$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_2]$.

Case 2. We now consider the case $\lambda \leq 0$. By the subadditivity of π and the fact that $\langle x + y \rangle = \langle \beta_i + \alpha_j \rangle$, for every $\lambda \in [-\bar{\varepsilon}, 0]$ we have

$$\begin{aligned}\Delta_\pi(x + \lambda, y + \lambda) &= \pi(x + \lambda) + \pi(y + \lambda) - \pi(\langle x + y + 2\lambda \rangle) \\ &= \pi(x + \lambda) + \pi(y + \lambda) - \pi(\langle \beta_i + \alpha_j + 2\lambda \rangle) \\ &\geq \pi(x + \lambda) + \pi(y + \lambda) - \pi(\beta_i + 2\lambda) - \pi(\alpha_j).\end{aligned}$$

Then, if we define

$$\psi_x(\lambda) := \pi(x + \lambda) + \pi(y + \lambda) - \pi(\beta_i + 2\lambda) - \pi(\alpha_j),$$

it is sufficient to show that there exists $\varepsilon_2 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \geq -c_2\lambda$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_2, 0]$.

Since i, j form an additive pair and because of the properties $\langle x+y \rangle = \langle \beta_i+\alpha_j \rangle$, $y = \phi_{ij}(x)$, and $\alpha_j = \phi_{ij}(\beta_i)$, we have $\psi_x(0) = (\pi(x)+\pi(y))-(\pi(\beta_i)+\pi(\alpha_j)) = \Delta_\pi(x, y)-\Delta_\pi(\beta_i, \alpha_j) = 0$. Furthermore, since $x + \lambda, \beta_i + 2\lambda \in W_i \subseteq D$ and $y + \lambda \in W_j \subseteq D$, we have

$$\psi'_x(\lambda) = \pi'(x + \lambda) + \pi'(y + \lambda) - 2\pi'(\beta_i + 2\lambda),$$

hence the left derivative at 0 is $\psi'_x(0) = \pi'(x)+\pi'(y)-2\pi'(\beta_i) = 2(\pi'(x)-\pi'(\beta_i)) = -2|\pi'(x)-\pi'(\beta_i)| \leq -2\rho$, where the second equation is due to Lemma 5 (i), the third equation holds because π' is increasing over W_i, and the inequality follows from the definition of ρ.

An argument similar to that used in Case 1 shows that there exists $\varepsilon_2 \in (0, \bar{\varepsilon}]$ such that $\psi'_x(\lambda) \leq -\rho$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_2, 0]$. Since $\psi_x(0) = 0$, we have $\psi_x(\lambda) = \int_0^\lambda \psi'_x(\nu) d\nu \geq -\rho\lambda = -c_2\lambda$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_2, 0]$ (where the inequality has been reversed because $\lambda \leq 0$). □

We show a linear lower bound also for the case of an additive pair of the type $i, \bar{y}$ with $i \in \{1, \ldots, n\}$ and $\bar{y} \in B$.

Lemma 16. *There exist $\varepsilon_3 \in (0, \bar{\varepsilon}]$ and $c_3 > 0$ such that the following holds: if $i \in \{1, \ldots, n\}$ and $\bar{y} \in B$ form an additive pair, then $\Delta_\pi(x, \bar{y}+\lambda) \geq c_3|\lambda|$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_3, \varepsilon_3]$. (If $\bar{y} = 0$, then λ should be taken in $[0, \varepsilon_3]$; if $\bar{y} = 1$, then λ should be taken in $[-\varepsilon_3, 0]$.)*

Proof. Define ρ as in (11). As shown in the previous proof, $\rho > 0$. We will show that the claimed result holds with $c_3 := \rho/2$ and some ε_3 to be determined later. Since the number of indices in $\{1, \ldots, n\}$ and points $\bar{y} \in B$ is finite, it will be sufficient to show the existence of ε_3 for a fixed pair. Therefore in the following we fix $i \in \{1, \ldots, n\}$ and $\bar{y} \in B$ forming an additive pair.

By Lemma 11 (ii), there exists $k \in \{1, \ldots, n\}$ such that $\langle x + \bar{y} \rangle = \phi_{ik}(x)$ for every $x \in W_i$. In particular, if we write $W_i = [\alpha_i, \beta_i]$ and $W_k = [\alpha_k, \beta_k]$, then $\langle \alpha_i+\bar{y} \rangle = \alpha_k$ and $\langle \beta_i+\bar{y} \rangle = \beta_k$.

Case 1 We first assume that π'' is negative on W_i and $\lambda \geq 0$. Note that in this case we can assume $\bar{y} \neq 1$. By (9), $\bar{y} + \lambda \in [0, 1]$ for every $\lambda \in [0, \bar{\varepsilon}]$. By the subadditivity of π, for every $x \in V_i$ and $\lambda \in [0, \bar{\varepsilon}]$ we have

$$\begin{aligned}\Delta_\pi(x, \bar{y} + \lambda) &= \pi(x) + \pi(\bar{y} + \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle) \\ &\geq \pi(x) + \pi(\langle \alpha_i + \bar{y} + \lambda \rangle) - \pi(\alpha_i) - \pi(\langle x + \bar{y} + \lambda \rangle).\end{aligned}$$

Then, if we define

$$\psi_x(\lambda) := \pi(x) + \pi(\langle \alpha_i + \bar{y} + \lambda \rangle) - \pi(\alpha_i) - \pi(\langle x + \bar{y} + \lambda \rangle),$$

it is sufficient to show that there exists $\varepsilon_3 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \geq c_3 \lambda$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_3]$.

Since $i, \bar{y}$ form an additive pair, $\psi_x(0) = (\pi(x) - \pi(\langle x + \bar{y} \rangle)) - (\pi(\alpha_i) - \pi(\langle \alpha_i + \bar{y} \rangle)) = \Delta_\pi(x, \bar{y}) - \Delta_\pi(\alpha_i, \bar{y}) = 0$.

We would now like to differentiate $\psi_x(\lambda)$ with respect to λ, but we have to be careful because of the $\langle \cdot \rangle$ operator in the definition of $\psi_x(\lambda)$. However, we observe that since $\langle \alpha_i + \bar{y} \rangle = \alpha_k$, $\langle \alpha_i + \bar{y} + \lambda \rangle = \alpha_k + \lambda$ for every $\lambda \in [0, \bar{\varepsilon}]$. Moreover, $\langle x + \bar{y} + \lambda \rangle$ is either equal to $x + \bar{y} + \lambda$ for every $x \in V_i$ and $\lambda \in [0, \bar{\varepsilon}]$, or equal to $x + \bar{y} + \lambda - 1$ for every $x \in V_i$ and $\lambda \in [0, \bar{\varepsilon}]$. (This follows from (8), as $\langle x + \bar{y} \rangle \in V_k$.) In either case, we have

$$\psi'_x(\lambda) = \pi'(\langle \alpha_i + \bar{y} + \lambda \rangle) - \pi'(\langle x + \bar{y} + \lambda \rangle) = \pi'(\alpha_i + \lambda) - \pi'(x + \lambda)$$

(we used Lemma 5 (ii)) and thus the right derivative at 0 is $\psi'_x(0) = \pi'(\alpha_i) - \pi'(x) = |\pi'(\alpha_i) - \pi'(x)| \geq \rho$, where the second equation holds because π' is decreasing over W_i (as π'' is negative on W_i), and the inequality follows from the definition of ρ.

An argument similar to that used in the proof of the previous lemma shows that there exists $\varepsilon_3 \in (0, \bar{\varepsilon}]$ such that $\psi'_x(\lambda) \geq \rho/2$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_3]$. Since $\psi_x(0) = 0$, we have $\psi_x(\lambda) = \int_0^\lambda \psi'_x(\nu) d\nu \geq \rho\lambda/2 = c_3\lambda$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_3]$.

