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FORMULATIONS OF MIXED-INTEGER SETS DEFINED BY TOTALLY UNIMODULAR CONSTRAINT MATRICES

Direttore della Scuola: Ch.mo Prof. Bruno Chiarellotto
Supervisore: Ch.mo Prof. Michele Conforti

Dottorando: Marco Di Summa

DATA CONSEGNA TESI
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To my parents

Abstract

A *mixed-integer program* is an optimization problem where one is required to minimize (or maximize) a linear function over a subset of $\mathbb{R}^n$ defined by a system of linear inequalities, with the additional restriction that some of the variables must take an integer value. Mixed-integer programming is a fundamental area of operations research, as many real-world problems can be formulated as mixed-integer programs.

Solving mixed-integer programs is difficult in general. A common approach to tackle this kind of problems exploits the fact that (under mild assumptions) the convex hull of feasible solutions is a polyhedron, i.e. a subset of $\mathbb{R}^n$ defined by a system of linear inequalities. When the inequalities describing such a polyhedron are known explicitly, the mixed-integer program reduces to a linear program, which is a tractable problem. Unfortunately it is usually very hard to find a linear inequality description of the convex hull of feasible solutions of a mixed-integer program. However in some cases the introduction of additional variables allows one to give a simple description of such a convex hull by means of linear inequalities in a higher dimensional space. Such a description is called an *extended formulation*. If an extended formulation is known that is *compact* (i.e. it uses a polynomial number of variables and constraints), the original mixed-integer programming problem can be solved in polynomial time by means of linear programming algorithms.

In this dissertation we study the family of mixed-integer programs whose feasible regions are defined by linear systems with *totally unimodular* matrices (i.e. all subdeterminants are 0, 1 or -1) having at most two nonzero entries per row. This class of problems is interesting because many instances arising e.g. in the context of production planning can be formulated as mixed-integer programs of this type.

We illustrate a technique to construct an extended formulation for any problem in this family. The approach is based on the enumeration of all possible fractional parts that the variables take at the vertices of the convex hull of the feasible region. The explicit knowledge of such values allows us to model the problem as a *pure* integer program (i.e. all variables are prescribed to take an integer value) by means of additional variables. For such a pure integer reformulation the convex hull can be obtained very easily and thus an extended formulation for the original problem is derived.

We then discuss the compactness of our extended formulation: we give sufficient conditions ensuring that the formulation is compact. When one of these conditions holds, the mixed-integer program can be solved in polynomial time. We also show how our technique can be successfully applied to some interesting practical problems.

Next we consider the possibility of describing the convex hull of the feasible region in the original space of definition of the problem (i.e with no additional variables). Such a formulation is found explicitly for some special cases by using e.g. flow techniques or linear programming duality.

Finally a possible extension is discussed: we show how a generalization of our technique can lead to a compact extended formulation for a problem that does not belong to the family introduced above.

Most of the results presented in this thesis are joint work with Michele Conforti, Friedrich Eisenbrand and Laurence A. Wolsey.

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Sommario (Italian abstract)

Un *programma intero misto* è un problema di ottimizzazione in cui si richiede di minimizzare (o massimizzare) una funzione lineare su un sottoinsieme di $\mathbb{R}^n$ definito da un sistema di disequazioni lineari, con la condizione aggiuntiva che alcune delle variabili devono assumere un valore intero. La programmazione intera mista è un'area molto importante della ricerca operativa, poiché numerosi problemi di interesse pratico possono essere formulati come programmi interi misti.

