The Mixing Set with Divisible Capacities*

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Abstract. Given rational numbers $C_0, \dots, C_m$ and $b_0, \dots, b_m$, the mixing set with arbitrary capacities is the mixed-integer set defined by conditions

$$\begin{aligned} s + C_t z_t &\geq b_t, \quad 0 \leq t \leq m, \\ s &\geq 0, \\ z_t &\text{ integer}, \quad 0 \leq t \leq m. \end{aligned}$$

Such a set has applications in lot-sizing problems. We study the special case of divisible capacities, i.e. C_t / C_{t-1} is a positive integer for $1 \leq t \leq m$. Under this assumption, we give an extended formulation for the convex hull of the above set that uses a quadratic number of variables and constraints.

Keywords: mixed-integer programming, compact extended formulations, mixing sets.

1 Introduction

Given rational numbers $C_0, \dots, C_m$ and $b_0, \dots, b_m$, the mixing set with arbitrary capacities is the mixed-integer set defined by conditions

$$s + C_t z_t \geq b_t, \quad 0 \leq t \leq m, \tag{1}$$

$$s \geq 0, \tag{2}$$

$$z_t \text{ integer}, \quad 0 \leq t \leq m. \tag{3}$$

The above set generalizes the *mixing set*, which is a set of the type (1)–(3) with $C_t = 1$ for all $0 \leq t \leq m$. The mixing set, which was introduced and studied by Günlük and Pochet [9] and further investigated by Miller and Wolsey [12], has played an important role in studying production planning problems (in particular lot-sizing [17]).

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When the values of the capacities C_t are arbitrary, (1)–(3) constitutes a relaxation of lot-sizing problems where different batch sizes or velocities of the machines are allowed. Giving a linear inequality description of the convex hull of such a set seems to be difficult and indeed it is not known whether linear optimization over (1)–(3) can be carried out in polynomial time.

We consider here the special case of a set defined by (1)–(3) where the capacities form a sequence of divisible numbers: that is, C_t/C_{t-1} is a positive integer for $1 \leq t \leq m$. We call such a set the *mixing set with divisible capacities* and we denote it by DIV. Our main result is a compact extended formulation for the polyhedron $\text{conv}(DIV)$, the convex hull of DIV.

Here we use the following terminology. A *formulation* of a polyhedron P (in its original space) is a description of P as the intersection of a finite number of half-spaces. So it consists of a system of linear inequalities $Cx \geq d$ such that $P = \{x : Cx \geq d\}$. A formulation of P is *extended* whenever it gives a polyhedral description of the type $Q = \{(x, \mu) : Ax + B\mu \geq d\}$ in a space that uses variables (x, μ) and includes the original x-space, so that P is the projection of Q onto the x-space.

If P is the convex hull of a mixed-integer set (such as the convex hull of the set defined by (1)–(3)), we say that a formulation is *compact* if its size (i.e. the number of inequalities and variables of the system defining P or Q as above) is bounded by a polynomial function of the description of the mixed-integer set (in our case the size of the system (1)–(2)).

The assumption of divisibility of the coefficients was exploited by several authors to tackle integer sets that are otherwise untractable, such as integer knapsack problems. Under the divisibility assumption, Marcotte [11] gave a simple formulation of the integer knapsack set without upper bounds on the variables. Pochet and Wolsey [16] studied the same set where the knapsack inequality is of the "$\geq$" type. Pochet and Weismantel [13] provided a linear inequality description of the knapsack set where all variables are bounded. Other hard problems studied under the assumption of divisibility of the coefficients include network design [14], lot-sizing problems [4] and the integer Carathéodory property for rational cones [10].

The mixing set with divisible capacities DIV was studied recently by Zhao and de Farias [20], who gave a polynomial-time algorithm to optimize a linear function over DIV (see also Di Summa [6]).

A formulation of the polyhedron $\text{conv}(DIV)$ either in the original space or in an extended space was not known for the general case and such a formulation does not seem to be easily obtainable by applying known techniques for constructing compact extended formulations, such as taking unions of polyhedra [1, 4] or enumeration of fractional parts [12, 3, 18, 19].

A formulation of $\text{conv}(DIV)$ was only known for some special cases. For the set DIV with $C_t = 1$ for $0 \leq t \leq m$ (i.e. the mixing set), a linear inequality description of the convex hull in the original space was given by Günlük and Pochet [9] and a compact extended formulation was obtained by Miller and Wolsey [12]. For the set DIV with only two distinct values of the capacities, Van Vyve [18] and Constantino, Miller and Van Vyve [5] gave a linear inequality description of the convex hull of the set both in the original space and in an extended space. Zhao and de Farias [20] gave a linear inequality formulation of conv(DIV) in its original space under some special assumptions on the parameters $C_0, \dots, C_m$ and $b_0, \dots, b_m$.

Since a polynomial-time algorithm for the set DIV was already known, one might wonder why we are interested in giving a polyhedral description of DIV. However recall that mixed-integer sets of the type (1)–(3) appear as substructures in multi-item lot-sizing problems, thus a linear inequality description of conv(DIV) leads to strong formulations for such problems.

In order to study the set DIV, we rewrite (1)–(3) in a slightly different form, as we need to have $C_t \neq C_{t'}$ for $t \neq t'$. In other words, we group together the inequalities (1) associated with the same capacity C_t and write the set DIV as follows:

$$s + C_k z_t \geq b_t, \quad t \in I_k,\ 0 \leq k \leq n, \tag{4}$$

$$s \geq 0, \tag{5}$$

$$z_t \text{ integer}, \quad t \in I_0 \cup \dots \cup I_n, \tag{6}$$

where $I_0, \dots, I_n$ are pairwise disjoint sets of indices and C_k/C_{k-1} is an integer *greater than one* for $1 \leq k \leq n$.