Case 2. We now assume that π'' is negative on W_i and $\lambda \leq 0$. Note that in this case we can assume $\bar{y} \neq 0$. By (9), $\bar{y} + \lambda \in [0, 1]$ for every $\lambda \in [-\bar{\varepsilon}, 0]$. By the subadditivity of π, for every $x \in V_i$ and $\lambda \in [-\bar{\varepsilon}, 0]$ we have

$$\begin{aligned}\Delta_\pi(x, \bar{y} + \lambda) &= \pi(x) + \pi(\bar{y} + \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle) \\ &\geq \pi(x) + \pi(\langle \beta_i + \bar{y} + \lambda \rangle) - \pi(\beta_i) - \pi(\langle x + \bar{y} + \lambda \rangle).\end{aligned}$$

Then, if we define

$$\psi_x(\lambda) := \pi(x) + \pi(\langle \beta_i + \bar{y} + \lambda \rangle) - \pi(\beta_i) - \pi(\langle x + \bar{y} + \lambda \rangle),$$

it suffices to show that there exists $\varepsilon_3 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \geq -c_3\lambda$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_3, 0]$.

Since $i, \bar{y}$ form an additive pair, $\psi_x(0) = (\pi(x) - \pi(\langle x + \bar{y} \rangle)) - (\pi(\beta_i) - \pi(\langle \beta_i + \bar{y} \rangle)) = \Delta_\pi(x, \bar{y}) - \Delta_\pi(\beta_i, \bar{y}) = 0$.

Similar to Case 1, one shows that

$$\psi'_x(\lambda) = \pi'(\langle \beta_i + \bar{y} + \lambda \rangle) - \pi'(\langle x + \bar{y} + \lambda \rangle) = \pi'(\beta_i + \lambda) - \pi'(x + \lambda),$$

and thus the left derivative at 0 is $\psi'_x(0) = \pi'(\beta_i) - \pi'(x) = -|\pi'(\beta_i) - \pi'(x)| \leq -\rho$, where we used the fact that π' is decreasing on W_i.

One now argues that there exists $\varepsilon_3 \in (0, \bar{\varepsilon}]$ such that $\psi'_x(\lambda) \le -\rho/2$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_3, 0]$. Since $\psi_x(0) = 0$, we have $\psi_x(\lambda) = \int_0^\lambda \psi'_x(\nu)d\nu \ge -\rho\lambda/2 = -c_3\lambda$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_3, 0]$ (where the inequality has been reversed because $\lambda \le 0$).

Case 3. We now assume that π'' is positive on W_i and $\lambda \ge 0$. By the subadditivity of π, for every $x \in V_i$ and $\lambda \in [0, \bar{\varepsilon}]$ we have

$$\begin{aligned}\Delta_\pi(x, \bar{y} + \lambda) &= \pi(x) + \pi(\bar{y} + \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle) \\ &\ge \pi(x) + \pi(\langle \beta_i + \bar{y} \rangle) - \pi(\beta_i - \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle).\end{aligned}$$

One defines

$$\psi_x(\lambda) := \pi(x) + \pi(\langle \beta_i + \bar{y} \rangle) - \pi(\beta_i - \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle)$$

and shows that there exists $\varepsilon_3 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \ge c_3\lambda$ for every $x \in V_i$ and $\lambda \in [0, \varepsilon_3]$. The proof is similar to that of Case 1.

Case 4. We finally assume that π'' is positive on W_i and $\lambda \le 0$. By the subadditivity of π, for every $x \in V_i$ and $\lambda \in [-\bar{\varepsilon}, 0]$ we have

$$\begin{aligned}\Delta_\pi(x, \bar{y} + \lambda) &= \pi(x) + \pi(\bar{y} + \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle) \\ &\ge \pi(x) + \pi(\langle \alpha_i + \bar{y} \rangle) - \pi(\alpha_i - \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle).\end{aligned}$$

One defines

$$\psi_x(\lambda) := \pi(x) + \pi(\langle \alpha_i + \bar{y} \rangle) - \pi(\alpha_i - \lambda) - \pi(\langle x + \bar{y} + \lambda \rangle)$$

and shows that there exists $\varepsilon_3 \in (0, \bar{\varepsilon}]$ such that $\psi_x(\lambda) \ge -c_3\lambda$ for every $x \in V_i$ and $\lambda \in [-\varepsilon_3, 0]$. The proof is similar to that of Case 2. $\square$

2.5 Defining the perturbation function

In this subsection we define a nonzero $\mathcal{C}^2$ function $\gamma : [0, 1] \to \mathbb{R}$ such that $\pi + t\gamma$ and $\pi - t\gamma$ are both minimal valid functions for some $t > 0$, thus proving that π is not extreme.

Recall that at the beginning of Section 2.4 we defined the nondegenerate closed intervals $V_1, \dots, V_n$ such that V_i was contained in the interior of W_i for every $i \in \{1, \dots, n\}$ and, for every $i, j \in \{1, \dots, n\}$, $\phi_{ij}|_{V_i}$ was a diffeomorphism of V_i to V_j. These intervals will contain the support of γ. However, later in the paper (proof of Lemma 26) we will need γ to take value 0 in a neighborhood of every endpoint of every V_i. For this reason, we now define subintervals $U_1, \dots, U_n$ of $V_1, \dots, V_n$ that will contain the support of γ. More precisely, similar to the construction of the V_i's, we define nondegenerate closed intervals $U_1, \dots, U_n$ such that U_i is contained in the interior of V_i for every $i \in \{1, \dots, n\}$ and, for every $i, j \in \{1, \dots, n\}$, $\phi_{ij}|_{U_i}$ is a diffeomorphism of U_i to U_j. We denote by ℓ_i and r_i the left and right endpoint (respectively) of U_i for $i \in \{1, \dots, n\}$; i.e., $U_i = [\ell_i, r_i]$. Note that since $\phi_{ij}|_{U_i}$ is a diffeomorphism of U_i to U_j, we have $\{\phi_{ij}(\ell_i), \phi_{ij}(r_i)\} = \{\ell_j, r_j\}$.

In order to construct the perturbation function γ, we need to define a $\mathcal{C}^1$ function $p : U_1 \to \mathbb{R}$ satisfying some properties, as described in the following lemma.

Lemma 17. *There exists a nonzero $\mathcal{C}^1$ function $p : U_1 \to \mathbb{R}$ such that $p(\ell_1) = p(r_1) = p'(\ell_1) = p'(r_1) = 0$ and*

$$\int_{\ell_i}^{r_i} p(\phi_{i1}(z))dz = 0 \quad \textit{for every } i \in \{1, \dots, n\}. \tag{12}$$

Proof. Let $\mathcal{F}$ denote the vector space of all $\mathcal{C}^1$ functions $p : U_1 \to \mathbb{R}$. For every $i \in \{1, \dots, n\}$, the left-hand side of equation (12) defines a linear mapping of $\mathcal{F}$ to $\mathbb{R}$. Thus (12) and the conditions $p(\ell_1) = p(r_1) = p'(\ell_1) = p'(r_1) = 0$ form a system of finitely-many homogenous linear equations in $\mathcal{F}$. Since $\mathcal{F}$ is infinite dimensional, this system has a nonzero solution (see, e.g, [9, Chapter 2, § 7, no. 5–6]). $\square$

Lemma 18. *If p is a function as in Lemma 17, then the function $\gamma : [0, 1] \to \mathbb{R}$ defined by*

$$\gamma(x) := \begin{cases} \displaystyle\int_{\phi_{1i}(\ell_1)}^{x} p(\phi_{i1}(z))dz & \text{if } x \in U_i \text{ with } i \in \{1, \dots, n\}, \\ 0 & \text{if } x \in [0, 1] \setminus (U_1 \cup \dots \cup U_n) \end{cases} \tag{13}$$

is a nonzero $\mathcal{C}^2$ function such that $\gamma'(x) = \gamma'(\phi_{ij}(x))$ for every $i, j \in \{1, \dots, n\}$ and $x \in V_i$.