Risolvere un programma intero misto è in generale difficile. Un approccio comunemente usato per affrontare problemi di questo tipo sfrutta il fatto che (sotto deboli ipotesi) l'inviluppo convesso delle soluzioni ammissibili è un poliedro, cioè un sottoinsieme di $\mathbb{R}^n$ definito da un sistema di disequazioni lineari. Quando le disequazioni che descrivono tale poliedro sono note esplicitamente, il programma intero misto può essere ricondotto ad un programma lineare, che è un problema trattabile. Purtroppo è generalmente molto complicato trovare una descrizione in termini di disequazioni lineari dell'inviluppo convesso delle soluzioni ammissibili di un programma intero misto. Tuttavia in certi casi l'introduzione di variabili aggiuntive permette di dare una semplice descrizione di questo inviluppo convesso tramite disequazioni lineari in uno spazio di dimensione superiore. Una tale descrizione è detta *formulazione estesa*. Se si conosce una formulazione estesa *compatta* (che usi cioè un numero polinomiale di variabili e vincoli), il programma intero misto iniziale può essere risolto in tempo polinomiale per mezzo di algoritmi per la programmazione lineare.

In questa tesi studieremo la famiglia di programmi interi misti le cui regioni ammissibili sono definite da sistemi lineari con matrici *totalmente unimodulari* (cioè tutti i sottodeterminanti valgono 0, 1 o -1) contenenti al massimo due elementi non nulli per riga. Questa famiglia è interessante perché molti problemi pratici (ad esempio nel campo della programmazione della produzione) possono essere formulati come programmi interi misti di questo tipo.

Illustreremo una tecnica che permette di costruire una formulazione estesa per un qualunque problema nella famiglia definita sopra. L'approccio che useremo si basa sull'enumerazione di tutte le parti frazionarie che le variabili assumono nei vertici dell'inviluppo convesso della regione ammissibile. La conoscenza esplicita di questi valori ci permetterà di modellare il problema come un programma intero *puro* (dove, cioè, tutte le variabili devono assumere un valore intero) per mezzo di variabili aggiuntive. Per tale riformulazione l'inviluppo convesso potrà essere ottenuto facilmente e deriveremo quindi una formulazione estesa per il problema iniziale.

Discuteremo poi la compattezza della nostra formulazione estesa: daremo condizioni sufficienti sotto le quali la formulazione è compatta. Quando una di queste condizioni è soddisfatta, il programma intero misto può essere risolto in tempo polinomiale. Mostreremo anche come la nostra tecnica possa essere applicata con successo ad alcuni problemi di interesse pratico.

In seguito analizzeremo la possibilità di descrivere l'inviluppo convesso della regione ammissibile nello spazio originale di definizione del problema (cioè senza l'introduzione di variabili aggiuntive). Per alcuni casi speciali riusciremo a trovare esplicitamente una tale formulazione usando ad esempio tecniche di flusso o la dualità della programmazione lineare.

Infine discuteremo una possibile estensione: mostreremo come una generalizzazione della nostra tecnica possa essere usata per trovare una formulazione estesa compatta per un problema che non appartiene alla famiglia introdotta sopra.

Gran parte dei risultati presentati in questa tesi sono stati ottenuti in collaborazione con Michele Conforti, Friedrich Eisenbrand e Laurence A. Wolsey.

Contents

Chapter 1

Introduction

A *mixed-integer linear program* (or simply *mixed-integer program*) is an optimization problem where one is required to minimize (or maximize) a linear function over a subset of $\mathbb{R}^n$ defined by a system of linear inequalities, with the additional restriction that some of the variables must take an integer value. Any mixed-integer program can then be formulated as

$$\min \quad cx \tag{1.1}$$

$$\text{subject to} \quad Ax \geq b, \tag{1.2}$$

$$x_i \text{ integer for } i \in I, \tag{1.3}$$

where A is an $m \times n$ matrix, b is a column vector in $\mathbb{R}^m$, c is a row vector in $\mathbb{R}^n$ and I is a nonempty subset of $\{1, \dots, n\}$. In the above problem, cx is the *objective function*, while the set defined by conditions (1.2)–(1.3) is the *feasible region*. Variables x_i for $i \in I$ are called the *integer variables*, while x_i for $i \notin I$ are the *continuous variables*. A subset of $\mathbb{R}^n$ that is the feasible region of a mixed-integer program is called a *mixed-integer set*.

