THE SWAP GRAPH OF THE FINITE SOLUBLE GROUPS

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Abstract. For a d-generated finite group G we consider the graph $\Delta_d(G)$ (swap graph) in which the vertices are the ordered generating d-tuples and in which two vertices $(x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ are adjacent if and only if they differ only by one entry. It was conjectured by Tennant and Turner that $\Delta_d(G)$ is a connected graph. We prove that this conjecture is true if G is a finite soluble group.

1. Introduction

Let G be a finite group and let $d(G)$ be the minimal number of generators of G. For any integer $d \geq d(G)$, let $V_d(G) = \{(g_1, \dots, g_d) \in G^d \mid \langle g_1, \dots, g_d \rangle = G\}$ be the set of all generating d-tuples of G. In [5] Tennant and Turner introduced the notion of "swap equivalence": the d-tuples γ_1 and $\gamma_2 \in V_d(G)$ are said to be swap equivalent if there is a sequence of elementary swaps passing through elements of $V_d(G)$ and leading from γ_1 to γ_2. An elementary swap is thought of as a transformation changing one element of the sequence to an arbitrary element of G. The property of this equivalence relation can be encoded in the "swap graph" $\Delta_d(G)$: two vertices $(x_1, \dots, x_d), (y_1, \dots, y_d) \in V_d(G)$ are adjacent in the swap graph if and only if they differ only by one entry. Tennant and Turner proposed the conjecture that $\Delta_d(G)$ is connected (swap conjecture). In [4] it is proved that the free metabelian group of rank 3 does not satisfy this conjecture, but no counterexample is known in the class of finite groups. In [1] it was proved that the conjecture is true if $d \geq d(G) + 1$. The case when $d = d(G)$ is much more difficult. Partial results have been obtained by the second author in [3], proving for example that a finite group G satisfies the swap conjecture if the derived subgroup of G has odd order or is nilpotent. Here we complete the investigation started in [3] obtaining a complete solution in the soluble case.

Theorem 1. *Let G be a finite soluble group. If $d \geq d(G)$, then the swap graph $\Delta_d(G)$ is connected.*

The proof depends on the solution of a combinatorial problem in linear algebra. Denote by $M_{p\times q}(F)$ the set of the $p \times q$ matrices with coefficients over the finite field F. Let r and n be integers such that $0 \leq r < n$ and let $A \in M_{r\times n}(F)$ with $\operatorname{rank}\big(A\big) = r$. Moreover let Ω_A be the set of matrices $B \in M_{(n-r)\times n}(F)$ with the property that

$$\det \begin{pmatrix} A \\ B \end{pmatrix} \neq 0.$$

We define a graph Γ_A whose vertices are the matrices in Ω_A and in which two vertices B_1 and B_2 are adjacent if and only if they differ only by one column. In [3]

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it is shown that, in order to settle the swap conjecture for the finite soluble groups, it would suffice to prove that the graph Γ_A is connected whenever $r = 0$ or r divides n and $(r, |F|) \neq (1, 2)$. In [3] the connectivity of Γ_A has been established only in the case that $|F| \geq 3$. Now we give a complete solution.

Theorem 2. *Let F be a finite field and let $A \in M_{r\times n}(F)$ be a matrix with $\operatorname{rank}(A) = r$, where $0 \leq r < n$. Then the graph Γ_A is not connected if and only if each of the following conditions is satisfied:*

(i) $|F| = 2$,
(ii) $r \geq 1$,
(iii) $n = r + 1$,
(iv) *A has no all-zero column.*

2. Proof of Theorem 2

We first prove that if conditions (i)–(iv) are not all satisfied, then Γ_A is connected. To this purpose, we fix two distinct nodes B and B' of Γ_A and show that there is a path connecting them. We use the notation $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ and $B' = (b'_1, \dots, b'_n)$ to indicate the columns of A, B and B'.

When $|F| \geq 3$, our proof strategy relies on the following lemma.