The main idea of our approach to construct a compact extended formulation for conv(DIV) can be summarized as follows: We consider the following expansion of s:

$$s = \alpha_0(s) + \sum_{j=1}^{n+1} \alpha_j(s) C_{j-1} \ ,$$

where $0 \leq \alpha_j(x) < \frac{C_j}{C_{j-1}}$ for $1 \leq j \leq n$, and $0 \leq \alpha_0(x) < C_0$. Furthermore $\alpha_j(x)$ is an integer for $1 \leq j \leq n+1$. We show that for fixed j, the number of possible values that $\alpha_j(s)$ can take over the set of vertices of conv(DIV) is bounded by a linear function of the number of constraints (1). To each of these possible values (say v), we associate an indicator variable that takes value 1 if $\alpha_j(s) = v$ and 0 otherwise. These indicator variables are the important additional variables of our compact extended formulation.

2 Expansion of a Number

Our arguments are based on the following expansion of a real number x:

$$x = \alpha_0(x) + \sum_{j=1}^{n+1} \alpha_j(x) C_{j-1} \ , \tag{7}$$

where $0 \le \alpha_j(x) < \frac{C_j}{C_{j-1}}$ for $1 \le j \le n$, and $0 \le \alpha_0(x) < C_0$. Furthermore $\alpha_j(x)$ is an integer for $1 \le j \le n+1$. Note that this expansion is unique. If we let

$$f_0(x) = \alpha_0(x), \quad f_k(x) = f_0(x) + \sum_{j=1}^{k} \alpha_j(x) C_{j-1} \ \text{ for } 1 \le k \le n \ ,$$

we have that

$$x = f_k(x) + \sum_{j=k+1}^{n+1} \alpha_j(x) C_{j-1} \ \text{ for } 0 \le k \le n \ . \tag{8}$$

Therefore for $0 \le k \le n$, $f_k(x)$ is the remainder of the division of x by C_k and it can be checked that

$$\alpha_k(x) = \left\lfloor \frac{f_k(x)}{C_{k-1}} \right\rfloor = \frac{f_k(x) - f_{k-1}(x)}{C_{k-1}} \quad \text{for } 1 \le k \le n \ ,$$
$$\alpha_{n+1}(x) = \left\lfloor \frac{x}{C_n} \right\rfloor = \frac{x - f_n(x)}{C_n} \ .$$

We also define $\Delta_k(x)$ as the integer quotient of the division of x by C_k, i.e.

$$\Delta_k(x) = \frac{x - f_k(x)}{C_k} = \sum_{j=k+1}^{n+1} \frac{C_{j-1}}{C_k} \alpha_j(x) \ \text{ for } 0 \le k \le n \ . \tag{9}$$

3 The Vertices of conv(DIV)

We consider the mixed-integer set DIV defined by (4)–(6) with the divisibility assumption. That is, $C_0 > 0$ and for $1 \le k \le n$, $C_k/C_{k-1} \ge 2$ is an integer. Also $I_j \cap I_k = \varnothing$ for $j \ne k$ and we set $b_l := 0$ where $l \notin I_0 \cup \cdots \cup I_n$. For $0 \le k \le n$, define $J_k = I_k \cup I_{k+1} \cup \cdots \cup I_n \cup \{l\}$.

We give an extended formulation for $\text{conv}(DIV)$ with $\mathcal{O}(mn)$ constraints and variables, where $m = |I_0| + \cdots + |I_n|$. The first step is studying the vertices of the polyhedron $\text{conv}(DIV)$. Several properties of the vertices of $\text{conv}(DIV)$ were given by Zhao and de Farias [20], who also described an algorithm to list all the vertices. We introduce here the properties that will be needed for our formulation.

Given s and an index $1 \le k \le n$, for $t \in J_0$ define

$$b_t^k = \begin{cases} b_t + C_k & \text{if } f_k(b_t) > f_k(s) \\ b_t & \text{if } f_k(b_t) \le f_k(s) \ . \end{cases}$$

Lemma 1. *Consider indices $0 \le k \le \ell$. Then, for $t \in I_\ell$, the inequality*

$$\Delta_k(s) + \frac{C_\ell}{C_k} z_t \ge \Delta_k\left(b_t^k\right) \tag{10}$$

is valid for $\text{conv}(DIV)$ *and implies inequality $s + C_\ell z_t \ge b_t$.*

Proof. Expanding s and b_t as in the first part of (9), inequality $s + C_\ell z_t \geq b_t$ can be rewritten as

$$\Delta_k(s) + \frac{C_\ell}{C_k} z_t \geq \Delta_k(b_t) + \frac{f_k(b_t) - f_k(s)}{C_k} \quad .$$

Since $\ell \geq k$, $\Delta_k(s) + \frac{C_\ell}{C_k} z_t$ is an integer. Therefore

$$\Delta_k(s) + \frac{C_\ell}{C_k} z_t \geq \Delta_k(b_t) + \left\lceil \frac{f_k(b_t) - f_k(s)}{C_k} \right\rceil = \Delta_k\left(b_t^k\right) \quad .$$

This also shows that (10) implies the original inequality $s + C_\ell z_t \geq b_t$. □

Note that (10) involves the term b_t^k and thus is not a linear inequality. We will show how to linearize this constraint, using the fact that for fixed k, the number b_t^k can take only two values.

Lemma 2. *Let* $(\bar{s}, \bar{z})$ *be any vector in* $\mathrm{conv}(DIV)$.

1. *Given indices* $1 \leq k \leq \ell$ *and* $t \in I_\ell$, *if* $\alpha_k(\bar{s}) \neq \alpha_k\left(b_t^{k-1}\right)$ *then* $\bar{s} + C_\ell \bar{z}_t \geq b_t + C_{k-1}$.
2. *Given an index* $k \geq 1$, *if* $\alpha_k(\bar{s}) \neq 0$ *then* $\bar{s} \geq C_{k-1}$.