When $I = \{1, \dots, n\}$, problem (1.1)–(1.3) is a *pure integer program* (or simply *integer program*). Thus we view integer programs as special types of mixed-integer programs. A problem of the form (1.1)–(1.2), with no integrality restrictions, is a *linear program*.

Linear and mixed-integer programming are fundamental areas of operations research. A large number of real-world problems can be formulated as linear or mixed-integer programs, such as problems arising in transportation, manufacturing, scheduling and many other fields (see e.g. [33, 49, 55]).

While linear programming is a tractable problem, mixed-integer programming is difficult in general, as the region defined by conditions (1.2)–(1.3) is usually very complicated to describe. In some special cases, the introduction of new variables in the problem allows one to give a simpler description of a mixed-integer set. A description of this type, which is given in a higher dimensional space, is called an *extended formulation* of the set (a more precise definition is given in Section 1.4).

In this work we study mixed-integer sets (1.2)–(1.3) whose constraint matrix A has some special structure that we will specify later. We present and discuss a technique that allows one to construct extended formulations for an arbitrary set having such a structure, and we also explore the possibility of describing the set in its original space of definition. Furthermore, possible extensions to other sets are discussed.

Before giving a more detailed outline of the thesis, we need to introduce some general concepts and known results that will be used throughout. Specifically, in Section 1.1 some useful facts about polyhedra are recalled. In Sections 1.2–1.3 we briefly discuss linear programming, integer programming and mixed-integer programming. In Section 1.4 we introduce the notion of extended formulation, which is a key concept of this work, and in particular we focus on the importance of extended formulations in mixed-integer programming. Some of the most well-known approaches to constructing extended formulations of a mixed-integer set are surveyed in Section 1.5. Finally, an outline of the contents of this dissertation is given in Section 1.6.

1.1 Polyhedra

This section briefly surveys the main definitions and results about polyhedra. A complete presentation of polyhedral theory, as well as the proofs of the theorems that are recalled here, can be found in [49] or [58].

We start with some well-known definitions about convexity.

Given a subset X of $\mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is a *convex combination* of the points in X if $x = \sum_{i=1}^p \delta_i x^i$ for some choice of an integer $p \geq 1$ and real numbers $\delta_1, \dots, \delta_p \geq 0$ satisfying $\sum_{i=1}^p \delta_i = 1$. A set is *convex* if it contains all convex combinations of its points.

The *convex hull* of X, denoted $\text{conv}(X)$, is the smallest convex set containing X: it consists of all possible convex combinations of its points.

A *polyhedron* is the intersection of a finite number of half-spaces. This definition immediately implies that every polyhedron is a convex set.

We discuss below two fundamental ways of describing a polyhedron. We then conclude the section by presenting a classical result of Balas.

1.1.1 External description of a polyhedron

Since a polyhedron is the intersection of a finite number of half-spaces, it follows that a polyhedron in $\mathbb{R}^n$ can be described as the set of points $x \in \mathbb{R}^n$ satisfying a linear system of inequalities $Ax \geq b$, where A is an $m \times n$ matrix and b is an m-vector: this is called an *external description* of the polyhedron.

When an external description of a polyhedron is given, some of the inequalities of the system $Ax \geq b$ may be redundant, that is, their removal do not modify the set of solutions to the system. We say that an external description of a polyhedron is *minimal* if it does not contain any redundant inequalities. We illustrate below a fundamental result of polyhedral theory concerning the number of inequalities in an external description of a polyhedron, but before doing this, some standard terminology has to be recalled.

Let P be a polyhedron in $\mathbb{R}^n$. Given an inequality $cx \geq \delta$ which is satisfied by all points in P, the set of points $F := \{x \in P : cx = \delta\}$ is called a *face* of P.[1] We then say that inequality

[1] Some authors require F to be nonempty.

$cx \geq \delta$ *induces* or *defines* face F. A face of P is a *proper face* if it is nonempty and does not coincide with the whole polyhedron P. A *facet* of P is a proper face of P which is not contained in any other proper face of P.

