LOT-SIZING WITH STOCK UPPER BOUNDS AND FIXED CHARGES

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Abstract. Here we study the discrete lot-sizing problem with an initial stock variable and an associated variable upper bound constraint. This problem is of interest in its own right, and is also a natural relaxation of the constant capacity lot-sizing problem with upper bounds and fixed charges on the stock variables. We show that the convex hull of solutions of the discrete lot-sizing problem is obtained as the intersection of two simpler sets, one a pure integer set and the second a mixing set with a variable upper bound constraint. For these two sets we derive both inequality descriptions and polynomial-size extended formulations of their respective convex hulls.

Finally we carry out some limited computational tests on single-item constant capacity lot-sizing problems with upper bounds and fixed charges on the stock variables in which we use the extended formulations derived above to strengthen the initial mixed-integer programming formulations.

Key words. Mixed-integer programming, discrete lot-sizing, stock fixed costs, mixing sets.

AMS subject classifications. 90C11, 90C57.

1. Introduction.

Much recent research has been concerned with the development of convex hull descriptions, exact cutting plane algorithms and tight and compact extended formulations for mixed-integer sets including simple 0-1 sets, mixing sets and polynomially solvable single item lot-sizing problems. Here we pursue this work deriving formulations for the convex hull of solutions of

(i) a pure integer set $K = \{(w, z) \in \{0, 1\} \times \mathbb{Z}^n : uw + z_t \geq b_t,\ 0 \leq z_t - z_{t-1} \leq 1,\ 1 \leq t \leq n\}$,

(ii) a mixing set in which the single continuous variable satisfies a variable upper bound constraint, and

(iii) the single item discrete lot-sizing problem with constant capacities with a variable upper bound constraint on the initial stock variable. Here the convex hull turns out to be the intersection of the convex hulls of the sets (i) and (ii).

For the convex hull of each of these sets, we provide both a linear-inequality description in the original space of variables and a polynomial-size (compact) linear-inequality description using additional variables (extended formulation).

Apart from an interest in the structure of the valid inequalities for these sets with a view to further generalizations, the motivation for this work is to solve lot-sizing problems with fixed costs on the stocks, for which all three sets studied above are relaxations. Computationally we show that relaxation (ii) combined with the default cutting planes generated by a standard commercial solver allow us to solve very rapidly a variety of single item lot-sizing problems with fixed costs on the stocks. In particular, uncapacitated instances that were solved in [1] using two classes of specialized cutting planes and specialized separation algorithms in several minutes can be solved in a few seconds, and constant capacity instances can also be tackled effectively.

Uncapacitated lot-sizing problems with upper bounds on stocks have been tackled by numerous authors, in particular valid inequalities have been proposed by Atamtürk and Küçükyavuz [1], Pochet and Wolsey [9] and Wolsey [13]. However problems with fixed costs on stocks have received little attention until recently, see Ortega and Van Vyve [11] for the problem with unlimited upper bounds on the stock, and more recently Atamtürk and Küçükyavuz [1] for arbitrary bounds.

Mixing sets, defined by Günlük and Pochet [6], were studied as a natural relaxation of constant capacity lot-sizing problems. Generalizations, motivated by variants of lot-sizing and also by node covering problems, have been tackled in several papers including among others [10, 4, 3].

Optimizing over most pure integer sets is NP-hard. Exceptions related to the set K are the $(1, k)$-configurations studied by Padberg [8], and special 0-1 knapsack sets, see Weismantel [12].

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The rest of the paper is organized as follows. In §§2–3 we analyze the single-item discrete lot-sizing model with a variable upper bound on the initial stock. We model this problem as a mixed-integer program and we describe the convex hull of the feasible region by means of linear inequalities, both in the original space and via an extended formulation. This target is achieved by first studying two relaxations of the feasible region in §2. In §4 we study the single-item constant-capacity lot-sizing problem with variable upper bounds on the stock. After formulating the problem as a mixed-integer program, we use the results for the discrete lot-sizing model to construct relaxations for the feasible region and we demonstrate the strength of the relaxations by carrying out some computational experiments. We conclude in §5 with some final remarks.

2. Discrete lot-sizing with variable upper bound on the initial stock. In this section we study the single-item discrete lot-sizing problem with a variable upper bound on the initial stock. The problem is to plan production and inventory levels for a horizon of n periods so that all demands are satisfied and the total cost is minimized. In each period the production is either 0 or at full capacity C, say $C = 1$ wlog. For every period $t = 1, \dots, n$, the demand d_t, the production cost p_t and the per unit holding cost h_t are given. In this model the initial inventory level is not given, but it is a variable to be determined. A fixed charge c is incurred if one chooses to have a positive initial inventory level, and this level cannot exceed a given upper bound u. A mixed-integer formulation of this problem is the following:

$$
\begin{aligned}
\min \quad & cw + h_0 s_0 + \textstyle\sum_{t=1}^{n}(p_t y_t + h_t s_t) && && (1)\\
\text{subject to} \quad & s_{t-1} + y_t = d_t + s_t, && 1 \le t \le n, && (2)\\
& 0 \le s_0 \le uw,\ w \in \{0,1\}, && && (3)\\
& s_t \ge 0,\ y_t \in \{0,1\}, && 1 \le t \le n. && (4)
\end{aligned}
$$

In the above formulation, variable s_0 represents the variable initial inventory level, whereas s_t for $1 \le t \le n$ is the stock at the end of period t. Variable w is equal to 1 if the initial inventory level is strictly positive, while the binary variable y_t ($1 \le t \le n$) indicates whether production takes place in period t.

Wlog one can assume that $0 \le d_t \le 1$ for $1 \le t \le n$, as we now explain. Assume that $d_t > 1$ for some index t and define $\tau := \max\{t : d_t > 1\}$. We first assume that $\tau > 1$. Note that since $y_\tau \le 1$, then $s_{\tau-1} \ge d_\tau - 1$ in every feasible solution. If we redefine $d_\tau \leftarrow 1$, $d_{\tau-1} \leftarrow d_{\tau-1} + (d_\tau - 1)$ and $s_{\tau-1} \leftarrow s_{\tau-1} - (d_\tau - 1)$, we obtain an equivalent system where variable $s_{\tau-1}$ can be still constrained to be nonnegative. Iterating this procedure, one of the following two alternatives will eventually hold: either $d_t \le 1$ for all t, or $d_1 > 1$ and $d_t \le 1$ for all $t > 1$. In the latter case, we can redefine $d_1 \leftarrow 1$, $u \leftarrow u - (d_1 - 1)$ and $s_0 \leftarrow s_0 - (d_1 - 1)$ (if now $u < 0$, the problem is infeasible). This shows that we can assume $0 \le d_t \le 1$ for all t.

After eliminating variables s_t for $1 \le t \le n$ from formulation (1)–(4), the feasible region (2)–(4) becomes:

$$
\begin{aligned}
& s_0 + \textstyle\sum_{i=1}^{t} y_i \ge \sum_{i=1}^{t} d_i, && 1 \le t \le n,\\
& 0 \le s_0 \le uw,\ w \in \{0,1\},\\
& y_t \in \{0,1\}, && 1 \le t \le n.
\end{aligned}
$$

If we now define $s = s_0$, $z_t = \sum_{i=1}^{t} y_i$ for $1 \le t \le n$, $z_0 = 0$ and $b_t = \sum_{i=1}^{t} d_i$ for $1 \le t \le n$, the feasible region can be rewritten as follows:

$$
\begin{aligned}
& s + z_t \ge b_t, && 1 \le t \le n, && (5)\\
& 0 \le s \le uw,\ w \in \{0,1\}, && && (6)\\
& 0 \le z_t - z_{t-1} \le 1,\ z_t \in \mathbb{Z}, && 1 \le t \le n. && (7)
\end{aligned}
$$

In the following we describe the convex hull of solutions satisfying (5)–(7). To do so, we first study two relaxations in §§2.1–2.2. The convex hull of (5)–(7) is then derived in §3.

2.1. A mixing relaxation. To construct a first relaxation of (5)–(7), we drop the constraints $z_t - z_{t-1} \le 1$ for $1 \le t \le n$ and obtain the following mixed-integer set, which we call M:

$$\begin{aligned} s + z_t \ge b_t, & \qquad 1 \le t \le n, \\ 0 \le s \le uw,\ w \in \{0,1\}, & \\ z_t \ge 0,\ z_t \in \mathbb{Z}, & \qquad 1 \le t \le n. \end{aligned}$$

The set M can be regarded as a variant of the mixing set, a basic mixed-integer set first studied by Günlük and Pochet [6]. The only differences are the 0-1 variable w and the inequalities $s \le uw$ and $z_t \ge 0$ for $1 \le t \le n$.

Günlük and Pochet [6] gave a linear-inequality description for the convex hull of the mixing set in its original space of variable, whereas Miller and Wolsey [7] provided a compact extended formulation. In the following two propositions we provide similar descriptions for the convex hull of M.

Throughout the paper, for $1 \le t \le n$ we denote by f_t be the fractional part of b_t, i.e. $f_t = b_t - \lfloor b_t \rfloor$. Furthermore, we define

$$f'_t = \begin{cases} f_t & \text{if } f_t > 0, \\ 1 & \text{if } f_t = 0. \end{cases}$$

THEOREM 2.1. *The convex hull of M is described by the linear inequalities*

$$0 \le s \le uw,\ 0 \le w \le 1,\ z_t \ge 0 \quad \textit{for } 1 \le t \le n, \tag{8}$$

$$z_t \ge \lceil b_t \rceil (1-w) + \lceil b_t - u \rceil^+ w \quad \textit{for } 1 \le t \le n, \tag{9}$$

and the following two groups of inequalities:

$$s + \textstyle\sum_{i=1}^k (f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) \ge 0, \tag{10}$$

$$s + \textstyle\sum_{i=1}^k (f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) + (1 - f'_{t_1})(z_{t_k} - \lceil b_{t_k} \rceil + w) \ge 0, \tag{11}$$

where $k \ge 1$ and $t_1, \dots, t_k$ is a sequence of indices such that $f'_{t_1} > \dots > f'_{t_k}$ ($f'_{t_{k+1}} = 0$).

Proof. We first show that all the above inequalities are valid for M (and thus for $\text{conv}(M)$).

The validity of inequalities $0 \le s \le uw$, $0 \le w \le 1$ and $z_t \ge 0$ for $1 \le t \le n$ is obvious.

If $w = 0$, inequalities $s + z_t \ge b_t$, $s \le uw$ and the integrality of z_t imply $z_t \ge \lceil b_t \rceil$. If $w = 1$, the same conditions along with the nonnegativity of z_t imply $z_t \ge \lceil b_t - u \rceil^+$. This shows that inequalities (9) are valid for M.

Inequalities (10) are valid for the mixing set (see [6]), thus also for M.

We now consider inequalities (11). If $w = 1$, (11) reduces to one of the inequalities that are valid for the mixing set (see [6]). If $w = 0$, (11) can be obtained as a combination of inequality $z_{t_k} \ge \lceil b_{t_k} \rceil$ (which holds if $w = 0$) and the inequality (10) corresponding to the same sequence of indices $t_1, \dots, t_k$.

Therefore inequalities (8)–(11) are valid for $\text{conv}(M)$. Let P be the polyhedron defined by these inequalities. Since P is contained in the linear relaxation of M (the inequality $s + z_t \ge b_t$ is implied by $w \le 1$ and (11) for $k = 1$ and $t_1 = t$), to conclude we only have to prove that the extreme points of P have integer w- and z-components.

We first show that w is integer in every extreme point of P. Suppose that this is not true, i.e. there exists an extreme point $(\bar s, \bar z, \bar w) \in P$ with $0 < \bar w < 1$. Below we show that the point $(\hat s, \hat z, \hat w) = \left(\frac{\bar s}{\bar w}, \frac{\bar z_t - (1-\bar w)\lceil b_t \rceil}{\bar w}, 1\right)$ belongs to P. Since $(\bar s, \bar z, \bar w) = \bar w(\hat s, \hat z, \hat w) + (1 - \bar w)(0, \lceil b_t \rceil, 0)$ and $(0, \lceil b_t \rceil, 0) \in P$, we have that $(\bar s, \bar z, \bar y)$ is not an extreme point of P.

It is readily checked that $0 \le \hat s \le u\hat w$ and $0 \le \hat w \le 1$.

For $1 \le t \le n$, $\hat z_t = \frac{\bar z_t - (1-\bar w)\lceil b_t \rceil}{\bar w} \ge \lceil b_t - u \rceil^+ \ge 0$, where the first inequality holds because $(\bar s, \bar z, \bar w)$ satisfies (9).