Proof. We prove the first statement. By Lemma 1, $(\bar{s}, \bar{z})$ satisfies (10) for the pair of indices $k-1, \ell$, that is,

$$\Delta_{k-1}(s) + \frac{C_\ell}{C_{k-1}} z_t \geq \Delta_{k-1}\left(b_t^{k-1}\right) \quad .$$

By (9), the above inequality can be rewritten as

$$\sum_{j=k}^{n+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j(s) + \frac{C_\ell}{C_{k-1}} z_t \geq \sum_{j=k}^{n+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j\left(b_t^{k-1}\right) \quad ,$$

or equivalently as

$$\sum_{j=k+1}^{n+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j(s) + \frac{C_\ell}{C_{k-1}} z_t - \sum_{j=k+1}^{n+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j\left(b_t^{k-1}\right) \geq \alpha_k\left(b_t^{k-1}\right) - \alpha_k(s) \quad . \quad (11)$$

Since $\left\{\frac{C_{j-1}}{C_{k-1}}, k < j \leq n+1\right\}$ is a sequence of divisible integers and since $\ell \geq k$, the left-hand side of the above inequality is an integer multiple of C_k/C_{k-1}. Since the right-hand side is an integer satisfying $-C_k/C_{k-1} < \alpha_k\left(b_t^{k-1}\right) - \alpha_k(s) < C_k/C_{k-1}$, this shows that if $\alpha_k(\bar{s}) \neq \alpha_k\left(b_t^{k-1}\right)$, then (11) cannot be tight for $(\bar{s}, \bar{z})$, thus

$$\Delta_{k-1}(\bar{s}) + \frac{C_\ell}{C_{k-1}} \bar{z}_t \geq \Delta_{k-1}\left(b_t^{k-1}\right) + 1 \quad .$$

Since $b_t^{k-1} = b_t + C_{k-1}$ if $f_{k-1}(b_t) > f_{k-1}(\bar{s})$ and $b_t^{k-1} = b_t$ if $f_{k-1}(b_t) \leq f_{k-1}(\bar{s})$, this shows that in both cases

$$\frac{f_{k-1}(\bar{s})}{C_{k-1}} + \Delta_{k-1}(\bar{s}) + \frac{C_\ell}{C_{k-1}}\bar{z}_t \geq \Delta_{k-1}(b_t) + \frac{f_{k-1}(b_t)}{C_{k-1}} + 1 \ .$$

Multiplying the above inequality by C_{k-1} gives $\bar{s} + C_\ell \bar{z}_t \geq b_t + C_{k-1}$.

The proof of the second statement is an immediate consequence of expansion (7). □

Lemma 3. *If $(\bar{s}, \bar{z})$ is a vertex of* $\mathrm{conv}(DIV)$*, then the following two properties hold:*

1. $\alpha_0(\bar{s}) = \alpha_0(b_t)$ *for some* $t \in J_0$.
2. *For* $1 \leq k \leq n$, $\alpha_k(\bar{s}) = \alpha_k\left(b_t^{k-1}\right)$ *for some* $t \in J_k$.

Proof. Let $(\bar{s}, \bar{z})$ be a vertex of $\mathrm{conv}(DIV)$. Since $\bar{z}$ is an integral vector, if 1. is violated then there is $\varepsilon \neq 0$ such that $(\bar{s} \pm \varepsilon, \bar{z}) \in \mathrm{conv}(DIV)$, a contradiction.

Assume that 2. is violated, i.e. there is an index k such that $\alpha_k(\bar{s}) \neq \alpha_k\left(b_t^{k-1}\right)$ for all $t \in J_k$. In particular, for $t = l$ we have $\alpha_k(\bar{s}) \neq 0$. Consider the vector v_{k-1} defined as follows:

$$s = -C_{k-1}, \quad z_t = \frac{C_{k-1}}{C_\ell},\, t \in I_\ell,\, \ell \leq k-1, \quad z_t = 0,\, t \in I_\ell,\, \ell > k-1 \ .$$

By Lemma 2 we have that $s \geq C_{k-1}$ and $\bar{s} + C_\ell \bar{z}_t \geq b_t + C_{k-1}$ for $t \in I_\ell$, $\ell \geq k$. This shows that the vectors $(\bar{s}, \bar{z}) \pm v_{k-1}$ belong to $\mathrm{conv}(DIV)$. Hence $(\bar{s}, \bar{z})$ is not a vertex of $\mathrm{conv}(DIV)$. □

We now introduce extra variables to model the possible values taken by s at a vertex of $\mathrm{conv}(DIV)$. The new variables are the following:

– Δ_0, $w_{0,t}$ for $t \in J_0$;
– Δ_k, $w^{\downarrow}_{k,t}, w^{\uparrow}_{k,t}$ for $1 \leq k \leq n$ and $t \in J_k$.

The role of the above variables is as follows:

– Variables Δ_k are the integer quotients of the division of s by C_k. That is, $\Delta_k = \Delta_k(s)$ as defined in (9).
– Variable $w_{0,t} = 1$ whenever $\alpha_0(s) = \alpha_0(b_t)$ and $w_{0,t} = 0$ otherwise.
– Variable $w^{\downarrow}_{k,t} = 1$ whenever $\alpha_k(s) = \alpha_k(b_t)$ and $w^{\uparrow}_{k,t} = 1$ whenever $\alpha_k(s) = \alpha_k(b_t + C_{k-1})$; $w^{\downarrow}_{k,t} = w^{\uparrow}_{k,t} = 0$ otherwise.