Let $\text{aff}(P)$ be the affine hull of P, i.e. the smallest affine variety containing P. The dimension of P, denoted $\dim(P)$, is the dimension of $\text{aff}(P)$ as an affine variety. P is *full-dimensional* if $\text{aff}(P) = \mathbb{R}^n$.

To state the next result, we assume that an external description of P is given as a system of linear inequalities and equations $Ax \geq b$, $A'x = b'$, where the system $Ax \geq b$ does not contain any pair of inequalities of the type $ax \geq \beta$, $-ax \geq -\beta$ (such a pair could be replaced with equation $ax = \beta$).

Theorem 1.1 *Let $Ax \geq b$, $A'x = b'$ be a minimal external description of P, where the system $Ax \geq b$ does not contain any pair of inequalities of the type $ax \geq \beta$, $-ax \geq -\beta$. Then:*

(i) *$A'x = b'$ consists of $n - \dim(P)$ linearly independent equations such that $\text{aff}(P) = \{x \in \mathbb{R}^n : A'x = b'\}$;*

(ii) *each inequality in the system $Ax \geq b$ induces a distinct facet of P and each facet of P is induced by a distinct inequality of the system $Ax \geq b$.*

The above theorem shows that all minimal external descriptions of a given polyhedron use the same number of equations and inequalities.

1.1.2 Internal description of a polyhedron

Given a polyhedron $P \subseteq \mathbb{R}^n$, a nonempty face F of P is *minimal* if no proper face of P is strictly contained in F. It can be proven that all minimal faces of P are affine varieties of the same dimension.

When the minimal faces of a polyhedron P consist of single points, they are called *vertices* or *extreme points* of P. In this case P is called a *pointed* polyhedron. An equivalent definition of vertex can be given: a point $v \in P$ is a vertex of P if and only if there do not exist $x^1, x^2 \in P \setminus \{v\}$ such that $v = \frac{1}{2}x^1 + \frac{1}{2}x^2$. Every polyhedron has only a finite number of vertices. However, such a number may be exponential in the number of variables and inequalities used to give an external description of the polyhedron.

A *ray* of a nonempty polyhedron P is a vector $r \in \mathbb{R}^n$ such that $x + r \in P$ for all $x \in P$. If there do not exist two rays r^1, r^2 of P such that $r = \frac{1}{2}r^1 + \frac{1}{2}r^2$ and $r^1 \neq \lambda r^2$ for all $\lambda \geq 0$, then r is called an *extreme ray* of P. It can be proven that P has an extreme ray if and only if it is a pointed polyhedron. Also, every polyhedron has only a finite number of extreme rays. Similarly to extreme points, such a number might be exponentially large.

The set of rays of P form a *convex cone* $\mathcal{C}(P)$, i.e. $\mathcal{C}(P)$ is nonempty (as $\mathbf{0}$, the all-zero vector, is a ray of P) and $\lambda_1 r^1 + \lambda_2 r^2 \in \mathcal{C}(P)$ for all $r^1, r^2 \in \mathcal{C}(P)$ and $\lambda_1, \lambda_2 \geq 0$. $\mathcal{C}(P)$ is called the *recession cone* (or *characteristic cone*) of P. If $P = \varnothing$, the standard definition is $\mathcal{C}(P) := \{\mathbf{0}\}$. It can be proven that $\mathcal{C}(P)$ is a polyhedron: if P is defined by the linear system $Ax \geq b$, then $\mathcal{C}(P)$ is defined by $Ax \geq \mathbf{0}$. It is easy to see that every system of the form

$Ax \geq \mathbf{0}$ defines a cone, which is therefore called a *polyhedral cone*. A polyhedral cone has either a unique vertex (called *apex*) or no vertices at all. In the former case, the apex is $\mathbf{0}$.