Proof. Since p and ϕ_{i1} for $i \in \{1, \dots, n\}$ are $\mathcal{C}^1$ functions, the integral is well defined and γ is of class $\mathcal{C}^2$ in the interior of each U_i, with

$$\gamma'(x) = p(\phi_{i1}(x)), \quad \gamma''(x) = p'(\phi_{i1}(x))\phi'_{i1}(x). \tag{14}$$

Since γ is set to 0 outside of the U_i's, in order to ensure that γ is of class $\mathcal{C}^2$ over $[0, 1]$ it is sufficient to check that, for every $i \in \{1, \dots, n\}$, the restriction $\gamma|_{U_i}$ and its first and second derivatives take value 0 at ℓ_i and r_i. The fact that $\gamma(\ell_i) = \gamma(r_i) = 0$ follows from the definition of γ, equation (12), and the observation that $\phi_{1i}(\ell_1) \in \{\ell_i, r_i\}$; the fact that the first and second derivatives of $\gamma|_{U_i}$ vanish at ℓ_i and r_i follows from equations (14), the observation that $\{\phi_{i1}(\ell_i), \phi_{i1}(r_i)\} = \{\ell_1, r_1\}$, and the properties $p(\ell_1) = p(r_1) = p'(\ell_1) = p'(r_1) = 0$ guaranteed by Lemma 17.

We now verify that $\gamma'(x) = \gamma'(\phi_{ij}(x))$ for every $i, j \in \{1, \dots, n\}$ and $x \in V_i$. This is clear if $x \in V_i \setminus U_i$, as in this case $\phi_{ij}(x) \in V_j \setminus U_j$ and thus $\gamma'(x) = \gamma'(\phi_{ij}(x)) = 0$ by definition of γ. Otherwise, if $x \in U_i$, then

$$\gamma'(x) = p(\phi_{i1}(x)) = p(\phi_{j1}(\phi_{ij}(x))) = \gamma'(\phi_{ij}(x)),$$

where the first equation follows from (14), the second equation is due to Lemma 7 (iii), and the last one follows again from (14) and the fact that $\phi_{ij}(x) \in U_j$.

Finally, the first equation in (14) shows that γ' is a nonzero function over each U_i because p is a nonzero function over U_1 (Lemma 17), and therefore γ is nonconstant over each U_i. In particular, γ is a nonzero function. $\square$

From now on, we let γ denote a function constructed as in Lemma 18. Define $\Delta_\gamma(x, y) := \gamma(x) + \gamma(y) - \gamma(\langle x + y \rangle)$ for every $x, y \in [0, 1]$. Note that, unlike $\Delta_\pi(x, y)$, $\Delta_\gamma(x, y)$ may be negative for some $x, y \in [0, 1]$.

Lemma 19. *For every $\bar{x}, \bar{y} \in [0, 1]$, if $\Delta_\pi(\bar{x}, \bar{y}) = 0$ then $\Delta_\gamma(\bar{x}, \bar{y}) = 0$.*

Proof. In this proof we use the notation $U := U_1 \cup \dots \cup U_n$. We will exploit several times the fact that, by construction of the U_i's and by Lemma 6 (i), given $x \in D$ we have $\pi'(x) \in \pi'(U)$ if and only if $x \in U$.

Assume that $\Delta_\pi(\bar{x}, \bar{y}) = 0$. There are some cases to analyze.

Case 1. Assume that $\bar{x}, \bar{y}, \langle \bar{x} + \bar{y} \rangle \in D$. Then, by Lemma 5, $\pi'(\bar{x}) = \pi'(\bar{y}) = \pi'(\langle \bar{x} + \bar{y} \rangle)$. It follows that either all or none of $\bar{x}, \bar{y}, \langle \bar{x} + \bar{y} \rangle$ are in U. If none of $\bar{x}, \bar{y}, \langle \bar{x} + \bar{y} \rangle$ are in U, then $\gamma(\bar{x}) = \gamma(\bar{y}) = \gamma(\langle \bar{x} + \bar{y} \rangle) = 0$ and therefore $\Delta_\gamma(\bar{x}, \bar{y}) = 0$. Thus we assume that all of $\bar{x}, \bar{y}, \langle \bar{x} + \bar{y} \rangle$ are in U, and let $i, j, k \in \{1, \ldots, n\}$ be indices such that $\bar{x} \in U_i$, $\bar{y} \in U_j$ and $\langle \bar{x} + \bar{y} \rangle \in U_k$. Since $\Delta_\pi(\bar{x}, \bar{y}) = 0$, Lemma 8 implies that $\bar{y} = \phi_{ij}(\bar{x})$. Then, by Lemma 10 (i), i, j form a nondegenerate additive pair. Furthermore, by Lemma 11 (i), $\langle x + \phi_{ij}(x) \rangle = \phi_{ik}(x)$ for every $x \in U_i$.

Since $\pi'(\bar{x}) = \pi'(\bar{y}) = \pi'(\langle \bar{x} + \bar{y} \rangle)$, by definition of ϕ_{i1} (see (4)) we have $\phi_{i1}(\bar{x}) = \phi_{j1}(\bar{y}) = \phi_{k1}(\langle \bar{x} + \bar{y} \rangle)$; let us denote by $\bar{t}$ this common value, and note that $\bar{t} \in U_1$. Then, by (13),

$$
\begin{aligned}
&\gamma(\bar{x}) + \gamma(\bar{y}) - \gamma(\langle \bar{x} + \bar{y} \rangle) \\
&\qquad = \int_{\phi_{1i}(\ell_1)}^{\bar{x}} p(\phi_{i1}(z))dz + \int_{\phi_{1j}(\ell_1)}^{\bar{y}} p(\phi_{j1}(z))dz - \int_{\phi_{1k}(\ell_1)}^{\langle \bar{x}+\bar{y} \rangle} p(\phi_{k1}(z))dz \\
&\qquad = \int_{\ell_1}^{\bar{t}} p(t) \left(\phi'_{1i}(t) + \phi'_{1j}(t) - \phi'_{1k}(t) \right) dt,
\end{aligned}
$$

where in the first (respectively, second, third) integral we used the change of variable $t = \phi_{i1}(z)$ (resp., $t = \phi_{j1}(z)$, $t = \phi_{k1}(z)$).

Therefore it suffices to prove that $\phi'_{1i}(t) + \phi'_{1j}(t) - \phi'_{1k}(t) = 0$ for every $t \in U_1$. To show this, fix $t \in U_1$ and define $x := \phi_{1i}(t) \in U_i$. Then

$$
\langle \phi_{1i}(t) + \phi_{1j}(t) \rangle = \langle x + \phi_{ij}(x) \rangle = \phi_{ik}(x) = \phi_{1k}(t),
$$

where the first and the last equation are due to Lemma 7 (iii). This implies in particular that $\langle \phi_{1i}(t) + \phi_{1j}(t) \rangle \in W_k$ for every $t \in U_1$ and thus $\langle \phi_{1i}(t) + \phi_{1j}(t) \rangle$ is either equal to $\phi_{1i}(t) + \phi_{1j}(t)$ for all $t \in U_1$ or equal to $\phi_{1i}(t) + \phi_{1j}(t) - 1$ for all $t \in U_1$. In both cases we can differentiate the equality $\langle \phi_{1i}(t) + \phi_{1j}(t) \rangle = \phi_{1k}(t)$, thus obtaining $\phi'_{1i}(t) + \phi'_{1j}(t) = \phi'_{1k}(t)$ for every $t \in U_1$, as desired.