Consider the following conditions:

$$s = C_0 \varDelta_0 + \sum_{i \in J_0} \alpha_0(b_i) w_{0,i}, \tag{12}$$

$$\varDelta_{k-1} = \frac{C_k}{C_{k-1}} \varDelta_k + \sum_{i \in J_k} \left(\alpha_k(b_i) w^{\downarrow}_{k,i} + \alpha_k(b_i + C_{k-1}) w^{\uparrow}_{k,i} \right), \quad 1 \le k \le n, \tag{13}$$

$$w_{0,i} \ge 0,\ i \in J_0;\ \sum_{i \in J_0} w_{0,i} = 1, \tag{14}$$

$$w^{\downarrow}_{k,i}, w^{\uparrow}_{k,i} \ge 0, : i \in J_k,\ 1 \le k \le n;\ \sum_{i \in J_k} \left(w^{\downarrow}_{k,i} + w^{\uparrow}_{k,i} \right) = 1, \quad 1 \le k \le n, \tag{15}$$

$$\sum_{\substack{i \in J_0: \\ \alpha_0(b_i) \ge \alpha_0(b_t)}} w_{0,i} \ge w^{\downarrow}_{1,t}, \quad t \in J_1, \tag{16}$$

$$\sum_{\substack{i \in J_k: \\ f_k(b_i) \ge f_k(b_t)}} w^{\downarrow}_{k,i} + \sum_{\substack{i \in J_k: \\ \alpha_k(b_i + C_{k-1}) \ge \alpha_k(b_t)+1}} w^{\uparrow}_{k,i} \ge w^{\downarrow}_{k+1,t}, \quad t \in J_{k+1},\ 1 \le k \le n-1, \tag{17}$$

$$\varDelta_k, w_{0,i}, w^{\downarrow}_{k,i}, w^{\uparrow}_{k,i} \text{ integer}, \quad i \in J_k,\ 0 \le k \le n. \tag{18}$$

Lemma 4. *If* $(\bar{s}, \bar{z})$ *is a vertex of* $\operatorname{conv}(DIV)$*, then* $(\bar{s}, \bar{z})$ *can be completed to a vector* $\left(\bar{s}, \bar{z}, \bar{\varDelta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ *satisfying* (12)–(18).

Proof. Given vertex $(\bar{s}, \bar{z})$, let i_0 be any index in J_0 such that $\alpha_0(b_{i_0}) = \alpha_0(\bar{s})$ (i_0 exists by Lemma 3). Take $\bar{w}_{0,i_0} = 1$ and $\bar{w}_{0,i} = 0$ for $i \ne i_0$.
Now fix $k \ge 1$ and define

$$T_k(\bar{s}) = \{ i \in J_k : \alpha_k(\bar{s}) = \alpha_k(b_i),\ f_{k-1}(\bar{s}) \ge f_{k-1}(b_i) \} \ .$$

If $T_k(\bar{s}) \ne \varnothing$, then define i_k as any element in $T_k(\bar{s})$ such that $f_{k-1}(b_{i_k})$ is maximum and take $\bar{w}^{\downarrow}_{k,i_k} = 1$. Otherwise ($T_k(\bar{s}) = \varnothing$) define i_k as any index in J_k such that $\alpha_k(\bar{s}) = \alpha_k(b_{i_k} + C_{k-1})$ (i_k exists by Lemma 3) and take $\bar{w}^{\uparrow}_{k,i_k} = 1$. Finally take $\bar{\varDelta}_k = \varDelta_k(\bar{s})$ for $0 \le k \le n$.

We prove that the point thus constructed satisfies (12)–(18). To see that (12) is satisfied, note that

$$C_0 \bar{\varDelta}_0 + \sum_{i \in J_0} \alpha_0(b_i) \bar{w}_{0,i} = C_0 \varDelta_0(\bar{s}) + \alpha_0(b_{i_0}) = C_0 \varDelta_0(\bar{s}) + f_0(b_{i_0}) = \bar{s} \ .$$

To prove (13), note that the following chain of equations holds:

$$\frac{C_k}{C_{k-1}} \bar{\varDelta}_k + \sum_{i \in J_k} \left(\alpha_k(b_i) \bar{w}^{\downarrow}_{k,i} + \alpha_k(b_i + C_{k-1}) \bar{w}^{\uparrow}_{k,i} \right)$$

$$= \frac{C_k}{C_{k-1}} \varDelta_k(\bar{s}) + \alpha_k(\bar{s}) = \varDelta_{k-1}(\bar{s}) = \bar{\varDelta}_{k-1} \ .$$

To see that (16) is verified, suppose that $\bar{w}^{\downarrow}_{1,t} = 1$ for an index $t \in J_1$. Then necessarily $t = i_1 \in T_1(\bar{s})$ and thus $f_0(\bar{s}) \geq f_0(b_t)$, that is, $\alpha_0(\bar{s}) \geq \alpha_0(b_t)$. Then $\alpha_0(b_{i_0}) = \alpha_0(\bar{s}) \geq \alpha_0(b_t)$ and (16) is satisfied.

We now consider (17) for $k \geq 1$. Suppose that $w^{\downarrow}_{k+1,t} = 1$ for an index $t \in J_{k+1}$. Then necessarily $t = i_{k+1} \in T_{k+1}(\bar{s})$. Therefore $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_t)$ and $f_k(\bar{s}) \geq f_k(b_t)$. This implies $\alpha_k(\bar{s}) \geq \alpha_k(b_t)$. We distinguish two cases.

1. Assume $\alpha_k(\bar{s}) \geq \alpha_k(b_t) + 1$. If $T_k(\bar{s}) \neq \varnothing$ then $\bar{w}^{\downarrow}_{k,i} = 1$ for an index $i \in J_k$ such that $\alpha_k(b_i) = \alpha_k(\bar{s}) \geq \alpha_k(b_t) + 1$. Then $f_k(b_i) \geq f_k(b_t)$. If $T_k(\bar{s}) = \varnothing$ then $\bar{w}^{\uparrow}_{k,i} = 1$ for an index $i \in J_k$ such that $\alpha_k(b_i + C_{k-1}) = \alpha_k(\bar{s}) \geq \alpha_k(b_t) + 1$. In both cases (17) is satisfied.
2. Now assume $\alpha_k(\bar{s}) = \alpha_k(b_t)$. In this case inequality $f_k(\bar{s}) \geq f_k(b_t)$ implies $f_{k-1}(\bar{s}) \geq f_{k-1}(b_t)$, thus $t \in T_k(\bar{s}) \neq \varnothing$. Then the choice of i_k shows that $\alpha_k(b_{i_k}) = \alpha_k(\bar{s}) = \alpha_k(b_t)$ and $f_{k-1}(b_{i_k}) \geq f_{k-1}(b_t)$, thus $f_k(b_{i_k}) \geq f_k(b_t)$ and (17) is satisfied.