The following theorem summarizes fundamental results that are due to Minkowski [47], Motzkin [48] and Weyl [67]:

Theorem 1.2 (Minkowski-Weyl theorem) *A subset P of $\mathbb{R}^n$ is a polyhedron if and only if there exist a finite set of points $\{v^1, \ldots, v^p\}$ and a finite set of vectors $\{r^1, \ldots, r^q\}$ such that*

$$P = \Big\{x \in \mathbb{R}^n : \quad x = \textstyle\sum_{i=1}^p \delta_i v^i + \sum_{j=1}^q \lambda_j r^j, \\ \textstyle\sum_{i=1}^p \delta_i = 1,\ \delta_i \geq 0, \qquad 1 \leq i \leq p, \\ \lambda_j \geq 0, \qquad 1 \leq j \leq q \Big\}.$$

Furthermore, if P is a pointed polyhedron then $\{v^1, \ldots, v^p\}$ can be chosen as the set of extreme points of P and $\{r^1, \ldots, r^q\}$ as the set of extreme rays of P.

A description of a polyhedron P as in the above theorem is called an *internal description* of P. We say that P is *generated* by points $v^1, \ldots, v^p$ and rays $r^1, \ldots, r^q$. Every pointed polyhedron is generated by its extreme points and extreme rays.

Note that the description of P given by Theorem 1.2 uses additional variables $\delta_1, \ldots, \delta_p$ and $\lambda_1, \ldots, \lambda_q$. This is an example of extended formulation, a concept that will be discussed in Section 1.4.

1.1.3 Union of polyhedra

We conclude this section by presenting a result due to Balas [4], which can be viewed as an extension of Minkowski-Weyl theorem.

Suppose that we know the external descriptions of k polyhedra $P_1, \ldots, P_k$ in $\mathbb{R}^n$ and we are interested in finding a description of the convex hull of $P_1 \cup \cdots \cup P_k$. The result below provides such a description.

Theorem 1.3 *For $1 \leq i \leq k$, let $P_i := \{x \in \mathbb{R}^n : A^i x \geq b^i\}$ be polyhedra in $\mathbb{R}^n$ having the same recession cone. Then the set $P := \operatorname{conv}(P_1 \cup \cdots \cup P_k)$ is a polyhedron and*

$$P = \Big\{x \in \mathbb{R}^n : \quad x = \textstyle\sum_{i=1}^k w^i, \\ A^i w^i \geq b^i \delta_i, \qquad 1 \leq i \leq k, \\ \textstyle\sum_{i=1}^k \delta_i = 1,\ \delta_i \geq 0, \quad 1 \leq i \leq k \Big\}.$$

This version of the theorem is not the most general one (see [4, 18]), but is sufficient for our purpose.

We remark that if P is a bounded polyhedron, then Theorem 1.2 can be obtained by applying the above result to the polyhedra $P_i := \{v^i\}$ for $1 \leq i \leq p$. (One could write a variant of Theorem 1.3 that subsumes the Minkowski-Weil theorem for unbounded polyhedra too.) Also, if $k = 1$ the description given above is essentially the original external description of the polyhedron $P_1 = P$. Therefore Theorem 1.3 provides in a sense an "intermediate" formulation of a polyhedron P, which coincides with the external or internal description in the extreme cases.

1.2 Linear programming

Recall that a linear program is a problem of the form

$$\min \quad cx \tag{1.4}$$
$$\text{subject to} \quad Ax \geq b. \tag{1.5}$$

where A is an $m \times n$ matrix, b is a column vector in $\mathbb{R}^m$ and c is a row vector in $\mathbb{R}^n$. Note that the feasible region of a linear program is a polyhedron.

Linear programming is a well-developed area of operations research. The systematic study of this subject was initiated by Dantzig and von Neumann. Here we only recall a few basic aspects that will be useful in the remainder of the thesis. A comprehensive presentation of the theory of linear programming can be found e.g. in [58].

Given a linear program (1.4)–(1.5), exactly one of the following alternatives holds:

(i) the problem is infeasible (i.e. no point in $\mathbb{R}^n$ satisfies $Ax \geq b$);

(ii) the problem has an optimal solution;

(iii) the problem is unbounded (i.e. system $Ax \geq b$ is feasible and there exists $r \in \mathbb{R}^n$ such that $Ar \geq 0$ and $cr < 0$).