Case 2. Assume that $\bar{x}, \bar{y} \in D$ and $\langle \bar{x} + \bar{y} \rangle \in B$. By Lemma 5 (i), $\pi'(\bar{x}) = \pi'(\bar{y})$. It follows that either $\bar{x}, \bar{y}$ are both in U or neither is. If $\bar{x}, \bar{y} \notin U$, then $\gamma(\bar{x}) = \gamma(\bar{y}) = \gamma(\langle \bar{x} + \bar{y} \rangle) = 0$. Thus we assume that $\bar{x}, \bar{y}$ are in U, and let $i, j \in \{1, \ldots, n\}$ be indices such that $\bar{x} \in U_i$ and $\bar{y} \in U_j$. Since $\Delta_\pi(\bar{x}, \bar{y}) = 0$, Lemma 8 implies that $\bar{y} = \phi_{ij}(\bar{x})$. Thus, by Lemma 10 (i), i, j form a degenerate additive pair.

Since $\pi'(\bar{x}) = \pi'(\bar{y})$, by (4) we have $\phi_{i1}(\bar{x}) = \phi_{j1}(\bar{y})$; let us denote by $\bar{t} \in U_1$ this common value. Then, by a calculation similar to that carried out in the previous case, and using $\gamma(\langle \bar{x} + \bar{y} \rangle) = 0$,

$$
\gamma(\bar{x}) + \gamma(\bar{y}) - \gamma(\langle \bar{x} + \bar{y} \rangle) = \int_{\ell_1}^{\bar{t}} p(t) \left(\phi'_{1i}(t) + \phi'_{1j}(t) \right) dt.
$$

Therefore it suffices to prove that $\phi'_{1i}(t) + \phi'_{1j}(t) = 0$ for every $t \in U_1$. To show this, note that $\langle x + \phi_{ij}(x) \rangle$ is constant for $x \in U_i$, as i, j form a degenerate additive pair. This implies that $x + \phi_{ij}(x)$ is constant for $x \in U_i$. Under the change of variable $t := \phi_{i1}(x)$, this condition becomes: $\phi_{1i}(t) + \phi_{1j}(t)$ is constant for $t \in U_1$. If we derive, we obtain $\phi'_{1i}(t) + \phi'_{1j}(t) = 0$ for every $t \in U_1$.

Case 3. Assume that $\bar{x}, \langle \bar{x} + \bar{y} \rangle \in D$ and $\bar{y} \in B$. By Lemma 5 (ii), $\pi'(\bar{x}) = \pi'(\langle \bar{x} + \bar{y} \rangle)$. It follows that either $\bar{x}, \langle \bar{x} + \bar{y} \rangle$ are both in U or neither is. If $\bar{x}, \langle \bar{x} + \bar{y} \rangle \notin U$, then $\gamma(\bar{x}) = \gamma(\bar{y}) = \gamma(\langle \bar{x} + \bar{y} \rangle) = 0$. Thus we assume that $\bar{x}, \langle \bar{x} + \bar{y} \rangle$ are in U, and let $i \in \{1, \ldots, n\}$ be the index such that $\bar{x} \in U_i$. Since $\Delta_\pi(\bar{x}, \bar{y}) = 0$, Lemma 10 (ii) implies that $i, \bar{y}$ form an additive pair. Furthermore, by Lemma 11 (ii), there exists $k \in \{1, \dots, n\}$ such that $\langle x + \bar{y} \rangle = \phi_{ik}(x)$ for all $x \in U_i$.

Since $\pi'(\bar{x}) = \pi'(\langle \bar{x} + \bar{y} \rangle)$, by (4) we have $\phi_{i1}(\bar{x}) = \phi_{k1}(\langle \bar{x} + \bar{y} \rangle)$; let $\bar{t} \in U_1$ denote this common value. Then, by a calculation similar to that carried out in the first case, and using $\gamma(\bar{y}) = 0$,

$$\gamma(\bar{x}) + \gamma(\bar{y}) - \gamma(\langle \bar{x} + \bar{y} \rangle) = \int_{\ell_1}^{\bar{t}} p(t) \left(\phi'_{1i}(t) - \phi'_{1k}(t) \right) dt.$$

Therefore it suffices to prove that $\phi'_{1i}(t) - \phi'_{1k}(t) = 0$ for every $t \in U_1$. To show this, we first observe that the condition $\langle x + \bar{y} \rangle = \phi_{ik}(x)$ for all $x \in U_i$ implies that $x - \phi_{ik}(x)$ is constant for all $x \in U_i$. Under the change of variable $t := \phi_{i1}(x)$, this condition becomes: $\phi_{1i}(t) - \phi_{1k}(t)$ is constant for $t \in U_1$. If we derive, we obtain $\phi'_{1i}(t) - \phi'_{1k}(t) = 0$ for every $t \in U_1$.

Case 4. Assume that $\bar{y}, \langle \bar{x} + \bar{y} \rangle \in D$ and $\bar{x} \in B$. Since $\Delta_\pi(\bar{y}, \bar{x}) = \Delta_\pi(\bar{x}, \bar{y}) = 0$, by the previous case we have $\Delta_\gamma(\bar{y}, \bar{x}) = 0$, hence $\Delta_\gamma(\bar{x}, \bar{y}) = 0$.

Case 5. Assume that at most one of $\bar{x}, \bar{y}, \langle \bar{x} + \bar{y} \rangle$ is in D, i.e., at least two of them are in B. Then, by Lemma 9, $\bar{x}, \bar{y}, \langle \bar{x} + \bar{y} \rangle \notin U$ and therefore $\gamma(\bar{x}) = \gamma(\bar{y}) = \gamma(\langle \bar{x} + \bar{y} \rangle) = 0$.

This concludes the proof of the lemma. $\square$

2.6 Upper bounds on Δ_γ

In this subsection we prove three upper bounds for Δ_γ that correspond to the three lower bounds found for Δ_π in Lemmas 14–16.

Before stating the upper bounds, we observe that since γ is a function of class $\mathcal{C}^2$, and Δ_γ is defined as $\Delta_\gamma(x, y) := \gamma(x) + \gamma(y) - \gamma(\langle x + y \rangle)$ for every $x, y \in [0, 1]$, we have that Δ_γ is of class $\mathcal{C}^2$ at least over the open set $\{(x, y) \in (0, 1)^2 : \langle x + y \rangle \neq 0\}$. (We recall that a multivariate function is of class $\mathcal{C}^2$ if all its second partial derivatives exist and are continuous.) We claim that Δ_γ is of class $\mathcal{C}^2$ also in a neighborhood of every point $(\bar{x}, \bar{y}) \in (0, 1)^2$ such that $\langle \bar{x} + \bar{y} \rangle = 0$. This is because, for every (x, y) in some neighborhood of $(\bar{x}, \bar{y})$, $\gamma(\langle x + y \rangle) = 0$ by (13), and thus $\Delta_\gamma(x, y) = \gamma(x) + \gamma(y)$ in this neighborhood. Therefore Δ_γ is of class $\mathcal{C}^2$ over $(0, 1)^2$.

In the following proofs we will use the notation $V := V_1 \cup \dots \cup V_n$.