Lemma 3. *Let $|F| \geq 3$. Suppose that there exist an index $i \in \{1, \dots, n\}$ and $\mu = {}^t(\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n) \in F^{n-1}$ such that*

$$\begin{pmatrix} a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & a_n \\ b_1 & \cdots & b_{i-1} & b_{i+1} & \cdots & b_n \end{pmatrix} \cdot \mu = \begin{pmatrix} a_i \\ b'_i \end{pmatrix}. \tag{2.1}$$

Pick any index $j \neq i$ such that $\mu_j \neq 0$. Then there exists $y \in F^{n-r}$ such that B and $\widetilde{B}$ are connected nodes of Γ_A, where $\widetilde{B}$ is the matrix obtained from B by replacing b_i with b'_i and b_j with y.

Proof. Let C be the matrix obtained from B by replacing b_j with some (at the moment unknown) vector $y \in F^{n-r}$. Note that C and $\widetilde{B}$ differ only in column i. In the following we prove that it is possible to choose y such that $\det \begin{pmatrix} A \\ C \end{pmatrix} \neq 0$ and $\det \begin{pmatrix} A \\ \widetilde{B} \end{pmatrix} \neq 0$. This implies that $\widetilde{B}$ is a node of Γ_A adjacent to C, which is in turn adjacent to B, thus concluding the proof of the lemma.

Define

$$S = \{\lambda \in F^{n-1} \mid (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)\lambda = a_j\}. \tag{2.2}$$

Since the matrix obtained from $\begin{pmatrix} A \\ C \end{pmatrix}$ by removing its jth column has rank $n-1$, we have that $\det \begin{pmatrix} A \\ C \end{pmatrix} \neq 0$ if and only if there is no $\lambda \in S$ such that

$$(b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n)\lambda = y. \tag{2.3}$$

Since $\mu_j \neq 0$, the matrix obtained from $\begin{pmatrix} A \\ \widetilde{B} \end{pmatrix}$ by removing its jth column has rank $n-1$. Then $\det \begin{pmatrix} A \\ \widetilde{B} \end{pmatrix} \neq 0$ if and only if there is no $\lambda \in S$ such that

$$(b_1, \ldots, b_{i-1}, b'_i, b_{i+1}, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n)\lambda = y. \tag{2.4}$$

(For notational convenience, we assumed here that $i < j$; if $i > j$, the argument is the same.) Therefore it will be sufficient to argue that there is at least one vector $y \in F^{n-r}$ such that (2.3) and (2.4) are not satisfied for any $\lambda \in S$.

Since $\operatorname{rank}(A) = r$, $|S| = |F|^{n-r-1}$. On the other hand, there are $|F|^{n-r}$ possible choices for y in F^{n-r}. It follows that there are at least $p := |F|^{n-r} - 2|F|^{n-r-1}$ choices of y such that (2.3) and (2.4) are not satisfied for any $\lambda \in S$. Since $|F| \geq 3$, we have $p > 0$ and the proof of the lemma is complete. □

The above lemma allows us to prove that Theorem 2 holds if $|F| \geq 3$, as shown below.

Lemma 4. *If $|F| \geq 3$ then Γ_A is a connected graph.*

Proof. Given two nodes B, B' of Γ_A, we prove that there is a path connecting B and B'. We proceed as follows: we assume that B and B' coincide in the first h columns, where $h \in \{0, \ldots, n-1\}$, and show that there exists a node $\widetilde{B}$ connected to B such that $\widetilde{B}$ and B' coincide in $h+1$ columns; by iterating this procedure, we eventually find a path connecting B and B'.

Choose any index $i > h$. If (2.1) does not hold for any $\mu \in F^{n-1}$, we construct $\widetilde{B}$ by replacing b_i with b'_i in B: $\widetilde{B}$ coincides with B' in $h+1$ columns and it is adjacent to B in Γ_A, as required.

So we assume that (2.1) holds for some $\mu \in F^{n-1}$. Since

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_h \\ b_h \end{pmatrix}, \begin{pmatrix} a_i \\ b'_i \end{pmatrix}$$

are all columns of B', they are linearly independent; thus there exists $j > h$ (with $j \neq i$) such that $\mu_j \neq 0$. We can then apply Lemma 3 and obtain a matrix $\widetilde{B}$ that is a node of Γ_A connected to B coinciding with B' in $h+1$ columns. This concludes the proof of Theorem 2 when $|F| \geq 3$. □

We now assume $|F| = 2$. In this case the above approach fails because in the last part of the proof of Lemma 3 we have $p = |F|^{n-r} - 2|F|^{n-r-1} = 0$. However, the following variant of Lemma 3 holds. (For every $i \in \{1, \ldots, n\}$ we denote by e_i the unit vector in F^n with a 1 in position i.)