For $1 \le t \le n$, $(\hat{s}, \hat{z}, \hat{w})$ satisfies (9) if and only if $\frac{\bar{z}_t - (1-\bar{w})\lceil b_t \rceil}{\bar{w}} \ge \lceil b_t - u \rceil^+$, which holds as $(\bar{s}, \bar{z}, \bar{w})$ satisfies (9).

It can be easily verified that $(\hat{s}, \hat{z}, \hat{w})$ satisfies (10) (resp. (11)) if and only if $(\bar{s}, \bar{z}, \bar{w})$ satisfies (10) (resp. (11)).

Therefore $w \in \{0, 1\}$ in all the extreme points of P. To conclude, we show that z is integral in all the extreme points of P. Let $(\bar{s}, \bar{z}, \bar{w})$ be an extreme point of P. We consider the two cases $\bar{w} = 0$ and $\bar{w} = 1$.

If $\bar{w} = 0$, then $\bar{s} = 0$ and $\bar{z}$ is an extreme point of the polyhedron obtained by intersecting P with the hyperplanes $w = 0$ and $s = 0$, namely

$$z_t \ge 0,\ z_t \ge \lceil b_t \rceil, \qquad 1 \le t \le n, \tag{12}$$

$$\textstyle\sum_{i=1}^k (f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) \ge 0, \tag{13}$$

$$\textstyle\sum_{i=1}^k (f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) + (1 - f'_{t_1})(z_{t_k} - \lceil b_{t_k} \rceil) \ge 0, \tag{14}$$

for all sequences of indices $t_1, \ldots, t_k$ as described above. It is readily checked that inequalities (13)–(14) are implied by (12), thus they do not play any role in the above linear system. Then, since $\bar{z}$ is a vertex of (12)–(14), we have that $\bar{z}_t = \max\{0, \lceil b_t \rceil\}$ for $1 \le t \le n$, thus $\bar{z}$ is an integral vector.

If $\bar{w} = 1$, then $(\bar{s}, \bar{z})$ is an extreme point of the polyhedron P_1 obtained by intersecting P with the hyperplane $w = 1$. A result appearing in [5] concerning the convex hull of the general mixing set $\{(x_0, x) \in \mathbb{R} \times \mathbb{Z}^n : b_t \le x_0 + x_t \le c_t,\ 1 \le t \le n\}$ allows one to show that $\bar{z}$ is an integral vector. □

We now present a compact extended formulation for $\mathrm{conv}(M)$ that is derived from the extended formulation that Miller and Wolsey [7] gave for the mixing set.

Theorem 2.2. *An extended formulation for* $\mathrm{conv}(M)$ *is given by the following linear system:*

$$s = \mu + \textstyle\sum_{\ell=0}^n f_\ell \delta_\ell,\ \sum_{\ell=0}^n \delta_\ell = 1, \tag{15}$$

$$\mu + \textstyle\sum_{\ell: f_\ell \ge f_t} \delta_\ell + z_t \ge \lfloor b_t \rfloor + 1, \qquad 1 \le t \le n, \tag{16}$$

$$z_t \ge \lceil b_t \rceil (1 - w) + \lceil b_t - u \rceil^+ w, \qquad 1 \le t \le n, \tag{17}$$

$$s \le uw,\ w + \textstyle\sum_{\ell: f_\ell = 0} \delta_\ell \ge 1, \tag{18}$$

$$0 \le w \le 1,\ z_t \ge 0, \qquad 1 \le t \le n, \tag{19}$$

$$\mu \ge 0,\ \delta_\ell \ge 0, \qquad 0 \le \ell \le n, \tag{20}$$

where $f_0 = 0$.

Proof. First we show that every point in M can be extended to a vector satisfying the above linear system. Let $(\bar{s}, \bar{z}, \bar{w})$ be a point in M. Constraints (15), (16) and (20) form an extended formulation for the mixing set (see [7]), thus there exists an integral vector $(\bar{\mu}, \bar{\delta})$ such that $(\bar{s}, \bar{z}, \bar{w}, \bar{\mu}, \bar{\delta})$ satisfies these constraints. All other inequalities, except $w + \sum_{\ell: f_\ell = 0} \delta_\ell \ge 1$, are part of the description of $\mathrm{conv}(M)$ given in Theorem 2.1. To see that $\bar{w} + \sum_{\ell: f_\ell = 0} \bar{\delta}_\ell \ge 1$, observe that if $\bar{w} = 0$, then $\bar{s} = 0$; together with equations (15) and the integrality of $(\bar{\mu}, \bar{\delta})$, this implies that $\bar{\delta}_\ell = 1$ for some ℓ for which $f_\ell = 0$.

Now we show that all the inequalities listed in Theorem 2.1 are implied by (15)–(20). In fact, the only nontrivial part is proving that inequalities (10)–(11) are implied by (15)–(20). Constraints (10) are valid for the mixing set (see [6]), so they are implied by inequalities (15), (16) and (20) (which form an extended formulation for the mixing set, as recalled above). To see that (11) is implied by (15)–(20), let $t_1, \ldots, t_k$ be a sequence of indices such that $f'_{t_1} > \cdots > f'_{t_k}$ and set $f'_{t_{k+1}} = 0$. If $f'_{t_1} = 1$, then (11) coincides with (10). Therefore we assume $f'_{t_1} < 1$. Note that this implies that $f'_{t_i} = f_{t_i}$ and $\lceil b_{t_i} \rceil = \lfloor b_{t_i} \rfloor + 1$ for $1 \le i \le k$, and in particular $f_{t_k} > 0$. The following chain of equations and inequalities concludes the proof (see below for a justification of the steps):

$$
\begin{aligned}
&s + \textstyle\sum_{i=1}^{k}(f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) + (1 - f'_{t_1})(z_{t_k} - \lceil b_{t_k} \rceil + w)\\
&= s + \textstyle\sum_{i=1}^{k}(f_{t_i} - f_{t_{i+1}})(z_{t_i} - \lfloor b_{t_i} \rfloor - 1) + (1 - f_{t_1})(z_{t_k} - \lfloor b_{t_k} \rfloor - 1 + w)\\
&\geq \mu + \textstyle\sum_{\ell=0}^{n} f_\ell \delta_\ell - \sum_{i=1}^{k}(f_{t_i} - f_{t_{i+1}})(\mu + \sum_{\ell: f_\ell \geq f_{t_i}} \delta_\ell) - (1 - f_{t_1})(\mu + \sum_{\ell: f_\ell \geq f_{t_k}} \delta_\ell - w)\\
&\geq \mu - f_{t_1}\mu - (1 - f_{t_1})(\mu + \textstyle\sum_{\ell: f_\ell \geq f_{t_k}} \delta_\ell - w) \geq -(1 - f_{t_1})(1 - \sum_{\ell: f_\ell = 0} \delta_\ell - w) \geq 0.
\end{aligned}
$$

The first inequality follows from the first equation (15) and from (16). The second inequality holds because

$$
\begin{aligned}
&\textstyle\sum_{\ell=0}^{n} f_\ell \delta_\ell - \sum_{i=1}^{k}(f_{t_i} - f_{t_{i+1}})(\mu + \sum_{\ell: f_\ell \geq f_{t_i}} \delta_\ell)\\
&\geq \textstyle\sum_{\ell=0}^{n} f_\ell \delta_\ell - \sum_{i=1}^{k}(f_{t_i} - f_{t_{i+1}})(\mu + \sum_{\ell=0}^{n} \delta_\ell)\\
&= \textstyle\sum_{\ell=0}^{n} f_\ell \delta_\ell - f_{t_1}\mu - f_{t_1} \sum_{\ell=0}^{n} \delta_\ell \geq -f_{t_1}\mu.
\end{aligned}
$$

The third inequality holds because of the second equation (15) (recall that $f_{t_k} > 0$). Finally, the last inequality follows from the second condition (18). □

We conclude this subsection with a brief discussion of the separation problem for the inequalities (8)–(11). Given a point $(\bar{s}, \bar{z}, \bar{w})$, it can be checked in $\mathcal{O}(n)$ time whether it violates any of the inequalities (8) or (9). Therefore we focus on inequalities (10)–(11).

PROPOSITION 2.3. *Given a point $(\bar{s}, \bar{z}, \bar{w})$, let $t_1, \ldots, t_k$ be a sequence of indices with $f'_{t_1} > \cdots > f'_{t_k}$ such that:*

(i) *$\lceil b_{t_k} \rceil - \bar{z}_{t_k} \geq \cdots \geq \lceil b_{t_1} \rceil - \bar{z}_{t_1} \geq (\beta - \bar{w})^+$, where $\beta = \lceil b_{t_k} \rceil - \bar{z}_{t_k}$;*

(ii) *$\lceil b_t \rceil - \bar{z}_t \leq \lceil b_{t_i} \rceil - \bar{z}_{t_i}$ for all t such that $f'_{t_{i+1}} < f'_t < f'_{t_i}$, $1 \leq i \leq k$ (with $f'_{t_{k+1}} = 0$);*

(iii) *$\lceil b_t \rceil - \bar{z}_t \leq (\beta - \bar{w})^+$ for all t such that $f'_t > f'_{t_1}$.*

Then, if $\beta \leq \bar{w}$, (10) is a most violated inequality, while if $\beta \geq \bar{w}$, (11) is a most violated inequality.

Proof. It can be verified that this choice of the sequence $t_1 < \cdots < t_k$ minimizes the left-hand side of (10)–(11). We also remark that the above statement is analogous to that presented in [9] for the mixing set, with the only exception that $\bar{w}$ is replaced by 1 in [9]. □

A sequence of indices as in the above proposition can be found in time $\mathcal{O}(n \log n)$: first reorder the indices so that $f'_1 > \cdots > f'_n$, then find the index t for which $\lceil b_t \rceil - \bar{z}_t$ is maximum (this will be index t_k), then the index $t < t_k$ such that $\lceil b_t \rceil - \bar{z}_t$ is maximum, and so forth.

2.2. A pure integer relaxation. A second relaxation of (5)–(7) is given by the following pure integer set, that we call K:

$$
\begin{aligned}
uw + z_t &\geq b_t, & 1 \leq t \leq n, \quad (21)\\
0 \leq z_t - z_{t-1} &\leq 1, & 1 \leq t \leq n, \quad (22)\\
w \in \{0, 1\},\ z_t &\in \mathbb{Z}, & 1 \leq t \leq n. \quad (23)
\end{aligned}
$$

The validity of (21) follows from (5) and from the inequality $s \leq uw$.

Note that after dropping the first $n - 1$ constraints of (21) and making the substitution $y_t = z_t - z_{t-1}$, one obtains as a relaxation the single row $(1, k)$-configuration $\{(w, y) \in \{0, 1\}^{n+1} : uw + \sum_{t=1}^{n} y_t \geq b_n\}$ (see [8]).

In the following, we find an extended formulation for the convex hull of K and then we project it onto the original space of variables, thus obtaining a linear-inequality description for conv(K) in the space of the (z, w) variables. This will be used in §3, where we will derive linear-inequality descriptions for the convex hull of (5)–(7).

2.2.1. Extended formulation of the pure integer relaxation. Define $K_0 = \{(z, w) \in K : w = 0\}$ and $K_1 = \{(z, w) \in K : w = 1\}$. Because $K = K_0 \cup K_1$, we first find linear-inequality descriptions for conv(K_0) and conv(K_1), and then derive an extended formulation for conv(K) using Balas' result on the convex hull of the union of polyhedra [2].