Lemma 20. *There exist $\delta_1 \in (0, \bar{\varepsilon}]$ and $d_1 > 0$ such that the following holds: if $i, j \in \{1, \dots, n\}$ form a nondegenerate additive pair, then $|\Delta_\gamma((x, y) + \lambda \mathbf{n}_{ij}(x))| \leq d_1 \lambda^2$ for every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\delta_1, \delta_1]$.*

Proof. Denote by $\mathbf{H}(x, y)$ the Hessian matrix of Δ_γ at a point $(x, y) \in (0, 1)^2$. For every vector $\mathbf{u} \in \mathbb{R}^2$, the second derivative of Δ_γ along direction $\mathbf{u}$ at a point $(x, y) \in (0, 1)^2$ is given by $\mathbf{u}^\top \mathbf{H}(x, y)\mathbf{u}$. Since Δ_γ is a $\mathcal{C}^2$ function over $(0, 1)^2$, and $V \times V$ is a compact subset of $(0, 1)^2$, there exists $M > 0$ such that $|\mathbf{u}^\top \mathbf{H}(x, y)\mathbf{u}| \leq M$ for every $(x, y) \in V \times V$ and $\mathbf{u} \in \mathbb{R}^2$ such that $\|\mathbf{u}\| = 1$.

Now fix $i, j \in \{1, \dots, n\}$ forming a nondegenerate additive pair. For every $(x, y) \in \overline{\Gamma}_{ij}$, we have $\Delta_\pi(x, y) = 0$ and thus, by Lemma 19, $\Delta_\gamma(x, y) = 0$.

We show that all first directional derivatives of Δ_γ are equal to zero at every $(x, y) \in \overline{\Gamma}_{ij}$. In order to prove this, since γ is a $\mathcal{C}^2$ function over $(0, 1)^2$, it suffices to show that the partial derivatives of γ vanish at every $(x, y) \in \overline{\Gamma}_{ij}$. By Lemma 11 (i), there exists $k \in \{1, \ldots, n\}$ such that $\langle x + y\rangle = \phi_{ik}(x)$ for every $x \in V_i$. By Lemma 18 we obtain

$$\frac{\partial \Delta_\gamma}{\partial x}(x, y) = \gamma'(x) - \gamma'(\langle x + y\rangle) = \gamma'(x) - \gamma'(\phi_{ik}(x)) = 0,$$
$$\frac{\partial \Delta_\gamma}{\partial y}(x, y) = \gamma'(y) - \gamma'(\langle x + y\rangle) = \gamma'(y) - \gamma'(\phi_{jk}(y)) = 0.$$

For every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, define $\psi_x(\lambda) := \Delta_\gamma((x, y) + \lambda \mathbf{n}_{ij}(x))$. The above discussion shows that $\psi_x(0) = 0$, $\psi'_x(0) = 0$, and $|\psi''_x(0)| \leq M$ for every $x \in V_i$, as $\psi'_x(0)$ and $\psi''_x(0)$ are respectively the first and the second derivative of Δ_γ along direction $\mathbf{n}_{ij}(x)$ at the point $(x, \phi_{ij}(x)) = (x, y)$.

Define a function $g : V_i \times [-\bar{\varepsilon}, \bar{\varepsilon}] \to \mathbb{R}$ by setting $g(x, \lambda) := \psi''_x(\lambda)$ for every $x \in V_i$ and $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$. Since Δ_γ is of class $\mathcal{C}^2$ over $(0, 1)^2$, g is a continuous function, and thus uniformly continuous, as $V_i \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ is a compact set. Then there exists $\delta_1 \in (0, \bar{\varepsilon}]$ such that $|g(\tilde{x}, \tilde{\lambda}) - g(x, \lambda)| \leq 1$ for every $(x, \lambda), (\tilde{x}, \tilde{\lambda}) \in V_i \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ with $\|(\tilde{x}, \tilde{\lambda}) - (x, \lambda)\| \leq \delta_1$. In particular, if we take $\tilde{x} = x$ and $\tilde{\lambda} = 0$, we have that $|\psi''_x(\lambda) - \psi''_x(0)| \leq 1$ for every $x \in V_i$ and $\lambda \in [-\delta_1, \delta_1]$. Since $|\psi''_x(0)| \leq M$ for every $x \in V_i$, this implies that $|\psi''_x(\lambda)| \leq M + 1$ for every $x \in V_i$ and $\lambda \in [-\delta_1, \delta_1]$.

Since $\psi_x(0) = 0$, $\psi'_x(0) = 0$ and $|\psi''_x(\lambda)| \leq M + 1$ for every $x \in V_i$ and $\lambda \in [-\delta_1, \delta_1]$, by Taylor's theorem with the remainder in mean-value form we obtain $|\psi_x(\lambda)| \leq (M + 1)\lambda^2/2$. The conclusion follows by taking $d_1 := (M + 1)/2$. □

Lemma 21. *There exist $\delta_2 \in (0, \bar{\varepsilon}]$ and $d_2 > 0$ such that the following holds: if $i, j \in \{1, \ldots, n\}$ form a degenerate additive pair, then $|\Delta_\gamma(x+\lambda, y+\lambda)| \leq d_2|\lambda|$ for every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\delta_2, \delta_2]$.*

Proof. As in the previous lemma, we use the fact that Δ_γ is a $\mathcal{C}^2$ function over $(0, 1)^2$, and $V \times V$ is a compact subset of $(0, 1)^2$. Then there exists $M > 0$ such that the first derivative of Δ_γ along direction $(1, 1)$ at a point (x, y) is at most M in absolute value for every $(x, y) \in V \times V$.

Now fix $i, j \in \{1, \ldots, n\}$ forming a degenerate additive pair. Then for every $(x, y) \in \overline{\Gamma}_{ij}$ we have $\Delta_\pi(x, y) = 0$ and thus, by Lemma 19, $\Delta_\gamma(x, y) = 0$. For every $(x, y) \in \overline{\Gamma}_{ij}$ and $\lambda \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, define $\psi_x(\lambda) := \Delta_\gamma(x + \lambda, y + \lambda)$. The above discussion shows that $\psi_x(0) = 0$ and $|\psi'_x(0)| \leq M$ for every $x \in V_i$, where the inequality follows from the fact that $\psi'_x(0)$ is the derivative of Δ_γ along direction $(1, 1)$ at the point $(x, \phi_{ij}(x)) = (x, y)$. If we define $g(x, \lambda) := \psi'_x(\lambda)$, an argument similar to that used in the proof of the previous lemma shows that there exists $\delta_2 \in (0, \bar{\varepsilon}]$ such that $|\psi'_x(\lambda)| \leq M + 1$ for every $x \in V_i$ and $\lambda \in [-\delta_2, \delta_2]$. This implies that $|\psi_x(\lambda)| \leq (M + 1)\lambda$ for every $x \in V_i$ and $\lambda \in [-\delta_2, \delta_2]$. The conclusion follows by taking $d_2 := M + 1$. □

Lemma 22. *There exist $\delta_3 \in (0, \bar{\varepsilon}]$ and $d_3 > 0$ such that the following holds: if $i \in \{1, \ldots, n\}$ and $\bar{y} \in B$ form an additive pair, then $|\Delta_\gamma(x, \bar{y}+\lambda)| \leq d_3|\lambda|$ for every $x \in V_i$ and $\lambda \in [-\delta_3, \delta_3]$. (If $\bar{y} = 0$, then λ should be taken in $[0, \delta_3]$; if $\bar{y} = 1$, then λ should be taken in $[-\delta_3, 0]$.)*