Lemma 5. *Let $|F| = 2$. Assume that (2.1) holds for some $i \in \{1, \ldots, n\}$ and $\mu \in F^{n-1}$. Pick any index $j \neq i$ such that $\mu_j = 1$ and assume that the vector $e_i + e_j$ does not belong to the space spanned by the rows of A. Then there exists $y \in F^{n-r}$ such that B and $\widetilde{B}$ are connected nodes of Γ_A, where $\widetilde{B}$ is the matrix obtained from B by replacing b_i with b'_i and b_j with y.*

Proof. By proceeding exactly as in the proof of Lemma 3 (and adopting the notation defined there), we find $p = |F|^{n-r} - 2|F|^{n-r-1} = 0$. Then it will be sufficient to argue that there exists $\lambda \in S$ such that the left-hand sides of (2.3) and (2.4) coincide.

Since e_i+e_j does not belong to the space spanned by the rows of A, the equations of the system defining S in (2.2) do not imply the equation $\lambda_i = 1$. This means that there exists $\lambda \in S$ such that $\lambda_i = 0$. For this choice of λ, the left-hand sides of (2.3) and (2.4) coincide. □

We need an additional lemma.

Lemma 6. *Assume that* $|F| = 2$ *but conditions* (ii)–(iv) *of Theorem 2 are not all satisfied. Fix* $h \in \{0, \dots, n-1\}$ *and let* B *and* B' *be two nodes of* Γ_A*, where* B *and* B' *coincide in at least* h *columns, say the columns with indices* $i_1, \dots, i_h$*. Assume that the matrix* $(a_{i_1}, \dots, a_{i_h})$ *has full rank if* $h \neq r$ *and has rank at least* $r-1$ *if* $h = r$*. Then* B *and* B' *are connected in* Γ_A*.*

Proof. The proof is by induction on h. The result is correct if $h = n-1$, as in this case B and B' differ in at most one column and thus are adjacent nodes of Γ_A.

We now prove the result for $0 \leq h \leq n-2$ assuming that it holds for larger values of h. To simplify notation, we assume that B and B' coincide in the *first* h columns, i.e., $i_1 = 1, \dots, i_h = h$. We distinguish four cases, depending on the rank of the matrix $(a_1, \dots, a_h)$.

Case 1: $\operatorname{rank}(a_1, \dots, a_h) = r$.

Choose any index $i > h$. Suppose first that (2.1) does not hold for any vector $\mu \in F^{n-1}$. Let $\widetilde{B}$ be the matrix obtained from B by replacing b_i with b'_i. $\widetilde{B}$ is a node of Γ_A adjacent to B that coincides with B' in the columns with indices $1, \dots, h, i$. Since $\operatorname{rank}(a_1, \dots, a_h, a_i) = r$, by induction $\widetilde{B}$ and B' are connected in Γ_A, and therefore so are B and B'.

We now suppose that there is a vector $\mu \in F^{n-1}$ such that (2.1) holds. Since B and B' coincide in the first h columns, there exists an index $j > h$, with $j \neq i$, such that $\mu_j = 1$. Since $\operatorname{rank}(a_1, \dots, a_h) = r$, there is no linear combination of the rows of A that gives $e_i + e_j$. Then Lemma 5 yields a matrix $\widetilde{B}$ that is a node of Γ_A connected to B. As both i and j are larger than h, $\widetilde{B}$ and B' coincide in the columns with indices $1, \dots, h, i$. Since $\operatorname{rank}(a_1, \dots, a_h, a_i) = r$, by induction $\widetilde{B}$ and B' are connected in Γ_A, and therefore so are B and B'.

Case 2: $\operatorname{rank}(a_1, \dots, a_h) = h \leq r-2$.