With w fixed at 0 or 1, the constraint matrix of inequalities (21)–(22) is totally unimodular. This implies that $\mathrm{conv}(K_0)$ is described by the inequalities

$$\begin{aligned} z_t &\geq \lceil b_t \rceil, & 1 \leq t \leq n, \\ w = 0,\ 0 \leq z_t - z_{t-1} &\leq 1, & 1 \leq t \leq n, \end{aligned}$$

and $\mathrm{conv}(K_1)$ is described by the inequalities

$$\begin{aligned} z_t &\geq \lceil b_t - u \rceil, & 1 \leq t \leq n, \\ w = 1,\ 0 \leq z_t - z_{t-1} &\leq 1, & 1 \leq t \leq n. \end{aligned}$$

Then, by Balas' result [2], as both sets are bounded, an extended formulation for $\mathrm{conv}(K)$ is given by the following inequalities:

$$\begin{aligned} &0 \leq \lambda \leq 1,\ w = w' + w'', \\ &z_t = z'_t + z''_t, && 1 \leq t \leq n, \\ &z'_t \geq \lceil b_t \rceil (1-\lambda), && 1 \leq t \leq n, \\ &z''_t \geq \lceil b_t - u \rceil \lambda, && 1 \leq t \leq n, \\ &w' = 0,\ w'' = \lambda, \\ &0 \leq z'_t - z'_{t-1} \leq 1 - \lambda, && 1 \leq t \leq n, \\ &0 \leq z''_t - z''_{t-1} \leq \lambda, && 1 \leq t \leq n. \end{aligned}$$

Since $w' = 0$, $w'' = \lambda$ and $w = w' + w'' = \lambda$, we can eliminate variables w', w'', λ and rewrite the above linear system as follows (the dual variables in parentheses to the left of the inequalities will be used later):

$$\begin{array}{lrl} & 0 \leq w \leq 1, & \\ (\tau_t) & z'_t + z''_t = z_t, & 1 \leq t \leq n, \\ (\rho'_t) & z'_t \geq \lceil b_t \rceil (1-w), & 1 \leq t \leq n, \\ (\rho''_t) & z''_t \geq \lceil b_t - u \rceil w, & 1 \leq t \leq n, \\ (u'_t) & z'_t - z'_{t-1} \geq 0, & 1 \leq t \leq n, \\ (v'_t) & -z'_t + z'_{t-1} \geq -(1-w), & 1 \leq t \leq n, \\ (u''_t) & z''_t - z''_{t-1} \geq 0, & 1 \leq t \leq n. \\ (v''_t) & -z''_t + z''_{t-1} \geq -w, & 1 \leq t \leq n. \end{array}$$

2.2.2. Convex hull of the pure integer relaxation in the original space. In order to obtain a linear-inequality description of the polyhedron $\mathrm{conv}(K)$ in its original space of variables, we project away the additional variables from the extended formulation derived above.

To this purpose, we associate dual multipliers to the inequalities as indicated to the left of the above system (there is no need to assign multipliers to the constraints $0 \leq w \leq 1$, as the only variable appearing here is an original variable). Apart from the constraints $0 \leq w \leq 1$, all facet-defining inequalities for $\mathrm{conv}(K)$ are of the form

$$\sum_{t=1}^{n} \tau_t z_t \geq \sum_{t=1}^{n} \left((\rho'_t \lceil b_t \rceil - v'_t)(1-w) + (\rho''_t \lceil b_t - u \rceil - v''_t) w \right), \tag{24}$$

where $(\tau, \rho', \rho'', u', v', u'', v'')$ is an extreme ray of the cone

$$\begin{aligned} \tau_t &= \rho'_t + u'_t - u'_{t+1} - v'_t + v'_{t+1}, && 1 \leq t \leq n, && (25) \\ \tau_t &= \rho''_t + u''_t - u''_{t+1} - v''_t + v''_{t+1}, && 1 \leq t \leq n, && (26) \\ &\rho'_t, \rho''_t, u'_t, v'_t, u''_t, v''_t \geq 0, && 1 \leq t \leq n, && (27) \end{aligned}$$

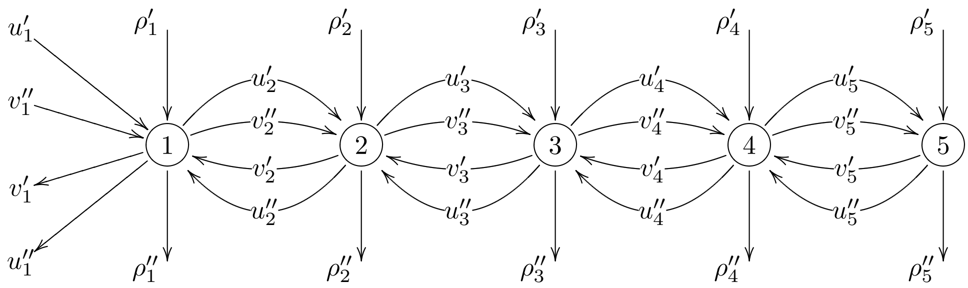

FIGURE 2.1. *The network $\mathcal{N}$ corresponding to linear system* (28)–(29) *with* $n = 5$.

with $u'_{n+1} = v'_{n+1} = u''_{n+1} = v''_{n+1} = 0$. After elimination of variables $\tau_1, \dots, \tau_n$, cone (25)–(27) reads:

$$\rho'_t + u'_t - u'_{t+1} - v'_t + v'_{t+1} = \rho''_t + u''_t - u''_{t+1} - v''_t + v''_{t+1}, \quad 1 \le t \le n, \tag{28}$$

$$\rho'_t, \rho''_t, u'_t, v'_t, u''_t, v''_t \ge 0, \qquad 1 \le t \le n, \tag{29}$$

while inequality (24) becomes

$$\sum_{t=1}^{n} (\rho'_t + u'_t - u'_{t+1} - v'_t + v'_{t+1}) z_t \ge \sum_{t=1}^{n} \left((\rho'_t \lceil b_t \rceil - v'_t)(1-w) + (\rho''_t \lceil b_t - u \rceil - v''_t) w \right),$$

which can be rewritten as follows:

$$\sum_{t=1}^{n} \rho'_t z_t + \sum_{t=1}^{n} (u'_t - v'_t)(z_t - z_{t-1}) \ge \sum_{t=1}^{n} \left((\rho'_t \lceil b_t \rceil - v'_t)(1-w) + (\rho''_t \lceil b_t - u \rceil - v''_t) w \right). \tag{30}$$

The linear system (28)–(29) defines the set of feasible flows in the network $\mathcal{N}$ depicted in Figure 2.1. Since there is a one-to-one correspondence between variables appearing in constraints (28)–(29) and arcs in $\mathcal{N}$, we will use, e.g., ρ'_t to denote both the variable and the corresponding arc. The network includes a dummy node (which has not been drawn) that is the tail of arcs u'_1, v''_1 and ρ'_t for $1 \le t \le n$, and the head of arcs u''_1, v'_1 and ρ''_t for $1 \le t \le n$.

By standard results on network flow problems, the extreme rays of the cone defined by (28)–(29) are $0, 1$ vectors (up to multiplication by a positive scalar) whose supports correspond to the directed cycles in $\mathcal{N}$ (note that cycles are allowed to contain the dummy node). Therefore we analyze the directed cycles in $\mathcal{N}$.

In the following we write x for the vector of variables $(\rho', \rho'', u', v', u'', v'')$. Sometimes we will identify vector x with the corresponding flow. Furthermore, we denote by $\operatorname{lhs}(x)$ (resp. $\operatorname{rhs}(x)$) the left-hand side (resp. right-hand side) of the inequality (30) corresponding to the vector x, and by $\mathcal{I}(x)$ the whole inequality.

In the following lemma we give some necessary conditions for a directed cycle $\bar{x}$ in $\mathcal{N}$ to generate a non-dominated inequality $\mathcal{I}(\bar{x})$. The words "we can assume that" in the lemma below mean that if $\mathcal{I}(\bar{x})$ is a non-redundant inequality, then either $\bar{x}$ satisfies the conditions described by the lemma, or there is another feasible flow satisfying the conditions and generating the same inequality. Recall that $b_1 \ge 0$ and $0 \le b_t - b_{t-1} \le 1$ for $2 \le t \le n$, as these relations will be used in the proof below. Furthermore, since our linear-inequality description of $\operatorname{conv}(K)$ will include the constraints $0 \le w \le 1$, we are allowed to use these inequalities when proving that some inequality $\mathcal{I}(\bar{x})$ is redundant.

LEMMA 2.4. *If $\bar{x}$ defines a directed cycle in $\mathcal{N}$ and inequality $\mathcal{I}(\bar{x})$ is non-redundant in the description of* $\operatorname{conv}(K)$, *then we can assume that:*

(i) $\bar{u}'_1 = 0$;

(ii) $\bar\rho'_t\bar u'_{t+1} = 0$ *for* $1 \le t \le n-1$*;*
(iii) $\bar\rho'_t\bar v'_t = 0$ *for* $2 \le t \le n$*;*
(iv) $\bar\rho''_t\bar u''_{t+1} = 0$ *for* $1 \le t \le n-1$*;*
(v) $\bar\rho''_t\bar v''_t = 0$ *for* $2 \le t \le n$*.*

Proof. Let $\bar x$ be as in the above statement.

(i) Assume $\bar u'_1 = 1$. Then $\bar\rho'_1 = 0$, otherwise $\bar x$ would not define a directed cycle. We construct a vector $\hat x$ by setting $\hat u'_1 = 0$, $\hat\rho'_1 = 1$ and all other components equal to the corresponding entries of $\bar x$. Note that $\hat x$ defines a directed cycle with $\mathrm{lhs}(\hat x) = \mathrm{lhs}(\bar x)$ and $\mathrm{rhs}(\hat x) - \mathrm{rhs}(\bar x) = \lceil b_1 \rceil (1-w) \ge 0$. Thus $\mathcal{I}(\bar x)$ is implied by $\mathcal{I}(\hat x)$ and inequality $w \le 1$.

(ii) Assume $\bar\rho'_t = \bar u'_{t+1} = 1$. Then $\bar\rho'_{t+1} = 0$. Define $\hat x$ by setting $\hat\rho'_t = \hat u'_{t+1} = 0$, $\hat\rho'_{t+1} = 1$ and all other components equal to the corresponding entries of $\bar x$. Vector $\hat x$ defines a directed cycle with $\mathrm{lhs}(\hat x) = \mathrm{lhs}(\bar x)$ and $\mathrm{rhs}(\hat x) - \mathrm{rhs}(\bar x) = (\lceil b_{t+1} \rceil - \lceil b_t \rceil)(1-w) \ge 0$.

(iii) Assume $\bar\rho'_t = \bar v'_t = 1$. Then $\bar\rho'_{t-1} = 0$. Define $\hat x$ by setting $\hat\rho'_t = \hat v'_t = 0$, $\hat\rho'_{t-1} = 1$ and all other components equal to the corresponding entries of $\bar x$. Vector $\hat x$ defines a directed cycle with $\mathrm{lhs}(\hat x) = \mathrm{lhs}(\bar x)$ and $\mathrm{rhs}(\hat x) - \mathrm{rhs}(\bar x) = (\lceil b_{t-1} \rceil - \lceil b_t \rceil + 1)(1-w) \ge 0$.

(iv) Assume $\bar\rho''_t = \bar u''_{t+1} = 1$. Then $\bar\rho''_{t+1} = 0$. Define $\hat x$ by setting $\hat\rho''_t = \hat u''_{t+1} = 0$, $\hat\rho''_{t+1} = 1$ and all other components equal to the corresponding entries of $\bar x$. Vector $\hat x$ defines a directed cycle with $\mathrm{lhs}(\hat x) = \mathrm{lhs}(\bar x)$ and $\mathrm{rhs}(\hat x) - \mathrm{rhs}(\bar x) = (\lceil b_{t+1} - u \rceil - \lceil b_t - u \rceil)w \ge 0$.

(v) Assume $\bar\rho''_t = \bar v''_t = 1$. Then $\bar\rho''_{t-1} = 0$. Define $\hat x$ by setting $\hat\rho''_t = v''_t = 0$, $\hat\rho''_{t-1} = 1$ and all other components equal to the corresponding entries of $\bar x$. Vector $\hat x$ defines a directed cycle with $\mathrm{lhs}(\hat x) = \mathrm{lhs}(\bar x)$ and $\mathrm{rhs}(\hat x) - \mathrm{rhs}(\bar x) = (\lceil b_{t-1} - u \rceil - \lceil b_t - u \rceil + 1)w \ge 0$. □

We now examine all the directed cycles $\bar x$ in $\mathcal{N}$ that satisfy conditions (i)–(v) of Lemma 2.4.

1. There are four types of directed cycles that do not contain the dummy node (non-specified entries are equal to zero):
(i) $\bar u'_t = \bar v'_t = 1$ for some $1 \le t \le n$;
(ii) $\bar u'_t = \bar u''_t = 1$ for some $1 \le t \le n$;
(iii) $\bar v''_t = \bar v'_t = 1$ for some $1 \le t \le n$;
(iv) $\bar v''_t = \bar u''_t = 1$ for some $1 \le t \le n$.
The corresponding inequalities (30) are respectively $0 \ge -(1-w)$, $z_t - z_{t-1} \ge 0$, $-z_t + z_{t-1} \ge -1$ and $0 \ge -w$. So we obtain the inequalities $0 \le w \le 1$ and $0 \le z_t - z_{t-1} \le 1$ for $1 \le t \le n$, which are part of the original description of K.