Proof. Here it is convenient to extend Δ_γ to $[0, 1] \times [-\bar{\varepsilon}, 1 + \bar{\varepsilon}]$ by defining a function $\widehat{\Delta}_\gamma : [0, 1] \times [-\bar{\varepsilon}, 1 + \bar{\varepsilon}] \to \mathbb{R}$, with $\widehat{\Delta}_\gamma(x, y) := \Delta_\gamma(x, \langle y\rangle)$ for every $x \in [0, 1]$ and $y \in [-\bar{\varepsilon}, 1 + \bar{\varepsilon}]$. We claim that $\widehat{\Delta}_\gamma$ is of class $\mathcal{C}^2$ in a neighborhood of every $(\bar{x}, \bar{y}) \in V \times B$. If $\bar{y} \notin \{0, 1\}$, this follows from the fact that Δ_γ is of class $\mathcal{C}^2$ in a neighborhood of $(\bar{x}, \bar{y})$. If, on the contrary,

$\bar{y} \in \{0,1\}$, then there exists an open neighborhood N of $(\bar{x}, \bar{y})$ such that every $(x,y) \in N$ satisfies $x \in (0,1)$, $\gamma(\langle y \rangle) = 0$, and $\langle x+y \rangle \in (0,1)$, where the latter condition follows from the fact that $\langle \bar{x} + \bar{y} \rangle = \bar{x} \in V \subseteq (0,1)$. Then $\widehat{\Delta}_\gamma(x,y) = \gamma(x) - \gamma(\langle x+y \rangle)$ for every $(x,y) \in N$. Since $x, \langle x+y \rangle \in (0,1)$ for every $(x,y) \in N$, $\widehat{\Delta}_\gamma$ is of class $\mathcal{C}^2$ in N.

The proof is now similar to that of Lemma 21: one observes that, since $\widehat{\Delta}_\gamma$ is of class $\mathcal{C}^2$ in a neighborhood of $V \times B$, and $V \times B$ is compact, there exists $M > 0$ such that the first derivative of Δ_γ along direction $(0,1)$ at a point $(x, \bar{y})$ is at most M in absolute value for every $(x, \bar{y}) \in V \times B$, and then argues that the desired result holds with $d_3 := M + 1$. □

2.7 Proving that π is not extreme

We can now prove Theorem 3. Specifically, we show that there exists $t > 0$ such that $\pi + t\gamma$ and $\pi - t\gamma$ are both minimal valid function, thus proving that π is not extreme.

For every $t > 0$, we define $\pi_t^+ := \pi + t\gamma$ and $\pi_t^- := \pi - t\gamma$. In order to show that π_t^+ and π_t^- are minimal valid functions for some $t > 0$, we prove that they satisfy all the conditions of Theorem 2: this is done in the next four lemmas.

Lemma 23. *There exists $\tau > 0$ such that π_t^+ and π_t^- are nonnegative functions for every $t \in [0, \tau]$.*

Proof. Define
$$\tau := \frac{\inf\{\pi(x) : x \in U\}}{\sup\{|\gamma(x)| : x \in U\}},$$
where $U := U_1 \cup \cdots \cup U_n$. We claim that the numerator is strictly positive. Assume by contradiction that this is not the case. Then, by continuity and nonnegativity of π on the compact set U, we have $\pi(\bar{x}) = 0$ for some $\bar{x} \in U$. Note that $\bar{x} \in D$, as $U \subseteq D$. Then, since $\bar{x}$ is a minimum point of π (because π is nonnegative), we have $\pi'(\bar{x}) = 0$. However, this contradicts with the fact that $\pi'(x) \neq 0$ for every $x \in W$ (see Lemma 6; recall that $U \subseteq W$). Thus the numerator is strictly positive. Since the denominator is also a positive number by Lemma 18, we have $\tau > 0$.

Take $t \in [0, \tau]$. If $x \in U$, then
$$|t\gamma(x)| \leq \tau \cdot \sup\{|\gamma(x)| : x \in U\} = \inf\{\pi(x) : x \in U\} \leq \pi(x),$$
and thus $\pi_t^+(x) \geq 0$ and $\pi_t^-(x) \geq 0$. If $x \in [0,1] \setminus U$, then $\gamma(x) = 0$ and thus $\pi_t^+(x) = \pi_t^-(x) = \pi(x) \geq 0$. Thus π_t^+ and π_t^- are nonnegative functions. □

Lemma 24. *For every $t \in \mathbb{R}$, $\pi_t^+(0) = \pi_t^+(1) = \pi_t^-(0) = \pi_t^-(1) = 0$.*

Proof. It is sufficient to observe that $\gamma(0) = \gamma(1) = 0$ by (13) and $\pi(0) = \pi(1) = 0$ because π satisfies the conditions of Theorem 2. □

Lemma 25. *For every $t \in \mathbb{R}$ and $x \in [0,1]$, $\pi_t^+(x) + \pi_t^+(\langle b - x \rangle) = \pi_t^-(x) + \pi_t^-(\langle b - x \rangle) = 1$.*

Proof. Since π satisfies the conditions of Theorem 2, we have $\pi(0) = 0$ and $\pi(0) + \pi(b) = 1$, hence $\pi(b) = 1$. Then the condition $\pi(x) + \pi(\langle b - x \rangle) = 1$ for every $x \in [0,1]$ can be restated as follows: $\Delta_\pi(x, \langle b - x \rangle) = 0$ for every $x \in [0,1]$. Thus, by Lemma 19, $\Delta_\gamma(x, \langle b - x \rangle) = 0$ for every $x \in [0,1]$.

Because $\Delta_\pi(x, \langle b - x \rangle) = \Delta_\gamma(x, \langle b - x \rangle) = 0$ for every $x \in [0,1]$, we have that, for every $t \in \mathbb{R}$, the additive slack of π_t^+ and π_t^- is zero on every pair of the type $(x, \langle b - x \rangle)$.

Since we assumed $b \in B$ at the beginning of Section 2, we have $\gamma(b) = 0$, and therefore $\pi_t^+(x) + \pi_t^+(\langle b - x\rangle) = \pi_t^-(x) + \pi_t^-(\langle b - x\rangle) = \pi(b) = 1$ for every $t \in \mathbb{R}$ and $x \in [0, 1]$. □

Lemma 26. *There exists $\sigma > 0$ such that π_t^+ and π_t^- are subadditive for every $t \in [0, \sigma]$.*

Proof. We denote by Δ_t^+ and Δ_t^- the additive slack of π_t^+ and π_t^-, respectively. Also, we use the notation $U := U_1 \cup \cdots \cup U_n$.

If $\Delta_\gamma(x, y) = 0$ for every $x, y \in [0, 1]$, then $\Delta_t^+(x, y) = \Delta_t^-(x, y) = \Delta_\pi(x, y) \geq 0$ for every $x, y \in [0, 1]$ and $t \in \mathbb{R}$, and thus π_t^+ and π_t^- are subadditive for every $t \in \mathbb{R}$.[1] Therefore in the following we assume that there exist $x^*, y^* \in [0, 1]$ such that $\Delta_\gamma(x^*, y^*) \neq 0$. Since Δ_γ is a continuous function, its value is nonzero in some neighborhood of (x^*, y^*) intersected with $[0, 1]^2$. By Lemma 19, this implies that Δ_π is nonzero in this neighborhood of (x^*, y^*).