Even though system (1.5) may define a polyhedron without vertices, every problem of the form (1.4)–(1.5) can be transformed into a linear program whose feasible region is a pointed polyhedron. So we assume without loss of generality that the feasible region (1.5) has at least one vertex (and thus it has at least one extreme ray).

A fundamental result in linear programming is the following.

Theorem 1.4 *If a linear program (1.4)–(1.5) has an optimal solution, then it has an optimal solution which is an extreme point of the feasible region. If a linear program is unbounded, then there is an extreme ray r of the feasible region such that $cr < 0$.*

Since a polyhedron has only a finite number of extreme points and extreme rays, a first approach to solve a linear program in a finite number of operations is simple enumeration. However, as mentioned in Section 1.1.2, the number of extreme points and extreme rays of a polyhedron might be exponentially large, thus such a technique cannot be used in practice.

The first algorithm proposed to solve linear programming problems, the *simplex method*, is a refinement of this approach. This method, which was introduced by Dantzig [19], consists in visiting some of the vertices of the feasible region, each time choosing the next vertex with a clever rule. This algorithm has a good performance in practice and is commonly used by commercial softwares. However, as shown by Klee and Minty [37], it is possible to construct linear programs that cause the simplex method to perform an exponential number of iterations.

The first polynomial time algorithm for linear programming, the *ellipsoid method*, was obtained by Khachiyan [36], who adapted to this problem a technique that was already used in nonlinear programming. Though Khachiyan's algorithm is not used in practice, it yielded the first proof that linear programming can be solved in polynomial time:

Theorem 1.5 *There is a polynomial time algorithm for solving linear programming (with rational input) that finds an optimal extreme point solution (if the problem has an optimal solution).*

Apart from the above result, the theoretical importance of the ellipsoid method comes from the fact that it does not require that the inequalities defining the feasible region be explicitly given. It is sufficient to have a polynomial time algorithm for the *separation problem*: given a point $\bar{x}$, either decide that $\bar{x}$ is feasible or find an inequality that is satisfied by all points in the feasible region and violated by $\bar{x}$. If the separation problem on a polyhedron is solvable in polynomial time, so is the linear optimization problem, even if the polyhedron has exponentially-many facets. In fact the two problems are equivalent, as shown by Grötschel, Lovász and Schrijver [28]:

Theorem 1.6 *Linear optimization is solvable in polynomial time if and only if so is the separation problem.*[2]

A good tradeoff between running time in the worst case and practical performance is achieved by *interior point methods*. The first algorithm of this type was introduced by Karmarkar [35]. Instead of moving on the boundary (like the simplex method), these algorithms follow a path in the interior of the feasible region that converges to an optimal solution of the problem.

We conclude this section by recalling a well-known result due to Farkas (see e.g. [58]), which will be used in a subsequent chapter.

Theorem 1.7 (Farkas' lemma) *A linear system $Ax \geq b$ is feasible if and only if $ub \leq 0$ for each $u \geq 0$ satisfying $uA = \mathbf{0}$.*

If some inequalities of the system $Ax \geq b$ are replaced by equations, the nonnegativity bounds on the corresponding components of u must be removed.

1.3 Integer and mixed-integer programming

Recall that a mixed-integer program is a problem of the form (1.1)–(1.3) with $I \neq \varnothing$, and a (pure) integer program is a problem of the same type with $I = \{1, \ldots, n\}$.

In contrast to linear programming, which can be solved efficiently, integer programming and mixed-integer programming are difficult problems: they are both $\mathcal{NP}$-complete problems [17]. Thus a polynomial time algorithm for solving these two problems in the general case is not known.

Given a mixed-integer set (1.2)–(1.3), the polyhedron defined by $Ax \geq b$ is called the *linear relaxation* (or *continuous relaxation*) of (1.2)–(1.3). The following fundamental result is due to Meyer [44]:

[2] This result holds under some mild technical assumptions (see [29] for the details).