Constraints (14)–(15) and (18) are clearly satisfied. □

We say that $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ is a *standard completion* of a vertex $(\bar{s}, \bar{z})$ if $\bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}$ are chosen as in the above proof. Then the above proof shows that every vertex of $\mathrm{conv}(DIV)$ has a standard completion satisfying (12)–(18).

Lemma 5. *If* $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ *satisfies* (12)–(18)*, then*

$$f_0(s) \geq f_0(b_t) \text{ if } \sum_{\substack{i \in J_0:\\ \alpha_0(b_i) \geq \alpha_0(b_t)}} w_{0,i} = 1, \qquad t \in J_0,$$

$$f_k(s) \geq f_k(b_t) \text{ if } \sum_{\substack{i \in J_k:\\ f_k(b_i) \geq f_k(b_t)}} w^{\downarrow}_{k,i} + \sum_{\substack{i \in J_k:\\ \alpha_k(b_i + C_{k-1}) \geq \alpha_k(b_t) + 1}} w^{\uparrow}_{k,i} = 1, \quad t \in J_k, k \geq 1.$$

Proof. Let $t \in J_0$ and assume that

$$\sum_{\substack{i \in J_0:\\ \alpha_0(b_i) \geq \alpha_0(b_t)}} \bar{w}_{0,i} = 1$$

holds. If $i \in J_0$ is the index such that $\bar{w}_{0,i} = 1$ then, by (12), $f_0(\bar{s}) = \alpha_0(b_i) \geq \alpha_0(b_t) = f_0(b_t)$.

We now fix $0 \leq k < n$ and assume by induction that the result holds for any index $t \in J_k$. We have to prove that if

$$\sum_{\substack{i \in J_{k+1}:\\ f_{k+1}(b_i) \geq f_{k+1}(b_t)}} w^{\downarrow}_{k+1,i} + \sum_{\substack{i \in J_{k+1}:\\ \alpha_{k+1}(b_i + C_k) \geq \alpha_{k+1}(b_t) + 1}} w^{\uparrow}_{k+1,i} = 1 \tag{19}$$

for some $t \in J_{k+1}$, then $f_{k+1}(\bar{s}) \geq f_{k+1}(b_t)$.

If $\bar{w}^{\uparrow}_{k+1,i} = 1$ for some index $i \in J_{k+1}$, then (13) and the above equation give $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_i + C_k) \geq \alpha_{k+1}(b_t) + 1$, thus $f_{k+1}(\bar{s}) \geq f_{k+1}(b_t)$.

If $\bar{w}^{\downarrow}_{k+1,i} = 1$ for some index $i \in J_{k+1}$, then (19) implies that $f_{k+1}(b_i) \geq f_{k+1}(b_t)$, thus $\alpha_{k+1}(b_i) \geq \alpha_{k+1}(b_t)$. Assume first that $\alpha_{k+1}(b_i) \geq \alpha_{k+1}(b_t) + 1$. Then $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_i) \geq \alpha_{k+1}(b_t) + 1$, thus $f_{k+1}(\bar{s}) \geq f_{k+1}(b_t)$.

Finally assume that $\bar{w}^{\downarrow}_{k+1,i} = 1$ for some $i \in J_{k+1}$ such that $\alpha_{k+1}(b_i) = \alpha_{k+1}(b_t)$. Since (19) implies $f_{k+1}(b_i) \geq f_{k+1}(b_t)$, we then have $f_k(b_i) \geq f_k(b_t)$. Inequality (17) for the index i implies that

$$\sum_{\substack{j \in J_k:\\ f_k(b_j) \geq f_k(b_i)}} \bar{w}^{\downarrow}_{k,j} + \sum_{\substack{j \in J_k:\\ \alpha_k(b_j + C_{k-1}) \geq \alpha_k(b_i)+1}} \bar{w}^{\uparrow}_{k,j} = 1 \ .$$

Then, by induction, $f_k(\bar{s}) \geq f_k(b_i)$. This, together with inequality $f_k(b_i) \geq f_k(b_t)$ proven above, shows that $f_k(\bar{s}) \geq f_k(b_t)$. Using $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_i) = \alpha_{k+1}(b_t)$, we conclude that $f_{k+1}(\bar{s}) \geq f_{k+1}(b_t)$. □

Lemma 5 and the same argument used in the final part of the proof of Lemma 4 prove the following:

Remark 6. If $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ is a standard completion of a vertex $(\bar{s}, \bar{z})$ of $\mathrm{conv}(DIV)$, then

$$f_0(s) \geq f_0(b_t) \Longleftrightarrow \sum_{\substack{i \in J_0:\\ \alpha_0(b_i) \geq \alpha_0(b_t)}} w_{0,i} = 1, \qquad t \in J_0,$$

$$f_k(s) \geq f_k(b_t) \Longleftrightarrow \sum_{\substack{i \in J_k:\\ f_k(b_i) \geq f_k(b_t)}} w^{\downarrow}_{k,i} + \sum_{\substack{i \in J_k:\\ \alpha_k(b_i + C_{k-1}) \geq \alpha_k(b_t)+1}} w^{\uparrow}_{k,i} = 1, \quad t \in J_k, k \geq 1.$$