We denote by Z the set of zeros of Δ_π, i.e., $Z := \{(x, y) \in [0, 1]^2 : \Delta_\pi(x, y) = 0\}$. Let $\varepsilon_i, c_i, \delta_i, d_i$ (with $i \in \{1, 2, 3\}$) be the constants given by Lemmas 14–16 and 20–22. By the above arguments, $\operatorname{dist}((x^*, y^*), Z) > 0$, where $\operatorname{dist}((x^*, y^*), Z)$ denotes the Euclidean distance of (x^*, y^*) from Z. Then there exists $\delta_0 > 0$ such that

$$\delta_0 \leq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3, \operatorname{dist}((x^*, y^*), Z)\}.$$

Moreover, since U_i is a closed interval contained in the interior of V_i for every $i \in \{1, \ldots, n\}$, by taking δ_0 sufficiently small we can assume that δ_0 satisfies the following additional property: for every $i \in \{1, \ldots, n\}$ and $x, y \in \mathbb{R}$, if $x \in U_i$ and $|x - y| < 2\delta_0$ then $y \in V_i$. Note that this implies that $U \subseteq [2\delta_0, 1 - 2\delta_0]$.

Fixed a value δ_0 satisfying the above properties, we denote by $Z(\delta_0)$ the set of points in $[0, 1]^2$ whose Euclidean distance from Z is smaller than δ_0. Note that $Z(\delta_0)$ is an open set and thus $[0, 1]^2 \setminus Z(\delta_0)$ is a compact set containing (x^*, y^*).

We define

$$\sigma := \min\left\{\frac{c_1}{d_1}, \frac{c_2}{d_2}, \frac{c_3}{d_3}, \frac{\inf\{\Delta_\pi(x, y) : (x, y) \in [0, 1]^2 \setminus Z(\delta_0)\}}{\sup\{|\Delta_\gamma(x, y)| : (x, y) \in [0, 1]^2 \setminus Z(\delta_0)\}}\right\}.$$

Since $d_1, d_2, d_3 > 0$, and the supremum is positive because $\Delta_\gamma(x^*, y^*) \neq 0$ and $(x^*, y^*) \in [0, 1]^2 \setminus Z(\delta_0)$, the number σ is well defined. Furthermore, the infimum in the above definition is positive, as Δ_π is a continuous function (by Lemma 4) whose set of zeros is $Z \subseteq Z(\delta_0)$. Since c_1, c_2, c_3 are also positive, we have $\sigma > 0$.

Fix any $t \in [0, \sigma]$. In order to prove that π_t^+ and π_t^- are subadditive, we show that

$$|t\Delta_\gamma(x, y)| \leq \Delta_\pi(x, y) \tag{15}$$

for every $x, y \in [0, 1]$.

Fix $x, y \in [0, 1]$. If $(x, y) \notin Z(\delta_0)$, then (15) is satisfied because

$$\begin{aligned} |t\Delta_\gamma(x, y)| &\leq \sigma \cdot \sup\{|\Delta_\gamma(x, y)| : (x, y) \in [0, 1]^2 \setminus Z(\delta_0)\} \\ &\leq \inf\{\Delta_\pi(x, y) : (x, y) \in [0, 1]^2 \setminus Z(\delta_0)\} \\ &\leq \Delta_\pi(x, y). \end{aligned}$$

Therefore in the following we assume that $(x, y) \in Z(\delta_0)$, and denote by λ the distance of (x, y) from Z (thus $\lambda < \delta_0$). Since Z is a closed set (as it is the set of zeros of the continuous function Δ_π), there exists $(\bar{x}, \bar{y}) \in Z$ whose distance from (x, y) is equal to λ.

For the next claim, recall that $V = V_1 \cup \cdots \cup V_n$.

[1] Actually, the case $\Delta_\gamma(x, y) = 0$ for every $x, y \in [0, 1]$ is not possible, as one can prove that in this situation $\gamma(x) = 0$ for every $x \in [0, 1]$, a contradiction to Lemma 18.

Claim. *If $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle \in [0,1] \setminus V$, then (x,y) satisfies* (15).

Proof of claim. Assume that $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle \in [0,1] \setminus V$. Note that $|x-\bar{x}| \le \lambda < \delta_0$. By the choice of δ_0, this implies that $x \notin U$, and thus $\gamma(x)=0$. Similarly, $\gamma(y)=0$. We show below that $\langle x+y \rangle \notin U$: this implies $\gamma(\langle x+y \rangle)=0$, thus $\Delta_\gamma(x,y)=0$, and (15) is satisfied.

Define $h := (x+y) - \langle x+y \rangle$ and $\bar{h} := (\bar{x}+\bar{y}) - \langle \bar{x}+\bar{y} \rangle$. Note that $h, \bar{h} \in \{0,1,2\}$, and $|h-\bar{h}| \le 1$, as $|(x+y)-(\bar{x}+\bar{y})| \le |x-\bar{x}|+|y-\bar{y}| < 2\delta_0 < 1$ (where the last inequality is an easy consequence of the choice of δ_0). If $h=\bar{h}$, then $|\langle x+y \rangle - \langle \bar{x}+\bar{y} \rangle| = |(x+y)-(\bar{x}+\bar{y})| < 2\delta_0$, hence, by the choice of δ_0, we have $\langle x+y \rangle \notin U$. If $h \neq \bar{h}$ (i.e., $|h-\bar{h}|=1$), then one verifies that $|\langle x+y \rangle - \langle \bar{x}+\bar{y} \rangle| > 1-2\delta_0$, which implies that $\langle x+y \rangle \notin [2\delta_0, 1-2\delta_0]$; since $U \subseteq [2\delta_0, 1-2\delta_0]$, this implies that $\langle x+y \rangle \notin U$. This concludes the proof of the claim. $\diamond$

We now distinguish some cases. We will use the fact that, by construction of the V_i's and by Lemma 6 (i), for every $x \in D$ we have $\pi'(x) \in \pi'(V)$ if and only if $x \in V$.

Case 1. Assume that $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle \in D$. Since $(\bar{x},\bar{y}) \in Z$, by Lemma 5 we have $\pi'(\bar{x}) = \pi'(\bar{y}) = \pi'(\langle \bar{x}+\bar{y} \rangle)$. It follows that either all or none of $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle$ are in V. If $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle \notin V$ then, by the Claim, (15) is satisfied. Thus we assume that $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle$ are in V.

Let $i,j \in \{1,\dots,n\}$ be indices such that $\bar{x} \in V_i$ and $\bar{y} \in V_j$. Since $\Delta_\pi(\bar{x},\bar{y})=0$, Lemma 8 implies that $\bar{y} = \phi_{ij}(\bar{x})$. Then, by Lemma 10 (i), i,j form a nondegenerate additive pair. Furthermore, $(\bar{x},\bar{y}) \in \overline{\Gamma}_{ij}$, as $\bar{x} \in V_i$. Since $\overline{\Gamma}_{ij} \subseteq \Gamma_{ij} \subseteq Z$, this shows that $(\bar{x},\bar{y})$ is a point of Γ_{ij} at minimum distance from (x,y). Then, since $(\bar{x},\bar{y})$ is not an endpoint of the curve Γ_{ij} and Γ_{ij} is a $\mathcal{C}^1$ curve, the vector $(x,y)-(\bar{x},\bar{y})$ is a scalar multiple of the normal direction $\mathbf{n}_{ij}(\bar{x})$: more precisely, $(x,y)-(\bar{x},\bar{y}) \in \{\lambda \mathbf{n}_{ij}, -\lambda \mathbf{n}_{ij}\}$. Since $\lambda < \delta_0 \le \min\{\varepsilon_1, \delta_1\}$, by combining Lemma 14 and Lemma 20 we obtain

$$|t\Delta_\gamma((\bar{x},\bar{y}) \pm \lambda \mathbf{n}_{ij})| \le td_1\lambda^2 \le \sigma d_1 \lambda^2 \le c_1\lambda^2 \le \Delta_\pi((\bar{x},\bar{y}) \pm \lambda \mathbf{n}_{ij}).$$

Then (15) is satisfied.