Since $\operatorname{rank}(A) = r$, there exist two indices i, j, with $i > h$ and $j > h$, such that $\operatorname{rank}(a_1, \dots, a_h, a_i, a_j) = h+2$. This implies that it is possible to construct a node C of Γ_A as follows: start from B, replace column b_j with b'_j, and then suitably modify the entries in columns b_t with $t \notin \{1, \dots, h, i, j\}$ in such a way that the resulting matrix C satisfies $\det \begin{pmatrix} A \\ C \end{pmatrix} \neq 0$. Since B and C coincide in the columns with indices $1, \dots, h, i$ and $\operatorname{rank}(a_1, \dots, a_h, a_i) = h+1$, the inductive hypothesis implies that B and C are connected in Γ_A. Now, C and B' coincide in the columns with indices $1, \dots, h, j$ and $\operatorname{rank}(a_1, \dots, a_h, a_j) = h+1$. By applying again induction, we conclude that C and B' are connected in Γ_A, and therefore so are B and B'.

Case 3: $\operatorname{rank}(a_1, \dots, a_h) = h = r-1$.

Suppose first that $h = n-2$. Then $n = h+2 = r+1$. Also, we have $r \geq 1$. Thus conditions (i)–(iii) of Theorem 2 are satisfied, and therefore (iv) must be violated; i.e., A has an all-zero column a_j. Note that $j > h$, as $(a_1, \dots, a_h)$ has full column-rank. Let i be the only index larger than h and distinct from j. If (2.1) does not hold for any $\mu \in F^{n-1}$, we construct $\widetilde{B}$ by replacing b_i with b'_i in B: $\widetilde{B}$ is a node of Γ_A adjacent to B that differs from B' in at most one column; thus $\widetilde{B}$ is connected to B' and therefore so is B. If (2.1) holds for some $\mu \in F^{n-1}$, then $\mu_j = 1$. Since a_j is an all-zero column, $e_i + e_j$ does not belong to the space spanned by the rows of A. By Lemma 5 we then conclude that there is a node $\widetilde{B}$ of Γ_A connected to B that differs from B' in at most one column; thus $\widetilde{B}$ is connected to B'.

We now suppose that $h \le n-3$. Let i be any index such that $i > h$ and $\operatorname{rank}(a_1, \ldots, a_h, a_i) = r$.

Claim. *There is an index $j \ne i$ such that $j > h$ and $\begin{pmatrix} a_j \\ b'_j \end{pmatrix}$ is not a linear combination of the columns*

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_h \\ b_h \end{pmatrix}, \begin{pmatrix} a_i \\ b_i \end{pmatrix}. \tag{2.5}$$

Proof of Claim. Assume by contradiction that the claim is false. Then, since $h \le n-3$, there are at least two distinct indices j, k, both distinct from i and larger than h, such that $\begin{pmatrix} a_j \\ b'_j \end{pmatrix}$ and $\begin{pmatrix} a_k \\ b'_k \end{pmatrix}$ are both linear combinations of the columns in (2.5). Note however that they cannot be linear combinations of the first h columns in (2.5). Then

$$\begin{pmatrix} a_j \\ b'_j \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \sum_{t=1}^{h} \mu_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_k \\ b'_k \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \sum_{t=1}^{h} \nu_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}$$

for some $\mu, \nu \in F^h$. It follows that $\begin{pmatrix} a_j \\ b'_j \end{pmatrix} + \begin{pmatrix} a_k \\ b'_k \end{pmatrix}$ belongs to the space generated by the first h columns of $\begin{pmatrix} A \\ B' \end{pmatrix}$ and thus $\det \begin{pmatrix} A \\ B' \end{pmatrix} = 0$, a contradiction. ◇

Therefore there is an index $j \ne i$ such that $j > h$ and $\begin{pmatrix} a_j \\ b'_j \end{pmatrix}$ is not a linear combination of the columns in (2.5). This implies that it is possible to construct a node C of Γ_A as follows: start from B, replace column b_j with b'_j, and then suitably modify the entries in columns b_t with $t \notin \{1, \ldots, h, i, j\}$. Since B and C coincide in the columns with indices $1, \ldots, h, i$ and $\operatorname{rank}(a_1, \ldots, a_h, a_i) = h+1$, the inductive hypothesis implies that B and C are connected in Γ_A. Now, C and B' coincide in the columns with indices $1, \ldots, h, j$ and $\operatorname{rank}(a_1, \ldots, a_h, a_j) \ge h = r-1$. By applying again induction, we conclude that C and B' are connected in Γ_A, and therefore so are B and B'.