2. The directed cycle defined by $\bar\rho'_t = \bar\rho''_t = 1$ for some $1 \le t \le n$ (other entries are equal to zero) generates the inequality

$$z_t \ge \lceil b_t \rceil (1-w) + \lceil b_t - u \rceil w. \tag{31}$$

3. We now consider any directed cycle containing arc ρ'_j for some $1 \le j \le n$, going through nodes $j, j-1, \dots, 1$ and terminating with either arc u''_1 or arc v'_1. Note that for $2 \le t \le j$, there are two arcs going from node t to node $t-1$: arcs u''_t and v'_t. Also observe that by Lemma 2.4 (iii), arc v'_j cannot be part of the directed cycle. We can then define a sequence of indices $0 = t_0 < t_1 < t_2 < \cdots < t_k = j$ such that the directed cycle contains arcs of type u''_t between nodes t_k and t_{k-1}, then arcs of type v'_t between nodes t_{k-1} and t_{k-2}, then again arcs of type u''_t, and so forth.
More formally, given any sequence $0 = t_0 < t_1 < t_2 < \cdots < t_k \le n$ (with $k \ge 1$), we consider the directed cycle defined by setting to 1 the following components of $\bar x$:
- $\bar\rho'_{t_k}$;
- $\bar u''_t$ for $t_{k-i-1} < t \le t_{k-i}$, $0 \le i \le k-1$, i even;
- $\bar v'_t$ for $t_{k-i-1} < t \le t_{k-i}$, $0 \le i \le k-1$, i odd.

The corresponding inequality $\mathcal{I}(\bar x)$ is

$$z_{t_k} - z_{t_{k-1}} + z_{t_{k-2}} - \cdots + (-1)^{k+1} z_{t_1} \ge \big(\lceil b_{t_k} \rceil - t_{k-1} + t_{k-2} - \cdots + (-1)^{k+1} t_1\big)(1-w). \tag{32}$$

4. We now consider any directed cycle containing arc ρ'_j for some $1 \le j \le n$, going through nodes $j, j-1, \dots, h$ (with $h \le j$) and terminating with arc ρ''_h. By Lemma 2.4 (iii)–(iv), neither

arc v'_j nor arc u''_{h+1} belongs to the directed cycle. Then the directed cycle is associated with a sequence of indices $h = t_1 < t_2 < \cdots < t_k = j$ with k odd and is defined by setting to 1 the following components of $\bar{x}$:

- $\bar{\rho}'_{t_k}$ and $\bar{\rho}''_{t_1}$;
- $\bar{u}''_t$ for $t_{k-i-1} < t \le t_{k-i}$, $0 \le i < k-1$, i even;
- $\bar{v}'_t$ for $t_{k-i-1} < t \le t_{k-i}$, $0 \le i < k-1$, i odd.

We show that inequality $\mathcal{I}(\bar{x})$ is not needed in the description of $\mathrm{conv}(K)$.

Assume first that $\lceil b_{t_k} - u\rceil - \lceil b_{t_1} - u\rceil - t_{k-1} + t_{k-2} - \cdots + t_1 \ge 0$. Define a feasible flow $\hat{x}$ by setting $\hat{\rho}'_{t_k} = \hat{\rho}''_{t_k} = 1$, $\hat{v}'_t = \hat{v}''_t = 1$ for $t_{k-i-1} < t \le t_{k-i}$, $0 \le i < k-1$, i odd, and all other entries equal to zero. Since $\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x})$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = (\lceil b_{t_k} - u\rceil - \lceil b_{t_1} - u\rceil - t_{k-1} + t_{k-2} - \cdots + t_1)w \ge 0$, inequality $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and inequality $w \ge 0$. It remains to observe that the flow defined by $\hat{x}$ decomposes into cycles that have already been examined in points 1–2 above.

Now assume that $\lceil b_{t_k} - u\rceil - \lceil b_{t_1} - u\rceil - t_{k-1} + t_{k-2} - \cdots + t_1 < 0$. Note that this implies $\lceil b_{t_k}\rceil - \lceil b_{t_1}\rceil - t_{k-1} + t_{k-2} - \cdots + t_1 \le 0$. Define a feasible flow $\hat{x}$ by setting $\hat{\rho}'_{t_1} = \hat{\rho}''_{t_1} = 1$, $\hat{u}'_t = \hat{u}''_t = 1$ for $t_{k-i-1} < t \le t_{k-i}$, $0 \le i < k-1$, i even, and all other entries equal to zero. Since $\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x})$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = -(\lceil b_{t_k}\rceil - \lceil b_{t_1}\rceil - t_{k-1} + t_{k-2} - \cdots + t_1)(1-w)$, inequality $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and inequality $w \le 1$. As above, $\hat{x}$ decomposes into directed cycles that have already been analyzed.

5. We now consider any directed cycle containing arc ρ'_j for some $1 \le j \le n$, going through nodes $j, j+1, \dots, h$ (with $h \ge j$) and terminating with arc ρ''_h. By conditions (ii) and (v) of Lemma 2.4, neither arc u'_{j+1} nor arc v''_h belongs to the directed cycle. Then the directed cycle is associated with a sequence of indices $j = t_1 < t_2 < \cdots < t_k = h$ with k odd and is defined by setting to 1 the following components of $\bar{x}$:

- $\bar{\rho}'_{t_1}$ and $\bar{\rho}''_{t_k}$;
- $\bar{v}''_t$ for $t_i < t \le t_{i+1}$, $1 \le i \le k-1$, i odd;
- $\bar{u}'_t$ for $t_i < t \le t_{i+1}$, $1 \le i \le k-1$, i even.

We show that inequality $\mathcal{I}(\bar{x})$ is not needed in the description of $\mathrm{conv}(K)$. Since the approach is almost identical to that used in point 4, we only state how to define $\hat{x}$.

If $\lceil b_{t_k}\rceil - \lceil b_{t_1}\rceil - t_{k-1} + t_{k-2} - \cdots + t_1 \ge 0$, we set $\hat{\rho}'_{t_k} = \hat{\rho}''_{t_k} = 1$, $\hat{v}'_t = \hat{v}''_t = 1$ for $t_i < t \le t_{i+1}$, $1 \le i \le k-1$, i odd, and all other entries equal to zero. Otherwise, we set $\hat{\rho}'_{t_1} = \hat{\rho}''_{t_1} = 1$, $\hat{u}'_t = \hat{u}''_t = 1$ for $t_i < t \le t_{i+1}$, $1 \le i \le k-1$, i even, and all other entries equal to zero.

6. It only remains to consider directed cycles containing arc v''_1, going through nodes $1, \dots, h$ for some $1 \le h \le n$ and terminating with arc ρ''_h. By Lemma 2.4 (v), arc v''_h is not part of the directed cycle. Thus there is a sequence of indices $0 = t_0 < t_1 < t_2 < \cdots < t_k = h$ with k even such that the directed cycle is defined by setting to 1 the following components of $\bar{x}$:

- $\bar{\rho}''_{t_k}$;
- $\bar{v}''_t$ for $t_i < t \le t_{i+1}$, $0 \le i \le k-1$, i even;
- $\bar{u}'_t$ for $t_i < t \le t_{i+1}$, $0 \le i \le k-1$, i odd.

We show that inequality $\mathcal{I}(\bar{x})$ is not needed in the description of $\mathrm{conv}(K)$. As above, we only describe $\hat{x}$.

If $\lceil b_{t_k}\rceil - t_{k-1} + t_{k-2} - \cdots + t_0 \ge 0$, we set $\hat{\rho}'_{t_k} = \hat{\rho}''_{t_k} = 1$, $\hat{v}'_t = \hat{v}''_t = 1$ for $t_i < t \le t_{i+1}$, $1 \le i \le k-1$, i even, and all other entries equal to zero. Otherwise, we set $\hat{u}'_t = \hat{u}''_t = 1$ for $t_i < t \le t_{i+1}$, $1 \le i \le k-1$, i odd, and all other entries equal to zero.

The above analysis proves the following result.

Theorem 2.5. *The convex hull of K is described by the linear inequalities $0 \le w \le 1$, $0 \le z_t - z_{t-1} \le 1$ for $1 \le t \le n$, inequalities* (31) *for $1 \le t \le n$ and inequalities* (32) *for all sequences of indices $1 \le t_1 < \cdots < t_k \le n$ (with $k \ge 1$).*

Remark 1. *Inequality* (32) *for $k = 1$ reads $z_t \ge \lceil b_t\rceil (1-w)$ (where $t = t_1$). Then, if $b_t \le u$, inequality* (31) *is redundant, whereas if $b_t > u$, inequality $z_t \ge \lceil b_t\rceil (1-w)$ is redundant. This implies that for every t we can gather the two inequalities into the single constraint $z_t \ge \lceil b_t\rceil (1-w) + \lceil b_t - u\rceil^+ w$.*

Remark 2. *If $b_1 > 0$ (i.e. $\lceil b_1\rceil = 1$), any inequality of type* (32) *with $k \ge 3$ odd is redundant. To see this, take any sequence $1 \le t_1 < t_2 < \cdots < t_k \le n$ with $k \ge 3$ odd. If $t_1 > 1$ then the inequality* (32) *associated with this sequence can be obtained by adding the inequality $z_1 \ge 1 - w$ to* *the inequality* (32) *generated by the sequence* $1, t_1, t_2, \ldots, t_k$. *Otherwise, if* $t_1 = 1$, *the inequality can be obtained by adding the inequality* $z_1 \geq 1-w$ *to the inequality* (32) *generated by the sequence* $t_2, \ldots, t_k$.

To conclude this subsection, we discuss the separation of inequalities (32).

PROPOSITION 2.6. *Given a point* $(\bar{z}, \bar{w})$, *a most violated inequality* (32) *can be found in time* $\mathcal{O}(n)$.

Proof. We first observe that the family of inequalities (32) can be described as follows: for every subset $S \subseteq \{1, \ldots, n-1\}$ and every index ℓ such that $\max\{t : t \in S\} < \ell \leq n$, we have the valid inequality

$$(z_\ell - \lceil b_\ell \rceil (1-w)) + \sum_{t \in S} (-z_t + z_{t-1} + (1-w)) \geq 0, \tag{33}$$

where $z_0 = 0$. Then, for fixed $\ell \in \{1, \ldots, n\}$, a most violated inequality (33) is given by $S = \{t : t < \ell,\ -\bar{z}_t + \bar{z}_{t-1} + (1-\bar{w}) < 0\}$.

An algorithm to find a most violated inequality (33) is then the following. First we construct the set $S' = \{t : -\bar{z}_t + \bar{z}_{t-1} + (1-\bar{w}) < 0\}$ and we compute the values

$$\sigma_\ell = \sum_{t \in S',\, t < \ell} (-\bar{z}_t + \bar{z}_{t-1} + (1-\bar{w})) \quad \text{for } 1 \leq \ell \leq n-1.$$

Then we find the index ℓ for which $(\bar{z}_\ell - \lceil b_\ell \rceil (1-\bar{w})) + \sigma_\ell$ is minimum. This index ℓ and the corresponding set $S = \{t \in S' : t < \ell\}$ give a most violated inequality (33). Note that the values σ_ℓ for $1 \leq \ell \leq n-1$ can be computed in time $\mathcal{O}(n)$, because

$$\sigma_\ell = \sigma_{\ell-1} - (-\bar{z}_\ell + \bar{z}_{\ell-1} + (1-\bar{w}))^- \quad \text{for } 2 \leq \ell \leq n-1.$$

Therefore the overall running time of the algorithm is $\mathcal{O}(n)$. ∎

3. The convex hull of the discrete lot-sizing set (5)–(7)**.** Recall that conditions (5)–(7) describe the feasible region of the single-item discrete lot-sizing problem with a variable upper bound on the initial stock. We denote this mixed-integer set by X. In this section we first find a compact extended formulation for $\operatorname{conv}(X)$ and then we project it onto the space of the (s, z, w) variables to obtain a linear-inequality description of $\operatorname{conv}(X)$ in its original space.

3.1. Extended formulation. Defining $X_0 = \{(s, z, w) \in X : w = 0\}$ and $X_1 = \{(s, z, w) \in X : w = 1\}$, one has $X = X_0 \cup X_1$. As for the set K in §2.2.1, we first find linear-inequality descriptions for $\operatorname{conv}(X_0)$ and $\operatorname{conv}(X_1)$ and then derive an extended formulation for $\operatorname{conv}(X)$ using Balas' result [2].