Theorem 1.8 *If all entries of A and b are rational numbers, then the convex hull of* (1.2)–(1.3) *is a polyhedron. Furthermore such a polyhedron and the linear relaxation of* (1.2)–(1.3) *have the same recession cone.*

Under the hypothesis of the above theorem, let P be the convex hull of (1.2)–(1.3). If a linear inequality description of the polyhedron P is known, then the optimization problem $\min\{cx : x \in P\}$ is a linear program. Using the above result and Theorem 1.5, one can prove that such a linear program is essentially equivalent to problem (1.1)–(1.3).

Theorem 1.9 *Assume that all entries of A and b are rational numbers and let P be the convex hull of the mixed-integer set* (1.2)–(1.3)*. Then one can solve the mixed-integer program* (1.1)–(1.3) *by applying an algorithm for linear programming to the problem* $\min\{cx : x \in P\}$*, provided that a linear inequality description of P is available.*

Unfortunately, the convex hull of (1.2)–(1.3) may be defined by a number of facet-defining inequalities which is exponential in the size of the original description of the problem, and it is usually very hard to characterize them. Thus the approach in the above theorem does not result (in general) in a polynomial time algorithm.

We do not discuss here the various techniques that are commonly used to solve pure and mixed-integer programs either exactly or approximately (see e.g. [49, 69]). We only spend some words on two important aspects of this field: valid inequalities and total unimodularity.

1.3.1 Valid inequalities

In the general case, the linear relaxation of a mixed-integer set X is only a superset of $\text{conv}(X)$. Thus the linear relaxation contains points that should be "cut off" in order to describe $\text{conv}(X)$. This leads to the following standard definitions.

Given a mixed-integer set $X \subseteq \mathbb{R}^n$, a *valid inequality* for X is a linear inequality which is satisfied by all points in X. It is readily checked that a linear inequality is valid for X if and only if it is valid for $\text{conv}(X)$. A *cutting plane* for X is a inequality that is valid for X but is violated by at least one point in the linear relaxation of X.

Given a mixed-integer set (1.2)–(1.3), different kinds of valid inequalities can be derived in several ways (see [18] for a survey of the various techniques). Methods based on cutting planes are commonly used to solve mixed-integer programs either exactly or approximately. Here we only recall two types of valid inequalities that will be used in the next chapters.

The *Chvátal-Gomory procedure* [27] can be used to generate valid inequalities for a pure integer set:

Theorem 1.10 (Chvátal-Gomory rounding) *Given a pure integer set* (1.2)–(1.3) *(thus $I = \{1, \dots, n\}$), take a combination of its inequalities: that is, for a nonnegative vector $u \in \mathbb{R}^m$, consider the valid inequality $uAx \geq ub$, which we denote by $ax \geq \beta$. If a is an integral vector, then the inequality $ax \geq \lceil \beta \rceil$ is valid for* (1.2)–(1.3).

Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ where A and b are rational, the set defined by all the inequalities that can be derived by using the above procedure is a polyhedron [57], denoted $P^{(1)}$ and called the *Chvátal-Gomory closure* (or *truncation*) of P. For each $k \geq 1$, $P^{(k+1)}$ is defined as the Chvátal-Gomory closure of $P^{(k)}$. Schrijver [57] proved that for every rational polyhedron there is an integer k such that $P^{(k)} = \text{conv}(P \cap \mathbb{Z}^n)$. (A similar result holds if P is a bounded polyhedron, independently of the rationality assumption [10].) Such a number k is the *Chvátal rank* of P.

We also need to introduce the *simple mixed-integer rounding inequality*, or *simple MIR-inequality* for short.

Theorem 1.11 (Simple MIR-inequality [49]) *Let X be the mixed-integer set defined by*

$$\begin{aligned} s + z &\geq b, \\ s &\geq 0, \\ z &\ \textit{integer}, \end{aligned}$$

for some real number b. The simple mixed-integer rounding inequality $s + f(b)z \geq f(b)(\lfloor b \rfloor + 1)$, where $f(b) := b - \lfloor b \rfloor$ denotes the fractional part of b, is valid for X.