Case 4: $h = r$ and $\operatorname{rank}(a_1, \ldots, a_r) = r-1$.

We assume without loss of generality that $\operatorname{rank}(a_1, \ldots, a_{r-1}) = r-1$, thus $a_r + \sum_{t=1}^{r-1} \nu_t a_t = 0$ for some $\nu \in F^{r-1}$.

Let i be an index such that $i > r$ and $\operatorname{rank}(a_1, \ldots, a_r, a_i) = r$. Suppose first that (2.1) does not hold for any $\mu \in F^{n-1}$. If we define $\widetilde{B}$ as the matrix obtained from B by replacing b_i with b'_i, then $\widetilde{B}$ is a node of Γ_A adjacent to B that coincides with B' in the columns with indices $1, \ldots, r, i$. Since $\operatorname{rank}(a_1, \ldots, a_r, a_i) = r$, by induction $\widetilde{B}$ is connected to B'.

Suppose now that (2.1) holds for some $\mu \in F^{n-1}$. We first assume that $\mu_r = 1$ and apply Lemma 5 with $j = r$. This is possible because since a_r is a linear combination of the first $r-1$ columns of A, no combination of the rows of A can give e_i+e_r. By Lemma 5, we find a node $\widetilde{B}$ of Γ_A connected to B that coincides with B' in the h columns with indices $1,\dots,r-1,i$. Furthermore, $\mathrm{rank}(a_1,\dots,a_{r-1},a_i)=r$. By Case 1, we are done.

We finally assume that $\mu_r=0$ in (2.1). Note that there is an index $j>r$ such that $\mu_j=1$. Let C be the matrix obtained from B by replacing column b_j with $c_j := b_j+b_r+\sum_{t=1}^{r-1}\nu_t b_t$. C is a node of Γ_A and it is adjacent to B. Now, recalling (2.1) and the fact that $\mu_j=1$,

$$\begin{pmatrix} a_i \\ b'_i \end{pmatrix} = \sum_{t\neq i}\mu_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} = \sum_{t\neq i,j}\mu_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \begin{pmatrix} a_j \\ c_j \end{pmatrix} + \begin{pmatrix} a_r \\ b_r \end{pmatrix} + \sum_{t=1}^{r-1}\nu_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}.$$

The above right-hand side is a linear combination of the columns of C (except the ith column) in which the coefficient of the rth column is 1, as $\mu_r=0$. Thus we are back to the case $\mu_r=1$ analyzed above. □

The proof that for $|F|=2$ the graph Γ_A is connected whenever conditions (i)–(iv) of Theorem 2 are not all satisfied now follows immediately from the above lemma with $h=0$.

To conclude, we assume that conditions (i)–(iv) of Theorem 2 are all satisfied and prove that Γ_A is not connected. Since $|F|=2$ and $\mathrm{rank}(A)=r=n-1$, the rows of A span a hyperplane defined by an equation of the form $\sum_{i\in I}x_i=0$, where I is a nonempty subset of $\{1,\dots,n\}$. Note that $|I|\geq 2$, otherwise the above equation would be of the form $x_i=0$ for some i and thus, by also using (ii), a_i would be the all-zero vector, contradicting condition (iv). The nodes of Γ_A are precisely the n-dimensional row vectors satisfying $\sum_{i\in I}x_i=1$. Fix any two distinct indices $i,j\in I$. It is immediate to verify that the nodes e_i and e_j are not connected in Γ_A.

3. Proof of Theorem 1

The proof of Theorem 1 uses exactly the same arguments as the proof of [3, Theorem 2]. We give only a sketch referring to [3, Section 4] for more details.