If $w = 0$, then also $s = 0$. Since, with s and w fixed at 0, the constraint matrix of the linear inequalities appearing in (5)–(7) is totally unimodular, the polyhedron $\operatorname{conv}(X_0)$ is described by the inequalities

$$\begin{aligned} z_t &\geq \lceil b_t \rceil, & 1 \leq t \leq n,\\ s &= 0,\ w = 0, & \\ 0 \leq z_t - z_{t-1} &\leq 1, & 1 \leq t \leq n. \end{aligned}$$

We now turn to X_1. This set is a mixing set with an upper bound on the continuous variable s. Though a linear-inequality description of its convex hull in the original space can be given explicitly [5], here we describe it using an extended formulation, as this will simplify the derivation of the inequalities defining $\operatorname{conv}(X)$.

To obtain an extended formulation for $\operatorname{conv}(X_1)$, we use the results of [3]: in that paper the authors describe a technique to derive an extended formulation for the convex hull of any mixed-integer set of the type $\{(x, y) \in \mathbb{R}^p \times \mathbb{Z}^q : Ax + By \geq \beta\}$, where $[A \mid B]$ is a totally unimodular matrix with at most two nonzero entries per row. Since X_1 is a set of this type, the results of [3] apply.

Define $b_0 = 0$, $b_{n+1} = u$ and let $g_1 > \cdots > g_m > g_{m+1} = 0$ be the $m+1$ distinct fractional parts of the numbers $b_0, b_1, \dots, b_{n+1}$. Set $g_0 = 1$. For $0 \le t \le n+1$, define $\varphi(t)$ as the unique index such that $f'_t = g_{\varphi(t)}$. Note that $0 \le \varphi(t) \le m$ for $0 \le t \le n+1$. Finally, set

$$\pi = \begin{cases} \varphi(n+1) - 1 & \text{if } \varphi(n+1) > 0 \text{ (i.e., } u \notin \mathbb{Z}), \\ m & \text{if } \varphi(n+1) = 0 \text{ (i.e., } u \in \mathbb{Z}). \end{cases}$$

Note the relation

$$g_{\pi+1} = u - \lfloor u \rfloor , \tag{34}$$

which will be used later.

From the results of [3] (modulo some minor modifications), an extended formulation for $\operatorname{conv}(X_1)$ is the following:

$$\begin{array}{rl} s = \sum_{\ell=0}^{m} (g_\ell - g_{\ell+1})\mu_\ell, & \\ \mu_\ell - \mu_{\ell-1} \ge 0, & 1 \le \ell \le m, \\ \mu_m - \mu_0 \le 1, & \\ \mu_{\varphi(t)} + z_t \ge \lceil b_t \rceil , & 1 \le t \le n, \\ \mu_0 \ge 0,\ \mu_\pi \le \lfloor u \rfloor , & \\ z_t \ge \lceil b_t - u \rceil , & 1 \le t \le n, \\ w = 1,\ 0 \le z_t - z_{t-1} \le 1, & 1 \le t \le n, \end{array}$$

As in §2.2.1, if one writes Balas' extended formulation for $\operatorname{conv}(X) = \operatorname{conv}(X_0 \cup X_1)$, several variables can be eliminated. The resulting formulation is:

$$\begin{array}{rcl} & 0 \le w \le 1, & \\ (\tau_t) & z_t = z'_t + z''_t, & 1 \le t \le n, \\ (\rho'_t) & z'_t \ge \lceil b_t \rceil (1 - w), & 1 \le t \le n, \\ (\rho''_t) & z''_t \ge \lceil b_t - u \rceil w, & 1 \le t \le n, \\ (\sigma) & \sum_{\ell=0}^{m} (g_\ell - g_{\ell+1})\mu_\ell = s, & \\ (r_0) & \mu_0 - \mu_m \ge -w, & \\ (r_\ell) & \mu_\ell - \mu_{\ell-1} \ge 0, & 1 \le \ell \le m, \\ (\gamma_0) & \mu_0 \ge 0, & \\ (\gamma_t) & \mu_{\varphi(t)} + z''_t \ge \lceil b_t \rceil w, & 1 \le t \le n, \\ (\gamma_{n+1}) & -\mu_\pi \ge -\lfloor u \rfloor w, & \\ (u'_t) & z'_t - z'_{t-1} \ge 0, & 1 \le t \le n, \\ (v'_t) & -z'_t + z'_{t-1} \ge -(1 - w), & 1 \le t \le n, \\ (u''_t) & z''_t - z''_{t-1} \ge 0, & 1 \le t \le n. \\ (v''_t) & -z''_t + z''_{t-1} \ge -w, & 1 \le t \le n. \end{array}$$

Since $m \le n+2$, we have obtained a compact extended formulation for $\operatorname{conv}(X)$ that uses $\mathcal{O}(n)$ variables and constraints. Up to a change of variables, this is essentially the formulation given by Miller and Wolsey [7] for the mixing set, with some modifications needed to model the upper bound on s.

3.2. Convex hull in the original space. We now project the set defined by the above linear system onto the space of the (s, z, w) variables. The approach is similar to that used for the polyhedron $\operatorname{conv}(K)$ in §2.2.2, and in fact the results found there will be used here.

For $0 \le \ell \le m+1$, define $T_\ell = \{t : \varphi(t) = \ell\}$. Apart from the constraint $0 \le w \le 1$, all facet-defining inequalities for $\mathrm{conv}(X)$ are of the form

$$\sigma s + \sum_{t=1}^n \tau_t z_t \ge \sum_{t=1}^n \Big[(\rho'_t \lceil b_t \rceil - v'_t)(1-w) + (\rho''_t \lceil b_t - u \rceil + \gamma_t \lceil b_t \rceil - v''_t) w\Big] - r_0 w - \gamma_{n+1} \lfloor u \rfloor w, \quad (35)$$

where $(\tau, \rho', \rho'', \sigma, r, \gamma, u', v', u'', v'')$ is an extreme ray of the cone

$$\tau_t = \rho'_t + u'_t - u'_{t+1} - v'_t + v'_{t+1}, \qquad 1 \le t \le n, \quad (36)$$
$$\tau_t = \rho''_t + \gamma_t + u''_t - u''_{t+1} - v''_t + v''_{t+1}, \qquad 1 \le t \le n, \quad (37)$$
$$r_\ell - r_{\ell+1} + \textstyle\sum_{t \in T_\ell} \gamma_t - (g_\ell - g_{\ell+1})\sigma = 0, \qquad 0 \le \ell \le m,\ \ell \ne \pi, \quad (38)$$
$$r_\ell - r_{\ell+1} + \textstyle\sum_{t \in T_\ell} \gamma_t - \gamma_{n+1} - (g_\ell - g_{\ell+1})\sigma = 0, \qquad \ell = \pi, \quad (39)$$
$$\gamma_0 \ge 0,\ \gamma_{n+1} \ge 0,\ \rho'_t, \rho''_t, \gamma_t, u'_t, v'_t, u''_t, v''_t, r_\ell \ge 0, \qquad 1 \le t \le n,\ 0 \le \ell \le m+1. \quad (40)$$

with $u'_{n+1} = v'_{n+1} = u''_{n+1} = v''_{n+1} = r_{m+1} = 0$. After elimination of variables $\tau_1, \dots, \tau_n$, cone (36)–(40) takes the form:

$$\rho'_t + u'_t - u'_{t+1} - v'_t + v'_{t+1} = \rho''_t + \gamma_t + u''_t - u''_{t+1} - v''_t + v''_{t+1}, \quad 1 \le t \le n, \quad (41)$$
$$r_\ell - r_{\ell+1} + \textstyle\sum_{t \in T_\ell} \gamma_t = (g_\ell - g_{\ell+1})\sigma, \qquad 0 \le \ell \le m,\ \ell \ne \pi, \quad (42)$$
$$r_\ell - r_{\ell+1} + \textstyle\sum_{t \in T_\ell} \gamma_t - \gamma_{n+1} = (g_\ell - g_{\ell+1})\sigma, \qquad \ell = \pi, \quad (43)$$
$$\gamma_0 \ge 0,\ \gamma_{n+1} \ge 0,\ \rho'_t, \rho''_t, \gamma_t, u'_t, v'_t, u''_t, v''_t, r_\ell \ge 0, \qquad 1 \le t \le n,\ 0 \le \ell \le m+1, \quad (44)$$

while the left-hand side of inequality (35) becomes

$$\sigma s + \sum_{t=1}^n (\rho'_t + u'_t - u'_{t+1} - v'_t + v'_{t+1}) z_t.$$

After manipulating the above expression, inequality (35) can be rewritten as follows:

$$\sigma s + \sum_{t=1}^n \rho'_t z_t + \sum_{t=1}^n (u'_t - v'_t)(z_t - z_{t-1}) \ge$$
$$\sum_{t=1}^n \Big[(\rho'_t \lceil b_t \rceil - v'_t)(1-w) + (\rho''_t \lceil b_t - u \rceil + \gamma_t \lceil b_t \rceil - v''_t) w\Big] - r_0 w - \gamma_{n+1} \lfloor u \rfloor w. \quad (45)$$

Note that for fixed $\bar\sigma \in \mathbb{R}$, the linear system (41)–(44) defines the set of feasible flows in the network $\mathcal{N}(\bar\sigma)$ depicted in Figure 3.1 (with a dummy node that has not been drawn explicitly), where the nodes on the bottom row $(\nu_0, \dots, \nu_m)$ have a requirement of $(g_\ell - g_{\ell+1})\bar\sigma$ for $0 \le \ell \le m$. We remark that for every node ν_ℓ with $\ell \ne \pi + 1$, there is at least one arc γ_t entering that node, whereas this is not necessarily true for node $\nu_{\pi+1}$. More specifically, assuming $\pi \ne m$, arc γ_t enters node $\nu_{\pi+1}$ if and only if b_t and u have the same fractional part.

In the following we denote by x the vector of variables $(\rho', \rho'', \sigma, r, \gamma, u', v', u'', v'')$. As in §2.2.2, we denote by $\mathrm{lhs}(x)$ (resp. $\mathrm{rhs}(x)$) the left-hand side (resp. right-hand side) of the inequality (45) corresponding to the vector x, and by $\mathcal{I}(x)$ the whole inequality.

Since (41)–(44) is a cone and we are interested in its extreme rays, it suffices to study the three cases $\bar\sigma = 0$, $\bar\sigma = -1$ and $\bar\sigma = 1$.

We start by considering the case $\bar\sigma = 0$. The following lemma shows that in this case every non-redundant inequality (45) is one of the inequalities listed in Theorem 2.5.

LEMMA 3.1. *Let $\bar x$ be an extreme ray of cone (41)–(44) with $\bar\sigma = 0$. If $\mathcal{I}(\bar x)$ is non-redundant in the description of* $\mathrm{conv}(X)$*, then it is valid for K.*

Proof. If $\bar\sigma = 0$, all node requirements in $\mathcal{N}(\bar\sigma)$ are equal to 0. Then (up to multiplication by a positive scalar) $\bar x$ is a 0,1 vector that defines a directed cycle in $\mathcal{N}(\bar\sigma)$. Note that if one removes nodes $\nu_0, \dots, \nu_m$, the resulting network is precisely that of §2.2.2 (Figure 2.1), and since $\bar\sigma = 0$,

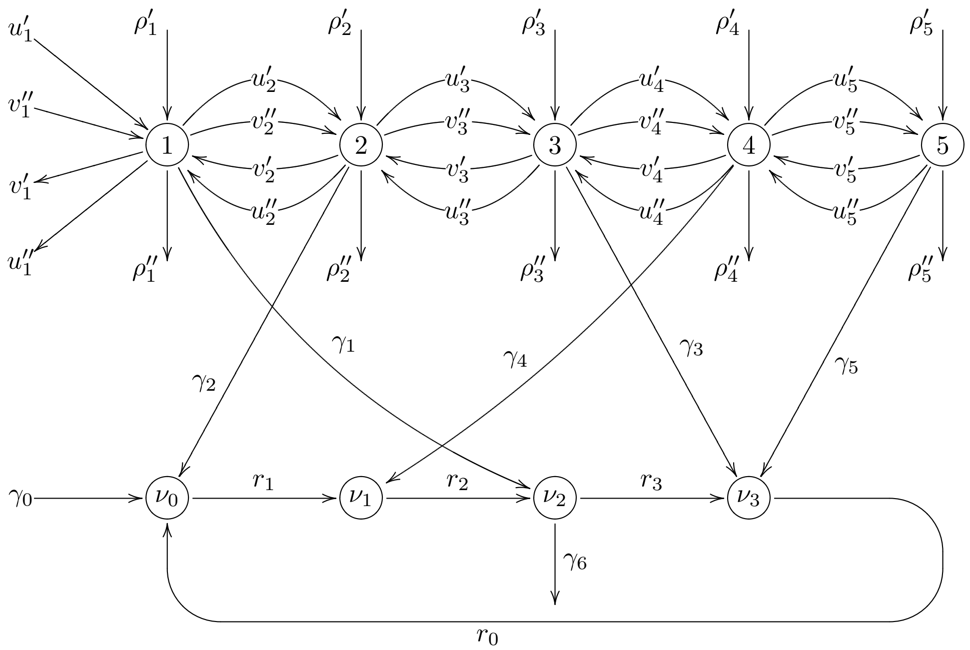

FIGURE 3.1. *The network $\mathcal{N}(\bar{\sigma})$ corresponding to a linear system of the type* (41)–(44) *with $n = 5$ and $m = 3$. Here $\varphi(1) = 2$, $\varphi(2) = 0$ (i.e. $b_2 \in \mathbb{Z}$), $\varphi(3) = 3$, $\varphi(4) = 1$, $\varphi(5) = 3$ and $\pi = 2$ (i.e. $\varphi(6) = 3$).*

inequality (45) reduces to (30). Thus we can assume that at least one of the arcs incident with some node ν_k $(0 \leq k \leq m)$ is part of the directed cycle, as otherwise we would find one of the inequalities listed in Theorem 2.5.