4 Linearizing (10)

Lemma 7. *Let* $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ *be a vector satisfying* (12)–(18)*. Then* (s, z) *satisfies inequality* $s + C_k z_t \geq b_t$ *if and only if* $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ *satisfies the inequality:*

$$\Delta_0 + \sum_{\substack{i \in J_0:\\ \alpha_0(b_i) \geq \alpha_0(b_t)}} w_{0,i} + z_t \geq \left\lfloor \frac{b_t}{C_0} \right\rfloor + 1 \quad \textit{if } t \in J_0, \tag{20}$$

$$\Delta_k + \sum_{\substack{i \in J_k:\\ f_k(b_i) \geq f_k(b_t)}} w^{\downarrow}_{k,i} + \sum_{\substack{i \in J_k:\\ \alpha_k(b_i + C_{k-1}) \geq \alpha_k(b_t)+1}} w^{\uparrow}_{k,i} + z_t \geq \left\lfloor \frac{b_t}{C_k} \right\rfloor + 1 \quad \textit{if } t \in J_k,\ k \geq 1. \tag{21}$$

Proof. We prove the following two facts: (i) if $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ is a standard completion of a vertex of $\mathrm{conv}(DIV)$, then (20)–(21) hold; (ii) if the vector $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ satisfies (12)–(18) along with (20) (if $t \in J_0$) or (21) (if $t \in J_k$ with $k \geq 1$), then it also satisfies $s + C_k z_t \geq b_t$.

By Lemma 1, inequality $s+C_\ell z_t \geq b_t$ is equivalent to $\Delta_k(s)+\frac{C_\ell}{C_k}z_t \geq \Delta_k\left(b_t^k\right)$ for $\ell \geq k$. In particular, for $\ell = k$ the latter inequality is in turn equivalent to the inequality $\Delta_k(s)+z_t+\delta \geq \Delta_k(b_t+C_k) = \left\lfloor \frac{b_t}{C_k} \right\rfloor + 1$, where δ is a 0, 1 variable that takes value 1 whenever $f_k(s) \geq f_k(b_t)$ and 0 otherwise.

If $t \in J_0$, by Remark 6 a standard completion $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\uparrow}, \bar{w}^{\downarrow}\right)$ of any vertex $(\bar{s}, \bar{z})$ of $\mathrm{conv}(DIV)$ satisfies

$$\sum_{\substack{i \in J_0:\\ \alpha_0(b_i) \geq \alpha_0(b_t)}} w_{0,i} = 1 \Longleftrightarrow f_0(s) \geq f_0(b_t) \ .$$

Then substituting the above expression for δ shows that $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\uparrow}, \bar{w}^{\downarrow}\right)$ satisfies (20). If $t \in J_k$ with $k \geq 1$, the proof that $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\uparrow}, \bar{w}^{\downarrow}\right)$ satisfies (21) is similar. This proves (i).

By Lemma 5, with the above definition of δ, one observes that $\delta = 0$ for every vector $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ satisfying (12)–(18) such that $f_k(s) < f_k(b_t)$. This implies (ii). □

The following result is readily checked:

Remark 8. Let $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ be a vector satisfying (12)–(18). Then (s, z) satisfies inequality $s \geq 0$ if and only if $\left(s, z, \Delta, w, w^{\uparrow}, w^{\downarrow}\right)$ satisfies the inequality

$$\Delta_n \geq 0 \ . \tag{22}$$

5 Strengthening (16)–(17)

Lemma 9. *The following inequalities are valid for the set defined by* (12)–(18) *and dominate* (16)–(17)*:*

$$\sum_{\substack{i \in J_0:\\ \alpha_0(b_i) \geq \alpha_0(b_t)}} w_{0,i} \geq \sum_{\substack{i \in J_1:\\ f_0(b_i) \geq f_0(b_t)}} w_{1,i}^{\downarrow}, \quad t \in J_1, \tag{23}$$

$$\sum_{\substack{i \in J_k:\\ f_k(b_i) \geq f_k(b_t)}} w_{k,i}^{\downarrow} + \sum_{\substack{i \in J_k:\\ \alpha_k(b_i+C_{k-1}) \geq \alpha_k(b_t)+1}} w_{k,i}^{\uparrow} \geq \sum_{\substack{i \in J_{k+1}:\\ f_k(b_i) \geq f_k(b_t)}} w_{k+1,i}^{\downarrow}, \quad t \in J_{k+1},\ 1 \leq k \leq n-1. \tag{24}$$

Proof. Fix $t \in J_{k+1}$ for $k \geq 1$ and define $L = \{i \in J_{k+1} : f_k(b_i) \geq f_k(b_t)\}$. Inequality (24) can be derived by applying the Chvátal-Gomory procedure to the following $|L|+1$ inequalities, which are all valid for (12)–(18):

$$\sum_{\substack{i \in J_k:\\ f_k(b_i) \geq f_k(b_j)}} w_{k,i}^{\downarrow} + \sum_{\substack{i \in J_k:\\ \alpha_k(b_i+C_{k-1}) \geq \alpha_k(b_j)+1}} w_{k,i}^{\uparrow} \geq w_{k+1,j}^{\downarrow}, \quad j \in L, \tag{25}$$

$$1 \geq \sum_{j \in L} w_{k+1,j}^{\downarrow}, \tag{26}$$

with multipliers $1/|L|$ for each of (25) and $1-1/|L|$ for (26). The derivation of (23) is similar. □

6 The Main Result

Let Q be the polyhedron in the space of variables $x = (s, z, \Delta, w, w^{\downarrow}, w^{\uparrow})$ defined by (12)–(15) together with (20)–(21), (22) and (23)–(24). We denote by $Ax \sim b$ the system comprising such equations and inequalities.