The first step is a reduction to a particular situation. We need to recall some terminology to describe this reduction. Let V be a finite dimensional vector space over a finite field and let H be a d-generated linear soluble group acting irreducibly and faithfully on V. (We include the possibility that H acts trivially on V, in which case $H=1$ and V is a 1-dimensional vector space over a finite field of prime order.) Let $F=\mathrm{End}_H(V)$, $r=\dim_F V$ and $m=r(d-1)+\theta$ where $\theta=1$ if V is a trivial H-module, $\theta=0$ otherwise. We consider the semidirect product $V^m\rtimes H$ where H acts in the same way on each of the m direct factors. Now fix $(h_1,\dots,h_d)\in H^d$ such that $H=\langle h_1,\dots,h_d\rangle$. We define a graph $\Gamma(V,h_1,\dots,h_d)$ in which the vertices are the ordered d-tuples $(w_1,\dots,w_d)$ in $(V^m)^d$ with $\langle h_1w_1,\dots,h_dw_d\rangle=V^m\rtimes H$ (it turns out that the set of these d-tuples is not empty and its cardinality is independent of the choice of $(h_1,\dots,h_d)$) and in which two vertices $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)$ are adjacent if and only if they differ only by one entry. Exactly as in the proof of [3, Theorem 2], using the concept of crown introduced by Gaschütz in [2] and an inductive argument, the following reduction statement can be proved:

Claim. *Theorem 1 holds if the graph $\Gamma(V, h_1, \ldots, h_d)$ is connected for every $(h_1, \ldots, h_d)$ generating a solvable irreducible subgroup of* $\mathrm{GL}(V)$.

The case $H = 1$ is easy: $V = F$ is a finite field of prime order and $\Gamma(V, h_1, \ldots, h_d)$ is the graph whose vertices are the ordered bases $(v_1, \ldots, v_d)$ of F^d and two bases are adjacent if and only if they differ only by one entry: it was already noticed in [3, Lemma 5] that this graph is connected (and it follows also from Theorem 2 taking $r = 0$ and $n = d$). Now assume that $h_1, \ldots, h_d$ generate a soluble irreducible non-trivial subgroup H of $\mathrm{GL}(V)$ and that $F = \mathrm{End}_H(V)$ has cardinality q. We may identify $H = \langle h_1, \ldots, h_d\rangle$ with a subgroup of the general linear group $\mathrm{GL}(r, q)$. In this identification h_i becomes an $r \times r$ matrix with coefficients in F: denote by x_i the matrix $1 - h_i$. Let $n = r \cdot d$ and, as before, $m = r(d - 1) = n - r$. The fact that $h_1, \ldots, h_d$ generate an irreducible subgroup of $\mathrm{GL}(r, q)$ implies that the $r \times n$ matrix $A = \begin{pmatrix} x_1 & \cdots & x_d \end{pmatrix}$ has rank r (see [3, Proposition 7]). Let now $w_i = (v_{i,1}, \ldots, v_{i,n}) \in V^m$. Every $v_{i,j}$ can be viewed as a $1 \times r$ matrix over F and we denote by y_i the $m \times r$ matrix with rows $v_{i,1}, \ldots, v_{i,r}$. It turns out (see [3, Proposition 7]) that

$$\langle h_1 w_1, \ldots, h_d w_d\rangle = V^m \rtimes H \text{ if and only if } \det \begin{pmatrix} x_1 & \cdots & x_d \\ y_1 & \cdots & y_d \end{pmatrix} \neq 0.$$

This implies that the graph $\Gamma(V, h_1, \ldots, h_d)$ is isomorphic to the graph Δ_A whose vertices are the block matrices $B = \begin{pmatrix} y_1 & \cdots & y_d \end{pmatrix}$ with the property that

$$\det \begin{pmatrix} A \\ B \end{pmatrix} \neq 0$$

and two of these block matrices are adjacent if and only if they differ only by one block. This graph Δ_A has the same vertices as the graph Γ_A which appears in the statement of Theorem 2, and clearly if B_1 and B_2 are adjacent in Γ_A, then they are also adjacent in Δ_A. But then either the connectivity of Δ_A follows from Theorem 2 or $n = r \cdot d = r + 1$ and $q = |F| = 2$. In the second case we would have $r = 1$ and consequently $|V| = q^r = 2$, but then $H \leq \mathrm{GL}(1, q) = 1$, against our assumption.

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