Assume first that $\bar{\gamma}_{n+1} = 0$. Since γ_{n+1} is the only arc leaving the set of nodes $\{\nu_0, \ldots, \nu_m\}$ and because we are assuming that at least one of the arcs incident with some node ν_k $(0 \leq k \leq m)$ is part of the directed cycle, $\bar{x}$ must satisfy $\bar{r}_\ell = 1$ for $0 \leq \ell \leq m$ and all other components are equal to zero. The corresponding inequality is $0 \geq -w$, i.e. $w \geq 0$.

Now assume that $\bar{\gamma}_{n+1} = 1$. We claim that if $\mathcal{I}(\bar{x})$ is non-redundant, then wlog $\bar{\gamma}_t = 0$ for $1 \leq t \leq n$. To prove this, we distinguish two cases.

1. Suppose that $\bar{\gamma}_t = 1$ for some $1 \leq t \leq n$ such that $\varphi(t) > \pi$. In reference to Figure 3.1, this means that the head of arc γ_t is to the right of the tail of arc γ_{n+1}. Since $\bar{x}$ defines a directed cycle in $\mathcal{N}(\bar{\sigma})$, then $\bar{\rho}_t'' = 0$ and $\bar{r}_\ell = 1$ for $\ell \in L := \{\varphi(t)+1, \ldots, m\} \cup \{0, \ldots, \pi\}$. Define a directed cycle $\hat{x}$ by setting $\hat{\gamma}_t = \hat{\gamma}_{n+1} = 0$, $\hat{r}_\ell = 0$ for $\ell \in L$, $\hat{\rho}_t'' = 1$ and all other components equal to the corresponding entries of $\hat{x}$. Since $\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x})$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = (\lceil b_t - u \rceil - (\lceil b_t \rceil - \lfloor u \rfloor - 1))w \geq 0$, inequality $\mathcal{I}(\bar{x})$ is redundant.

2. Suppose that $\bar{\gamma}_t = 1$ for some $1 \leq t \leq n$ such that $\varphi(t) \leq \pi$, so the head of arc γ_t is now to the left of (or coincides with) the tail of arc γ_{n+1}. Condition $\varphi(t) \leq \pi$ means that the fractional part of b_t is either larger than that of u or equal to zero (see the definitions of $\varphi(t)$ and π). In both cases, $\lceil b_t - u \rceil = \lceil b_t \rceil - \lfloor u \rfloor$. Now the same proof as above applies, except that in this case $L = \{\varphi(t)+1, \ldots, \pi\}$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = (\lceil b_t - u \rceil - (\lceil b_t \rceil - \lfloor u \rfloor))w = 0$.

Therefore wlog $\bar{\gamma}_t = 0$ for $1 \leq t \leq n$ if $\bar{\gamma}_{n+1} = 1$. It follows that the only nonzero components of $\bar{x}$ are $\bar{\gamma}_0 = \bar{\gamma}_{n+1} = 1$ and $\bar{r}_\ell = 1$ for $1 \leq \ell \leq \pi$. The corresponding inequality is $0 \geq -\lfloor u \rfloor w$, i.e. $w \geq 0$ (or $0 \geq 0$). □

We now consider the case $\bar{\sigma} = -1$.

LEMMA 3.2. *If $\bar{x}$ is an extreme ray of cone* (41)–(44) *with $\bar{\sigma} = -1$, then $\mathcal{I}(\bar{x})$ is the inequality $s \leq uw$.*

Proof. If $\bar{\sigma} = -1$, the requirements of nodes $\nu_0, \ldots, \nu_m$ are all negative. In other words, we can

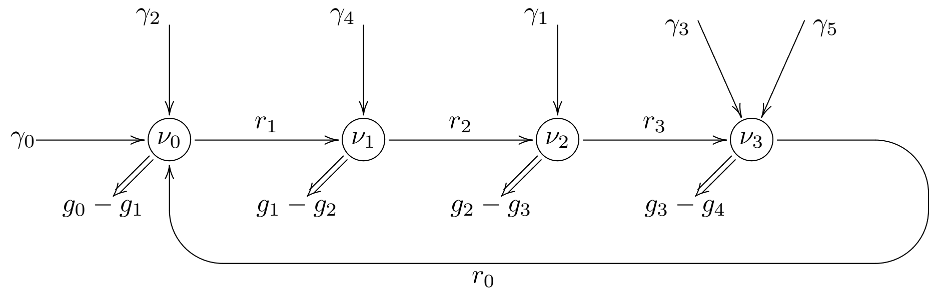

FIGURE 3.2. *The reduced network corresponding to that of Figure 3.1. Thick arrows represent node requirements.*

think of an inflow entering each of these nodes. In this case $\bar{x}$ defines an acyclic flow in the network. Since there is a positive inflow in the set of nodes $\nu_0, \dots, \nu_m$ and since γ_{n+1} is the only arc leaving this set of nodes, then $\bar{\gamma}_{n+1} > 0$. Then the absence of directed cycles in the support of $\bar{x}$ implies that the only other variables that can have a nonzero value are the r_ℓ's. Since the total inflow of nodes $\nu_0, \dots, \nu_m$ is equal to 1, we see that $\bar{\gamma}_{n+1} = 1$ and $\bar{r}_0 = \sum_{\ell=\pi+1}^{m}(g_\ell - g_{\ell+1}) = g_{\pi+1} = u - \lfloor u \rfloor$, where the last equality follows from (34). Inequality (45) then reads $-s \geq -uw$, i.e. $s \leq uw$. ☐

We now analyze the case $\bar{\sigma} = 1$. In this case the requirements of nodes $\nu_0, \dots, \nu_m$ are all positive and we can think of an outflow leaving each of these nodes. Similarly to the above case, an extreme ray $\bar{x}$ of cone (41)–(44) with $\bar{\sigma} = 1$ defines an acyclic flow in $\mathcal{N}(\bar{\sigma})$.

LEMMA 3.3. *If $\bar{x}$ is an extreme ray of cone* (41)–(44) *with $\bar{\sigma} = 1$, then $\bar{\rho}''_t = 0$ for $1 \leq t \leq n$ and $\bar{u}''_1 = \bar{v}'_1 = \bar{\gamma}_{n+1} = 0$.*

Proof. Since the total outflow of the network is strictly positive, there must be at least one arc that has the dummy node as tail and carries a positive flow. Then, since $\bar{x}$ defines an acyclic flow, none of the arcs having the dummy node as head can carry a positive flow. The conclusion follows. ☐

The next lemma is a key result for finding a linear-inequality description of $\operatorname{conv}(M)$. Since the proof technique is similar to that used in §2.2.2, but there are more technicalities, we postpone the proof to the appendix.

LEMMA 3.4. *Let $\bar{x}$ be an extreme ray of cone* (41)–(44) *with $\bar{\sigma} = 1$, and suppose that $\mathcal{I}(\bar{x})$ is non-redundant in the description of* $\operatorname{conv}(X)$. *Then we can assume that $\bar{\gamma}_t = \bar{\rho}'_t$ for $1 \leq t \leq n$.*

By Lemmas 3.3 and 3.4, we can assume that $\bar{u}'_t = \bar{u}''_t = \bar{v}'_t = \bar{v}''_t = \bar{\rho}''_t = 0$ for $1 \leq t \leq n$. Furthermore, for each $t = 1, \dots, n$, arcs ρ'_t and γ_t can be replaced with a single arc. Then the network can be reduced to that depicted in Figure 3.2 (recall that $\bar{\gamma}_{n+1} = 0$ by Lemma 3.3), and inequality (45) simplifies as follows:

$$s + \sum_{t=1}^{n} \gamma_t(z_t - \lceil b_t \rceil) + r_0 w \geq 0. \tag{46}$$

To conclude the derivation of the inequalities defining $\operatorname{conv}(X)$, we have to find the acyclic flows in this reduced network. There are two types of acyclic flows, depending on the value of variable γ_0.

1. Let $\bar{x}$ be an acyclic flow in the reduced network with $\bar{\gamma}_0 > 0$. Let $\bar{\gamma}_{t_0}, \dots, \bar{\gamma}_{t_k}$ $(k \geq 0)$ be those components of vector $\bar{\gamma}$ that have positive value, with $\varphi(t_0) \geq \cdots \geq \varphi(t_k)$. Since the flow is acyclic, its support cannot contain two distinct arcs $\gamma_t, \gamma_{t'}$ entering the same node, i.e. it is not possible that $\varphi(t) = \varphi(t')$. Thus $\varphi(t_0) > \cdots > \varphi(t_k)$. Also note that $t_0 = 0$. Now it can be checked that $\bar{\gamma}_{t_i} = g_{\varphi(t_i)} - g_{\varphi(t_{i+1})} = f'_{t_i} - f'_{t_{i+1}}$ for $0 \leq i \leq k$, where we set $g_{\varphi(t_{k+1})} = f'_{t_{k+1}} = 0$. Moreover, $\bar{r}_0 = 0$. The corresponding inequality (46) is

$$s + \sum_{i=1}^{k} (f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) \geq 0. \tag{47}$$

If $k \geq 1$, this is precisely inequality (10), whereas if $k = 0$ the inequality reads $s \geq 0$.

2. Let $\bar{x}$ be an acyclic flow in the reduced network with $\bar{\gamma}_0 = 0$. Let $\bar{\gamma}_{t_1}, \ldots, \bar{\gamma}_{t_k}$ $(k \geq 0)$ be those components of vector $\bar{\gamma}$ that have positive value, with $\varphi(t_1) \geq \cdots \geq \varphi(t_k)$. As above, one sees that in fact $\varphi(t_1) > \cdots > \varphi(t_k)$. It can be checked that $\bar{\gamma}_{t_i} = g_{\varphi(t_i)} - g_{\varphi(t_{i+1})} = f'_{t_i} - f'_{t_{i+1}}$ for $0 \leq i \leq k-1$, while $\bar{\gamma}_{t_k} = g_{\varphi(t_k)} + 1 - g_{\varphi(t_1)} = f'_{t_k} + 1 - f'_{t_1}$. Moreover, $\bar{r}_0 = 1 - g_{\varphi(t_1)} = 1 - f'_{t_1}$. The corresponding inequality (46) is (again, $f'_{t_{k+1}} = 0$)

$$s + \sum_{i=1}^{k} (f'_{t_i} - f'_{t_{i+1}})(z_{t_i} - \lceil b_{t_i} \rceil) + (1 - f'_{t_1})(z_{t_k} - \lceil b_{t_k} \rceil + w) \geq 0. \tag{48}$$

This is precisely inequality (11).

We can now state the main result of this section.

THEOREM 3.5. $\mathrm{conv}(X) = \mathrm{conv}(M) \cap \mathrm{conv}(K)$.