Lemma 10. *Let M be the submatrix of A indexed by the columns corresponding to variables $w, w^{\downarrow}, w^{\uparrow}$ and the rows corresponding to* (14)*–*(15) *and* (23)*–*(24)*. The matrix M is totally unimodular.*

Proof. We use a characterization of Ghouila-Houri [8], which states that a $0, \pm 1$ matrix $B = (b_{ij})$ is totally unimodular if and only if for every row submatrix B' of B, the set of row indices of B' can be partitioned into two subsets R_1, R_2 such that $\sum_{i \in R_1} b_{ij} - \sum_{i \in R_2} b_{ij} \in \{0, \pm 1\}$ for all column indices j.

We partition the rows of M into the submatrices $M_0, \ldots, M_n$ defined as follows:

- M_0 consists of the rows corresponding to equation (14) and inequalities (23) for $t \in J_1$;
- for $1 \leq k \leq n-1$, M_k consists of the rows corresponding to equation (15) and inequalities (24) for $t \in J_{k+1}$;
- M_n consists of the row corresponding to equation (15) for $k = n$.

For each odd k, we multiply by -1 the rows of M that belongs to M_k and the columns of M corresponding to variables $w^{\downarrow}_{k,t}, w^{\uparrow}_{k,t}$ for all $t \in J_k$. Then M becomes a 0-1 matrix.

For $1 \leq k \leq n-1$, we order the rows of M_k as follows: first the row corresponding to (15), then those corresponding to (24) according to a non-decreasing order of the values $f_k(b_t)$. The order for the rows of M_0 is analogous. Note that in every matrix M_k the support of any row, say the j-th row, contains that of the $(j+1)$-th row (in other words, the rows of M_k form a laminar family).

We now define a bipartition (R_1, R_2) of the rows of M: for each odd k, we include in R_1 the odd row indices of M_k and in R_2 the even row indices; for each even k, we include in R_1 the even row indices of M_k and in R_2 the odd row indices. One can check that the condition of the theorem of Ghouila-Houri is thus satisfied for $B' = M$. If B' is a row submatrix of M, the bipartition is defined similarly. $\square$

Theorem 11. *If $\bar{x} = (\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is a vertex of Q, then $(\bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is an integral vector. It follows that the inequalities defining Q provide an extended formulation for the polyhedron* $\mathrm{conv}(DIV)$ *with $\mathcal{O}(mn)$ variables and constraints, where $m = |I_0| + \cdots + |I_n|$.*

Proof. Note that the columns of A corresponding to variables s and z_t for $t \in I_k$ and $0 \leq k \leq n$ are unit columns (as s only appears in (12) and each variable z_t only appears in one of (20)–(21)).

Also note that in the subsystem of $Ax \sim b$ comprising (13)–(15), (22) and (23)–(24) (i.e. with (12) and (20)–(21) removed) variables $\Delta_0, \ldots, \Delta_n$ appear with nonzero coefficient only in (13) and (22). Furthermore the submatrix of A indexed by the rows corresponding to (13) and (22) and the columns corresponding to variables $\Delta_0, \ldots, \Delta_n$ is an upper triangular matrix with 1 on the diagonal.

Let $Cx = d$ be a nonsingular subsystem of tight inequalities taken in $Ax \sim b$ that defines a vertex $\bar{x} = \left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ of Q. The above observations show that (12)–(13), (20)–(21) and (22) must be present in this subsystem. Furthermore let C' be the submatrix of C indexed by the columns corresponding to variables $w, w^{\downarrow}, w^{\uparrow}$ and the rows that do not correspond to (12)–(13), (20)–(21) and (22). Then the computation of a determinant with Laplace expansion shows that $|\det(C)| = |\det(C')| \neq 0$.

Since C' is a nonsingular submatrix of the matrix M defined in Lemma 10, by Lemma 10 $|\det(C)| = |\det(C')| = 1$. Since all entries of A (except those corresponding to (12)) are integer and the right-hand side vector b is integral, by Cramer's rule we have that $(\bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is an integral vector. □

7 The Mixing Set with Divisible Capacities and Nonnegative Integer Variables

The mixing set with divisible capacities and nonnegativity bounds on the integer variables DIV^+ is the following:

$$\begin{aligned} s + C_k z_t &\geq b_t, && t \in I_k,\ 0 \leq k \leq n, \\ b_l \leq s &\leq b_u, \\ z_t &\geq 0 \text{ integer}, && t \in I_0 \cup \cdots \cup I_n, \end{aligned}$$

where the capacities C_k's and the sets I_k's are as in the previous sections.

Di Summa [6] gave a polynomial time algorithm to optimize a linear function over DIV^+. We discuss the problem of finding an extended formulation for the polyhedron $\text{conv}(DIV^+)$ which is compact.

We do not know how to incorporate the bounds $z_t \geq 0$ in a formulation of the type given for the polyhedron Q of Theorem 11, as the standard approach requires that the system, purged of the equations defining s and Δ_k, be defined by a totally unimodular matrix (see for instance [3, 12, 15, 18, 19]). However this is not the case, as discussed in the next paragraph. So we use an approach based on union of polyhedra in a manner described e.g. in [1, 4].