Case 2. Assume that $\bar{x}, \bar{y} \in D$ and $\langle \bar{x}+\bar{y} \rangle \in B$. Since $(\bar{x},\bar{y}) \in Z$, by Lemma 5 (i) we have $\pi'(\bar{x}) = \pi'(\bar{y})$. It follows that either both $\bar{x}$ and $\bar{y}$ are in V or neither is. If $\bar{x}, \bar{y} \notin V$ then, by the Claim, (15) is satisfied (as also $\langle \bar{x}+\bar{y} \rangle \notin V$). Thus we assume that $\bar{x}, \bar{y} \in V$.

Let $i,j \in \{1,\dots,n\}$ be indices such that $\bar{x} \in V_i$ and $\bar{y} \in V_j$. Since $\Delta_\pi(\bar{x},\bar{y})=0$, Lemma 8 implies that $\bar{y} = \phi_{ij}(\bar{x})$. Thus, by Lemma 10 (i), i,j form a degenerate additive pair. Furthermore, $(\bar{x},\bar{y}) \in \overline{\Gamma}_{ij}$. The same argument as that used in Case 1 shows that the vector $(x,y)-(\bar{x},\bar{y})$ is a scalar multiple of $\mathbf{n}_{ij}(\bar{x})$. Note that since i,j form a degenerate additive pair, $\Gamma_{ij} = \{(x,y) \in W_i \times W_j : x+y = \langle \bar{x}+\bar{y} \rangle\}$, thus Γ_{ij} is a segment whose normal direction is parallel to the vector $(1,1)$. Therefore $\mathbf{n}_{ij}(\bar{x}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then either $(x,y) = \left(\bar{x}+\frac{\lambda}{\sqrt{2}}, \bar{y}+\frac{\lambda}{\sqrt{2}}\right)$ or $(x,y) = \left(\bar{x}-\frac{\lambda}{\sqrt{2}}, \bar{y}-\frac{\lambda}{\sqrt{2}}\right)$. Since $\lambda/\sqrt{2} < \delta_0 \le \min\{\varepsilon_2, \delta_2\}$, by combining Lemma 15 and Lemma 21 we obtain

$$\begin{aligned}\left|t\Delta_\gamma\left(\bar{x} \pm \frac{\lambda}{\sqrt{2}}, \bar{y} \pm \frac{\lambda}{\sqrt{2}}\right)\right| &\le td_2\left|\frac{\lambda}{\sqrt{2}}\right| \le \sigma d_2 \left|\frac{\lambda}{\sqrt{2}}\right| \le c_2\left|\frac{\lambda}{\sqrt{2}}\right| \\ &\le \Delta_\pi\left(\bar{x} \pm \frac{\lambda}{\sqrt{2}}, \bar{y} \pm \frac{\lambda}{\sqrt{2}}\right).\end{aligned}$$

Then (15) is satisfied.

CASE 3. Assume that $\bar{x}, \langle \bar{x}+\bar{y} \rangle \in D$ and $\bar{y} \in B$. Since $(\bar{x},\bar{y}) \in Z$, by Lemma 5 (ii) we have $\pi'(\bar{x}) = \pi'(\langle \bar{x}+\bar{y} \rangle)$. It follows that either both $\bar{x}$ and $\langle \bar{x}+\bar{y} \rangle$ are in V or neither is. If $\bar{x}, \langle \bar{x}+\bar{y} \rangle \notin V$ then, by the Claim, (15) is satisfied. Thus we assume that $\bar{x}, \langle \bar{x}+\bar{y} \rangle \in V$.

Let $i \in \{1,\dots,n\}$ be the index such that $\bar{x} \in V_i$. Since $\Delta_\pi(\bar{x},\bar{y}) = 0$, Lemma 10 (ii) implies that $i, \bar{y}$ form an additive pair. This implies that the segment $S := W_i \times \{\bar{y}\}$ is contained in Z. Then $(\bar{x},\bar{y})$ is a point of S at minimum distance from (x,y). Since $(\bar{x},\bar{y})$ is not an endpoint of S (as $\bar{x} \in V_i$), the vector $(x,y)-(\bar{x},\bar{y})$ is orthogonal to S. Then either $(x,y) = (\bar{x},\bar{y}+\lambda)$ or $(x,y) = (\bar{x},\bar{y}-\lambda)$. Since $\lambda < \delta_0 \le \min\{\varepsilon_3, \delta_3\}$, by combining Lemma 16 and Lemma 22 we obtain

$$|t\Delta_\gamma(\bar{x},\bar{y}\pm\lambda)| \le td_3|\lambda| \le \sigma d_3|\lambda| \le c_3|\lambda| \le \Delta_\pi(\bar{x},\bar{y}\pm\lambda).$$

Then (15) is satisfied.

CASE 4. Assume that $\bar{y}, \langle \bar{x}+\bar{y} \rangle \in D$ and $\bar{x} \in B$. By the previous case, we have $|t\Delta_\gamma(y,x)| \le \Delta_\pi(y,x)$. Since $\Delta_\gamma(y,x) = \Delta_\gamma(x,y)$ and $\Delta_\pi(y,x) = \Delta_\pi(x,y)$, (15) is satisfied.

CASE 5. Assume that at most one of $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle$ is in D, i.e., at least two of them are in B. Then, by Lemma 9, $\bar{x}, \bar{y}, \langle \bar{x}+\bar{y} \rangle \notin V$. By the Claim, (15) is satisfied.

Since the above analysis covers all possible cases, the proof of the lemma is complete. □

Let σ and τ be numbers satisfying the conditions of Lemmas 23 and 26. Let t be any real number such that $0 < t \le \min\{\tau,\sigma\}$. By Lemmas 23–26 and Theorem 2, π_t^+ and π_t^- are minimal valid functions. This shows that π is not an extreme function, thus concluding the proof of Theorem 3.

3 Concluding remarks

Theorem 3 shows that if a continuous extreme function for the one-dimensional pure integer infinite group relaxation I_b is piecewise of class $\mathcal{C}^2$, then it is piecewise linear. Roughly speaking, this means that continuous piecewise smooth extreme functions are piecewise linear. It is not clear whether a similar result can be proven when the smoothness assumption on π is only slightly weakened, for instance by assuming that π is differentiable twice but its second derivative is not continuous, or that π belongs to some Hölder class $\mathcal{C}^{1,\alpha}$ with $\alpha \in (0,1]$. Our proof exploits the piecewise $\mathcal{C}^2$ hypothesis several times, so we do not know how to deal with a weaker assumption. On the other hand, there is no evidence that the result fails even under far weaker assumptions, for instance when π is assumed to be just piecewise differentiable. (Note however that the smoothness assumption cannot be relaxed to piecewise continuity: as discussed in the introduction, continuous extreme functions that are not piecewise linear have been discovered.)

Another possible strengthening of Theorem 3 would be the following: is it true that an extreme function cannot have a point in which the second derivative exists and is nonzero? We do not have a counterexample to this conjecture, but we do not see how it could be proven.

Finally, we mention that although we have defined piecewise linear functions to be continuous, some authors give a weaker definition that allows piecewise linear functions to be discontinuous at the breakpoints. Indeed, several extreme functions are known that are piecewise linear according to this weaker definition (see, e.g., [7, 15]). Similarly, a function could be defined to be piecewise of class $\mathcal{C}^2$ when it is of class $\mathcal{C}^2$ in the *interior* of each interval delimited by a pair of consecutive breakpoints, thus allowing these functions to be discontinuous. It would be interesting to understand whether the result in this paper can be extended to discontinuous functions.

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