Proof. It suffices to observe that every facet-defining inequality for $\mathrm{conv}(X)$ is valid for $\mathrm{conv}(M)$ or $\mathrm{conv}(K)$. ∎

4. Constant-capacity lot-sizing with stock upper bounds and fixed charges: Computation. Here we consider the constant-capacity lot-sizing problem with stock upper bounds and fixed charges $LS - CC - SVUB$:

$$\min \textstyle\sum_{t=0}^{n} (h_t s_t + c_t w_t) + \sum_{t=1}^{n} (p_t x_t + q_t y_t) \tag{49}$$

$$\text{subject to} \quad s_{t-1} + x_t = d_t + s_t, \qquad 1 \leq t \leq n, \tag{50}$$

$$0 \leq x_t \leq C y_t,\ y_t \in \{0,1\}, \qquad 1 \leq t \leq n, \tag{51}$$

$$0 \leq s_t \leq u_t w_t,\ w_t \in \{0,1\}, \qquad 0 \leq t \leq n. \tag{52}$$

In the above formulation, x_t is the amount produced in period t, s_t is the stock available at the end of period t, and y_t and w_t are binary variables used to model the fixed charges on production and stock respectively. The data of the problem is as follows: p_t and h_t are the per unit production and storage costs respectively, q_t and c_t are the fixed charges on production and stock respectively; finally, there are upper bounds u_t on the stock and a constant capacity C bounding the production level.

For each $k = 1, \ldots, n$, the discrete lot-sizing set X^k

$$\begin{array}{ll} s_{k-1} + C \sum_{i=k}^{t} y_i \geq \sum_{i=k}^{t} d_i, & k \leq t \leq n, \\ 0 \leq s_{k-1} \leq u_{k-1} w_{k-1}, & \\ y_t \in \{0,1\}, & k \leq t \leq n, \end{array}$$

studied in the previous section, is a relaxation of (50)–(52). As shown there, the set X^k admits itself two relaxations: a mixing relaxation M^k and a pure integer relaxation K^k.

The set $X^{WW} := \bigcap_{k=1}^{n} X^k$ is known as the Wagner-Whitin relaxation of (50)–(52). When the problem is uncapacitated, i.e. $C \geq \sum_{t=1}^{n} d_t$, the convex hull of X^{WW} was given by Van Vyve and Ortega [11] for the case of unbounded inventory level (i.e. large u_k) and by Atamtürk and Küçükyavuz [1] for the case of bounded inventory level. The inequalities describing $\mathrm{conv}\left(X^{WW}\right)$ given in those two papers are special cases of the inequalities of Theorem 2.1. This shows that in the uncapacitated case the intersection of the convex hull descriptions of the sets X^k given in Theorem 2.1 constitutes an integral polyhedron. However it is easy to construct examples showing this is not the case with constant capacities.

For the uncapacitated case Atamtürk and Küçükyavuz [1] have solved a variety of randomly generated instances of (49)–(52) using valid inequalities and polynomial-time separation algorithms. Here we report briefly on computational experiments on instances generated in the same way both for the uncapacitated and the capacitated version of the problem. All computations were performed using Xpress-MP (release 2007B) on a machine with 1.80 GHz Intel Core 2 Duo processor and 2 GB of RAM.

		Initial formulation			M^k
f	c	best sol.	best bd	gap	time
1000	2	106.1	83.5	21.4	2
	5	106.1	82.7	22.0	2
	10	106.2	83.8	21.0	1
	20	105.6	81.0	23.3	2
2000	2	108.3	75.7	30.0	3
	5	108.4	73.5	32.2	2
	10	109.5	73.9	32.5	1
	20	110.2	71.2	35.4	2
5000	2	109.6	62.2	43.2	1
	5	110.6	57.0	48.5	1
	10	110.2	57.0	48.3	2
	20	111.2	54.8	50.7	1

TABLE 4.1
Uncapacitated instances.

In Table 4.1 we report on some computational experience on uncapacitated instances. In order to compare our results with previous experiments, we generate the data as in Table 4 of [1]: demands, production costs and stock upper bounds are randomly generated integers in the following ranges: $d_t \in [0, 30]$, $p_t \in [4, 24]$, $u_t \in [30, 30(c+1)]$, where c is a parameter satisfying $c \in \{2, 5, 10, 20\}$. Furthermore, in all periods $h_t = 1$, $q_t = 10$ and $f_t = f \in \{1000, 2000, 5000\}$. The number of periods n is 180.

For each choice of the parameters f and c, we generated four instances. We first ran the solver (with its default settings) using the initial formulation (49)–(52) with a time limit of five minutes. In the left part of Table 4.1 we report the ratio of the best integer solution found to the optimal solution (in percentage), the ratio of the best lower bound to the optimal solution, and the relative gap between best solution and best bound (all these values are averages over the four instances). We remark that none of the instances was solved to optimality using the initial formulation. We then added the extended formulation of $\text{conv}(M^k)$ given in Theorem 2.2 for $k = 1, \ldots, n$. However, to avoid an excessive increase in the size of the formulation, for each k the set M^k was truncated to the first 15 periods following period k (the value 15 was chosen after some preliminary tests). Using these extended formulations, all the instances were solved to optimality in a few seconds, as reported in the right part of the table. For instances randomly generated in the way described above, the average solution time reported in [1] is over 400 seconds. Thus it appears that the problem can be solved effectively by considering just the Wagner-Whitin relaxation of (49)–(52) and the default cutting planes of the mixed-integer programming solver.

We also tried to solve the same instances using the extended formulations of both $\text{conv}(M^k)$ and $\text{conv}(K^k)$, which together give an extended formulation of $\text{conv}(X^k)$. However this results in an increase in the size of the formulation that affects negatively the overall performance of the solver.

In Table 4.2 we report our results on capacitated instances. The data is exactly as in the uncapacitated instances described above, except that the number of periods is 120 and there is a capacity of $C = 50$. As above, first we ran the solver for five minutes using only the initial formulation (49)–(52) (no instance was solved to optimality), and then we added the extended formulations of the sets $\text{conv}(M^k)$ for $k = 1, \ldots, n$. In this case each set M^k was truncated to the first 20 periods following period k. Even though the performance was not as good as in the uncapacitated case, most of the instances could be solved to optimality within the time limit of five minutes (in the last column we report the number of instances not solved within the time limit). Furthermore, the relative gap for the unsolved instances is always smaller than that obtained by running the solver using the initial formulation.

5. Concluding remarks. As pointed out above, the set (50)–(52) of solutions of the constant-capacity lot-sizing problem with stock variable upper bound and fixed charge $LS-CC-SVUB$ is a subset of $\bigcap_{k=1}^{n} X^k$, which is the Wagner-Whitin relaxation. Based on the structures of several other lot-sizing variants, it was an initial conjecture that $\text{conv}\left(\bigcap_{k=1}^{n} X^k\right) = \bigcap_{k=1}^{n} \text{conv}\left(X^k\right)$.

		Initial formulation			M^k		
f	c	best sol.	best bd	gap	time	gap	unslv
1000	2	100.2	99.6	0.7	11	0.0	0
	5	100.6	99.0	1.6	66	0.0	0
	10	100.9	98.7	2.1	29	0.0	0
	20	100.6	98.8	1.7	147	0.0	1
2000	2	101.6	98.5	3.0	27	0.0	0
	5	101.9	97.8	4.0	144	0.2	1
	10	102.2	97.7	4.3	177	0.0	0
	20	102.6	97.6	4.9	186	0.3	2
5000	2	102.1	97.6	4.4	52	0.0	0
	5	102.8	96.9	5.7	300	0.8	4
	10	103.0	95.8	7.0	300	2.2	4
	20	103.1	96.3	6.6	300	2.3	4

TABLE 4.2
Capacitated instances.

However this is not the case, so finding a tight compact extended formulation for conv $\left(\bigcap_{k=1}^{n} X^k\right)$ that is small enough to be computationally effective is still an open question. For the set (50)–(52) it follows from the description of the problem $LS-CC-SUB$ in Wolsey [13] that the problem $LS-CC-SVUB$ is in $\mathcal{P}$. However it is not clear whether the corresponding $\mathcal{O}(n^3)\times\mathcal{O}(n^3)$ extended formulation can be generalized to include the stock variable upper bound constraint, and in any case this formulation is in general too large to be practically useful.

Finally, as recalled above, Atamtürk and Küçükyavuz [1] have given two classes of valid inequalities for the uncapacitated version of the problem. It would be interesting to understand whether similar inequalities can be derived that exploit the capacity bound on production.

Appendix. Proof of Lemma 3.4.

Here we prove Lemma 3.4. To do so, we need two preliminary results.

LEMMA A.1. *Let $\bar{x}$ be an extreme ray of cone* (41)–(44) *with $\bar{\sigma}=1$, and suppose that $\mathcal{I}(\bar{x})$ is non-redundant in the description of* conv(X). *Then we can assume that:*

(i) $\bar{u}'_1 = 0$;

(ii) $\bar{\rho}'_t\bar{u}'_{t+1} = 0$ *for* $1 \le t \le n-1$;

(iii) $\bar{\rho}'_t\bar{v}'_t = 0$ *for* $2 \le t \le n$.

Proof. The proof is an adaptation of that of Lemma 2.4 (i)–(iii). To show that (i) holds, assume $\varepsilon := \bar{u}'_1 > 0$. Then $\bar{\rho}'_1 = 0$, otherwise $\bar{x}$ would not define an acyclic flow. We construct a vector $\hat{x}$ by setting $\hat{u}'_1 = 0$, $\hat{\rho}'_1 = \varepsilon$ and all other components equal to the corresponding entries of $\bar{x}$. Note that $\hat{x}$ defines an acyclic flow with $\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x})$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = \varepsilon\lceil b_1\rceil(1-w) \ge 0$. Thus $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and inequality $w \le 1$. The proofs of (ii)–(iii) can be obtained by a similar adaptation of the proof of Lemma 2.4. ∎

LEMMA A.2. *Let $\bar{x}$ be an extreme ray of cone* (41)–(44) *with $\bar{\sigma}=1$, and suppose that $\mathcal{I}(\bar{x})$ is non-redundant in the description of* conv(X). *Then we can assume that:*

(i) $\bar{\gamma}_t\bar{v}''_t = 0$ *for* $2 \le t \le n$;

(ii) $\bar{\gamma}_t\bar{u}''_{t+1} = 0$ *for* $1 \le t \le n-1$.

Proof. Let $\bar{x}$ be as in the above statement.

(i) Assume that both $\bar{\gamma}_t$ and $\bar{v}''_t$ are positive for some $2 \le t \le n$. Define $\varepsilon = \min\{\bar{\gamma}_t, \bar{v}''_t\}$ and let L be the set of indices $\ell \in \{0,\dots,m\}$ such that arc r_ℓ belongs to the unique directed path from node $\nu_{\varphi(t-1)}$ to node $\nu_{\varphi(t)}$. We define a vector $\hat{x}$ by setting $\hat{\gamma}_t = \bar{\gamma}_t - \varepsilon$, $\hat{v}''_t = \bar{v}''_t - \varepsilon$, $\hat{\gamma}_{t-1} = \bar{\gamma}_{t-1} + \varepsilon$, $\hat{r}_\ell = \bar{r}_\ell + \varepsilon$ for $\ell \in L$, and all other entries equal to the corresponding components of $\bar{x}$. Note that $\hat{x}$ defines a feasible flow in the network, and $\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x})$. We now show that $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) \ge 0$ whenever $w \ge 0$, thus proving that $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and the inequality $w \ge 0$. Since at most one of $\hat{\gamma}_t$ and $\hat{v}''_t$ is positive, this will complete the proof.

1. Suppose first that $t = 1$. Then $L = \{1,\dots,\varphi(t)\}$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = \varepsilon(1-\lceil b_1\rceil)w$, which is nonnegative if $w \ge 0$.

2. Now suppose that $t \ge 2$ and $\varphi(t-1) \le \varphi(t)$. Then $L = \{\varphi(t-1)+1,\dots,\varphi(t)\}$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = \varepsilon(\lceil b_{t-1}\rceil + 1 - \lceil b_t\rceil)w$, which is nonnegative if $w \ge 0$.

3. Finally assume that $t \geq 2$ and $\varphi(t-1) > \varphi(t)$. Note that this implies $\lceil b_{t-1} \rceil = \lceil b_t \rceil$. In this case $L = \{0, \ldots, \varphi(t)\} \cup \{\varphi(t-1)+1, \ldots, m\}$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = \varepsilon(\lceil b_{t-1} \rceil - \lceil b_t \rceil)w = 0$.