To this purpose, let $\{\beta_1, \ldots, \beta_q\}$ be the set of distinct values in the set $\{b_i : i \in I_0 \cup \cdots \cup I_n,\ b_l < b_i < b_u\}$. Assume $\beta_1 < \cdots < \beta_q$ and define $\beta_0 := b_l$ and $\beta_{q+1} := b_u$. For each $0 \leq \ell \leq q$, let $DIV(\ell)$ be the following set:

$$\begin{aligned} s + C_k z_i &\geq b_i, && i \in I_k : b_i > \beta_\ell,\ 0 \leq k \leq m, \\ \beta_\ell \leq s &\leq \beta_{\ell+1}, \\ z_i &\geq 0, && i \in I_k : b_i \leq \beta_\ell,\ 0 \leq k \leq m, \\ z_i &\text{ integer}, && i \in I_0 \cup \cdots \cup I_m. \end{aligned}$$

We will use the following fact:

$$\operatorname{conv}(DIV^+) = \operatorname{conv}\left(\bigcup_{\ell=1}^{q} DIV(\ell)\right) . \tag{27}$$

We now examine the problem of finding extended formulations which are compact for the polyhedra $\operatorname{conv}(DIV(\ell))$. Note that $DIV(\ell)$ is the cartesian product of the following two sets:

$$\begin{aligned} s + C_k z_i \geq b_i, &\quad i \in I_k : b_i > \beta_\ell,\ 0 \leq k \leq m, \\ \beta_\ell \leq s \leq \beta_{\ell+1}, & \\ z_i \text{ integer}, &\quad i \in I_k : b_i > \beta_\ell,\ 0 \leq k \leq m, \end{aligned}$$

and

$$\begin{aligned} z_i \geq 0, &\quad i \in I_k : b_i \leq \beta_\ell,\ 0 \leq k \leq m, \\ z_i \text{ integer}, &\quad i \in I_k : b_i \leq \beta_\ell,\ 0 \leq k \leq m. \end{aligned}$$

If we denote by $UDIV(\ell)$ the first of the above two sets, then $\operatorname{conv}(DIV(\ell)) = \operatorname{conv}(UDIV(\ell)) \times \{z : z_i \geq 0\}$.

Remark that $UDIV(\ell)$ is a mixing set with divisible capacities without non-negativity bounds on the integer variables, except that now we have an upper bound $s \leq \beta_{\ell+1}$. A compact extended formulation for $UDIV(\ell)$ can be derived by using the same ideas presented in this paper (but there are more technicalities) and can be found in [7].

Using (27) and a classical result of Balas [2], a compact extended formulation for $\operatorname{conv}(DIV^+)$ can be derived from the compact extended formulations of the q polyhedra $\operatorname{conv}(DIV(\ell))$.

7.1 An Instance with non-TU Matrix

We show an instance of DIV for which the formulation given by the inequalities describing Q in Theorem 11, purged of the equations defining s and Δ_k, is not defined by a totally unimodular matrix. The instance is the following:

$$\begin{aligned} s + z_1 &\geq 0.1, \\ s + 10z_2 &\geq 6.3, \\ s + 100z_3 &\geq 81.4, \\ s + 100z_4 &\geq 48.6, \\ s \geq 0;\ z_1, \ldots, z_4 &\text{ integer}. \end{aligned}$$

Note that $I_0 = \{1\}$, $I_1 = \{2\}$ and $I_3 = \{3, 4\}$.

Among the constraints defining the extended formulation of the convex hull of the above set, we consider the following four inequalities:

$$\begin{aligned} w^{\downarrow}_{1,2} + w^{\uparrow}_{1,2} + w^{\downarrow}_{1,3} + w^{\uparrow}_{1,3} + w^{\downarrow}_{1,4} + w^{\uparrow}_{1,4} &\geq w^{\downarrow}_{2,3} + w^{\downarrow}_{2,4}, \\ w_{0,3} + w_{0,4} &\geq w^{\downarrow}_{1,3} + w^{\downarrow}_{1,4}, \\ w^{\downarrow}_{1,4} + w^{\uparrow}_{1,4} &\geq w^{\downarrow}_{2,4}, \\ \Delta_1 + w^{\downarrow}_{1,2} + w^{\uparrow}_{1,2} + w^{\downarrow}_{1,4} + w^{\uparrow}_{1,4} + z_2 &\geq 1, \end{aligned}$$

which correspond respectively to (24) for $k = 1$ and $t = 3$, (23) for $t = 3$, (24) for $k = 1$ and $t = 4$, and (21) for $k = 1$ and $t = 2$.

The constraint matrix of the above four inequalities is not totally unimodular, as the determinant of the column submatrix corresponding to variables $w^{\downarrow}_{1,4}, w^{\downarrow}_{1,3}, w^{\downarrow}_{2,4}, w^{\uparrow}_{1,2}$ is -2.

8 Remarks and Open Questions

- The extended formulation presented here is based on the expansion $x = \alpha_0(x) + \sum_{j=1}^{n+1} \alpha_j(x) C_{j-1}$ of a real number x and then exploits the fact that, if $\bar{x}$ is a vertex of the polyhedron to be studied, then for fixed $0 \leq j \leq n+1$, there are few values that $\alpha_j(\bar{x})$ can take. This is essential for the extended formulation to be compact.
  This can be seen as a nontrivial extension of the technique used by Miller and Wolsey [12] in the single capacity mixing set (i.e. $n = 0$) to model a continuous variable x by taking $C_0 = 1$ and $x = \alpha_0(x) + \alpha_1(x)C_0$. Indeed, If one imposes in DIV the further restriction that s is integer (which removes all the complexity in the single capacity mixing set), the complexity of DIV remains essentially unchanged.
- CAP is the following mixed-integer set:

$$\begin{aligned} s_i + C_t z_t &\geq b_{it}, \quad && 1 \leq i \leq q,\ 0 \leq t \leq m, \\ s_i &\geq b_{\ell_i}, && 1 \leq i \leq q, \\ z_t &\text{ integer}, && 0 \leq t \leq m, \end{aligned}$$

  where again $C_0, \ldots, C_m$ is a sequence of divisible numbers. Note that the set DIV is a special case of CAP, obtained by taking $q = 1$. What is the complexity of optimizing a linear function over CAP? Does CAP admit a formulation that is computationally useful? These questions were investigated and answered by Miller and Wolsey [12] for the single capacity case.
- Our last question concerns the mixing set with arbitrary capacities, defined by (1)–(3) in the introduction of this paper. Again, what is the complexity of optimizing a linear function over (1)–(3)? In the case where the number of distinct capacities is small, does there exist an extended formulation which is compact?

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