(ii) Assume that both $\bar{\gamma}_t$ and $\bar{u}''_{t+1}$ are positive for some $1 \leq t \leq n-1$. Define $\varepsilon = \min\{\bar{\gamma}_t, \bar{u}''_{t+1}\}$ and let L be the set of indices $\ell \in \{0, \ldots, m\}$ such that arc r_ℓ belongs to the unique directed path from node $\nu_{\varphi(t+1)}$ to node $\nu_{\varphi(t)}$. We define a vector $\hat{x}$ by setting $\hat{\gamma}_t = \bar{\gamma}_t - \varepsilon$, $\hat{u}''_{t+1} = \bar{u}''_{t+1} - \varepsilon$, $\hat{\gamma}_{t+1} = \bar{\gamma}_{t+1} + \varepsilon$, $\hat{r}_\ell = \bar{r}_\ell + \varepsilon$ for $\ell \in L$, and all other entries equal to the corresponding components of $\bar{x}$. Note that $\hat{x}$ defines a feasible flow in the network, and $\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x})$. We now show that $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) \geq 0$ whenever $w \geq 0$, thus proving that $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and the inequality $w \geq 0$. Since at most one of $\hat{\gamma}_t$ and $\hat{u}''_{t+1}$ is positive, this will complete the proof.

1. Suppose first that $\varphi(t+1) \leq \varphi(t)$. Then $L = \{\varphi(t+1)+1, \ldots, \varphi(t)\}$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = \varepsilon(\lceil b_{t+1} \rceil - \lceil b_t \rceil)w$, which is nonnegative if $w \geq 0$.

2. Now assume $\varphi(t+1) > \varphi(t)$. Note that this implies $\lceil b_{t+1} \rceil = \lceil b_t \rceil + 1$. In this case $L = \{0, \ldots, \varphi(t)\} \cup \{\varphi(t+1)+1, \ldots, m\}$ and $\mathrm{rhs}(\hat{x}) - \mathrm{rhs}(\bar{x}) = \varepsilon(\lceil b_{t+1} \rceil - \lceil b_t \rceil - 1)w = 0$. □

We can now prove Lemma 3.4.

Proof of Lemma 3.4. For any feasible ray x of cone (41)–(44), define $T(x) = \{t : \gamma_t \neq \rho'_t\}$. If $\bar{x}$ is an extreme ray of cone (41)–(44) with $\bar{\sigma} = 1$, then it satisfies all the conditions listed in Lemmas A.1–A.2. In fact, to prove the lemma we only need to assume that $\bar{x}$ satisfies those conditions. Specifically, we show that if $\bar{x}$ is a feasible flow satisfying the conditions listed in Lemmas A.1–A.2 such that $T(\bar{x}) \neq \varnothing$, then $\mathcal{I}(\bar{x})$ is implied by the inequalities $0 \leq w \leq 1$ and an inequality $\mathcal{I}(\hat{x})$, where $\hat{x}$ is a feasible flow satisfying the properties listed in Lemmas A.1–A.2 along with the condition $|T(\hat{x})| = |T(\bar{x})| - 1$. The claim will follow.

Let $j = \min\{t : t \in T(\bar{x})\}$. We first assume that $\bar{\rho}'_j > \bar{\gamma}_j$. Then, since $\bar{\rho}'_t = \bar{\gamma}_t$ for $t < j$ and since $\bar{u}''_1 = \bar{v}'_1 = 0$ (by Lemma 3.3), the support of the flow must contain a directed path P from arc ρ'_j to an arc γ_h for some $h > j$. By Lemma A.1 (ii) and Lemma A.2 (i), neither arc u'_{j+1} nor arc v''_h belongs to P. Then P is associated with a sequence of indices $j = t_1 < t_2 < \cdots < t_k = h$ with k odd and consists of the following arcs:

- ρ'_{t_1} and γ_{t_k};
- v''_t for $t_i < t \leq t_{i+1}$, $1 \leq i \leq k-1$, i odd;
- u'_t for $t_i < t \leq t_{i+1}$, $1 \leq i \leq k-1$, i even.

Define $\varepsilon = \bar{\gamma}_{t_k} = \bar{\gamma}_h$. Since the flow is acyclic and $\bar{\rho}'_{t_1} > 0$, none of the arcs ρ'_t for $t_1 < t \leq t_k$ carries a positive flow. It follows that every arc of P carries a flow of value at least ε. We now show that inequality $\mathcal{I}(\bar{x})$ is not needed in the description of $\mathrm{conv}(X)$.

Assume first that $\lceil b_{t_k} \rceil - \lceil b_{t_1} \rceil - t_{k-1} + t_{k-2} - \cdots + t_1 \geq 0$. Starting from $\bar{x}$, we construct a feasible flow $\hat{x}$ by decreasing the flow on the arcs in $P \setminus \{\gamma_{t_k}\}$ by ε and increasing the flow on arc ρ'_{t_k} by ε. Note that

$$\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x}) + \varepsilon(z_{t_k} - z_{t_1}) - \varepsilon \sum_{i \text{ even}} (z_{t_{i+1}} - z_{t_i}) = \mathrm{lhs}(\bar{x}) + \varepsilon \sum_{i \text{ even}} (z_{t_i} - z_{t_{i-1}}),$$

and, after some manipulation,

$$\mathrm{rhs}(\hat{x}) = \mathrm{rhs}(\bar{x}) + \varepsilon(\lceil b_{t_k} \rceil - \lceil b_{t_1} \rceil)(1-w) + \varepsilon \sum_{i>0 \text{ even}} (t_i - t_{i-1})w.$$

Consider the inequality obtained by summing up $\mathcal{I}(\hat{x})$ and inequalities $-\varepsilon(z_t - z_{t-1}) \geq -\varepsilon$ for $t_{i-1} < t \leq t_i$, $1 \leq i \leq k-1$, i even. The left-hand side of this inequality is precisely $\mathrm{lhs}(\bar{x})$, while the right-hand side is $\mathrm{rhs}(\bar{x}) + \varepsilon(\lceil b_{t_k} \rceil - \lceil b_{t_1} \rceil - t_{k-1} + t_{k-2} - \cdots + t_1)(1-w)$. Then $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and the original inequalities defining X. It remains to observe that $|T(\hat{x})| = |T(\bar{x})| - 1$, as $\hat{\rho}'_{t_k} = \hat{\gamma}_{t_k}$ while $\bar{\rho}'_{t_k} = 0 < \bar{\gamma}_{t_k}$.

Now assume that $\lceil b_{t_k} \rceil - \lceil b_{t_1} \rceil - t_{k-1} + t_{k-2} - \cdots + t_1 \leq -1$. Let Q be the unique directed path from arc γ_{t_1} to node $\varphi(t_k)$. Starting from $\bar{x}$, we construct a feasible flow $\hat{x}$ by decreasing the flow on the arcs in $P \setminus \{\rho'_{t_1}\}$ by ε and increasing the flow along Q by ε. Note that

$$\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x}) - \varepsilon \sum_{i \text{ even}} (z_{t_{i+1}} - z_{t_i}) \quad \text{and}$$

$$\mathrm{rhs}(\hat{x}) = \mathrm{rhs}(\bar{x}) - \varepsilon(\lceil b_{t_k}\rceil - \lceil b_{t_1}\rceil)w + \varepsilon \sum_{i \text{ even}} (t_i - t_{i-1})w - \delta w,$$

where $\delta = 1$ if arc r_0 belongs to Q and $\delta = 0$ otherwise. Consider the inequality obtained by summing up $\mathcal{I}(\hat{x})$ and inequalities $\varepsilon(z_t - z_{t-1}) \geq 0$ for $t_i < t \leq t_{i+1}$, $1 \leq i \leq k-1$, i even. The left-hand side of this inequality is precisely $\mathrm{lhs}(\bar{x})$, while the right-hand side is $\mathrm{rhs}(\bar{x}) - \varepsilon(\lceil b_{t_k}\rceil - \lceil b_{t_1}\rceil - t_{k-1} + t_{k-2} - \cdots + t_1 + \delta)w$. Then $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and the original inequalities defining X. It remains to observe that $|T(\hat{x})| = |T(\bar{x})| - 1$, as $\hat{\rho}'_{t_1} = \hat{\gamma}_{t_1}$ while $\bar{\rho}'_{t_1} > \bar{\gamma}_{t_1}$.

We now suppose that $\bar{\rho}'_j < \bar{\gamma}_j$. Then, since $\bar{\rho}'_t = \bar{\gamma}_t$ for $t < j$ and since $\bar{u}'_1 = 0$ (by Lemma A.1), the support of the flow must contain a directed path P either from an arc ρ'_h with $h > j$ to arc γ_j, or from arc v''_1 to arc γ_j. In the former case, the proof is similar to that carried out above. Therefore we only consider the case when P is a path from arc v''_1 to arc γ_j.

By Lemma A.2 (i), arc v''_j does not belong to P. Then P is associated with a sequence of indices $0 = t_0 < t_1 < t_2 < \cdots < t_k = h$ with k even and consists of the following arcs:

- γ_{t_k};
- v''_t for $t_i < t \leq t_{i+1}$, $0 \leq i \leq k-1$, i even;
- u'_t for $t_i < t \leq t_{i+1}$, $0 \leq i \leq k-1$, i odd.

Define $\varepsilon = \bar{\gamma}_{t_k} = \bar{\gamma}_h$. Since $\bar{\rho}'_t = \bar{\gamma}_t$ for $t < j$, every arc of P carries a flow of value at least ε. We now show that the inequality $\mathcal{I}(\bar{x})$ is not needed in the description of $\mathrm{conv}(X)$.

Assume first that $\lceil b_{t_k}\rceil - t_{k-1} + t_{k-2} - \cdots + t_0 \geq 0$. Starting from $\bar{x}$, we construct a feasible flow $\hat{x}$ by decreasing the flow on the arcs in $P \setminus \{\gamma_{t_k}\}$ by ε and increasing the flow on arc ρ'_{t_k} by ε. Note that

$$\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x}) + \varepsilon \sum_{i \text{ odd}} (z_{t_i} - z_{t_{i-1}}) \quad \text{and}$$

$$\mathrm{rhs}(\hat{x}) = \mathrm{rhs}(\bar{x}) + \varepsilon \lceil b_{t_k}\rceil (1-w) + \varepsilon \sum_{i \text{ odd}} (t_i - t_{i-1})w.$$

Consider the inequality obtained by summing up $\mathcal{I}(\hat{x})$ and inequalities $-\varepsilon(z_t - z_{t-1}) \geq -\varepsilon$ for $t_{i-1} < t \leq t_i$, $0 \leq i \leq k-1$, i odd. The left-hand side of this inequality is precisely $\mathrm{lhs}(\bar{x})$, while the right-hand side is $\mathrm{rhs}(\bar{x}) + \varepsilon(\lceil b_{t_k}\rceil - t_{k-1} + t_{k-2} - \cdots + t_0)(1-w)$. Then $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and the original inequalities defining X. It remains to observe that $|T(\hat{x})| = |T(\bar{x})| - 1$, as $\hat{\rho}'_{t_k} = \hat{\gamma}_{t_k}$ while $\bar{\rho}'_{t_k} = 0 < \bar{\gamma}_{t_k}$.

Now assume that $\lceil b_{t_k}\rceil - \lceil b_{t_1}\rceil - t_{k-1} + t_{k-2} - \cdots + t_0 \leq -1$. Let Q be the unique directed path from arc γ_0 to node $\varphi(t_k)$. Starting from $\bar{x}$, we construct a feasible flow $\hat{x}$ by decreasing the flow on the arcs in P by ε and increasing the flow along Q by ε. Note that

$$\mathrm{lhs}(\hat{x}) = \mathrm{lhs}(\bar{x}) - \varepsilon \sum_{i \text{ odd}} (z_{t_{i+1}} - z_{t_i}) \quad \text{and}$$

$$\mathrm{rhs}(\hat{x}) = \mathrm{rhs}(\bar{x}) - \varepsilon \lceil b_{t_k}\rceil w + \varepsilon \sum_{i \text{ odd}} (t_i - t_{i-1})w.$$

Consider the inequality obtained by summing up $\mathcal{I}(\hat{x})$ and inequalities $\varepsilon(z_t - z_{t-1}) \geq 0$ for $t_i < t \leq t_{i+1}$, $0 \leq i \leq k-1$, i odd. The left-hand side of this inequality is precisely $\mathrm{lhs}(\bar{x})$, while the right-hand side is $\mathrm{rhs}(\bar{x}) - \varepsilon(\lceil b_{t_k}\rceil - t_{k-1} + t_{k-2} - \cdots + t_0)w$. Then $\mathcal{I}(\bar{x})$ is implied by $\mathcal{I}(\hat{x})$ and the original inequalities defining X. It remains to observe that $|T(\hat{x})| = |T(\bar{x})| - 1$, as $\hat{\rho}'_{t_k} = \hat{\gamma}_{t_k} = 0$ while $\bar{\rho}'_{t_k} = 0 < \bar{\gamma}_{t_k}$. □

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