1.3.2 Totally unimodular matrices

A matrix A is *totally unimodular* if every square submatrix of A has determinant 0, 1 or -1. Note that all entries of a totally unimodular matrix are 0, 1 or -1.

Totally unimodular matrices appear in several combinatorial optimization problems, see e.g. [49]. The main reason for the importance of this class of matrices comes from the following characterization, which is due to Hoffman and Kruskal [34]:

Theorem 1.12 *An $m \times n$ matrix A is totally unimodular if and only if for each vector $b \in \mathbb{Z}^m$, all vertices of the polyhedron $\{x \in \mathbb{R}^n : Ax \geq b,\ x \geq \mathbf{0}\}$ are integral.*

Since in the next chapter the variables of our problems will not be forced to be all nonnegative, we will actually use the result below rather than Theorem 1.12:

Theorem 1.13 *If A is an $m \times n$ totally unimodular matrix and b is an integral vector, then* $\text{conv}\{x \in \mathbb{Z}^n : Ax \geq b\} = \{x \in \mathbb{R}^n : Ax \geq b\}$.

In other words, if A is totally unimodular and b is integral, the convex hull of the pure integer set $\{x \in \mathbb{Z}^n : Ax \geq b\}$ and its linear relaxation $\{x \in \mathbb{R}^n : Ax \geq b\}$ are the same polyhedron. It follows that in this case pure integer programming can be solved in polynomial time by means of linear programming.

We will make constant use of totally unimodular matrices. In particular, we will need a characterization due to Ghouila-Houri [26]. To introduce it, the following definition is needed.

Given a $0, \pm 1$-matrix A, with entries a_{ij} for $1 \le i \le m$ and $1 \le j \le n$, an *equitable bicoloring* of the columns of A is a partition of $\{1, \dots, n\}$ into two classes R and B such that

$$\left| \sum_{j \in R} a_{ij} - \sum_{j \in B} a_{ij} \right| \le 1 \text{ for } 1 \le i \le m.$$

The two classes R and B are sometimes called colors, hence the term bicoloring (the names R, B stand for red and blue respectively).

Theorem 1.14 *[26] A $0, \pm 1$-matrix A is totally unimodular if and only if every column submatrix of A admits an equitable bicoloring of its columns.*

Note that since a matrix is totally unimodular if and only if so is its transpose, the above theorem admits a symmetric version in which the roles of rows and columns are interchanged.

1.4 Extended formulations

As discussed in Section 1.1.1, for a fixed polyhedron P the number of inequalities in any external description of P in its original space is bounded from below by the number of facets of P. Therefore, if P has a huge number of facets, it is impossible to give an external description of P having "small" size. Nonetheless, P may admit a description of smaller size in a higher dimensional space. To formalize this concept, we now give two definitions.

Given a set Q in the space $\mathbb{R}^{n+p}$ (that uses variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$), the *projection* of Q onto the space of the x-variables is the set of points $x \in \mathbb{R}^n$ that can be completed to a vector (x, y) of Q:

$$\text{proj}_x(Q) := \{x \in \mathbb{R}^n : \text{there exists } y \in \mathbb{R}^p \text{ such that } (x, y) \in Q\}.$$

The projection of a polyhedron is always a polyhedron (see also Section 1.4.2).

Given a polyhedron P in the space $\mathbb{R}^n$ (that uses variables x), an *extended formulation* of P is the external description of a polyhedron Q in a space $\mathbb{R}^{n+p}$ (that uses variables x and y) such that $P = \text{proj}_x(Q)$. In other words, an extended formulation of P is a linear system in the variables (x, y) that defines a polyhedron whose projection onto the space of the x-variables is exactly P. We call $\mathbb{R}^n$ the *original space* of variables and $\mathbb{R}^{n+p}$ the *extended space*.

Every polyhedron P admits infinitely-many extended formulations. The number of facets of an extended formulation of P can be very far from that of P. In particular, it may happen that a polyhedron with an exponential number of facets admits an extended formulation with only a polynomial number of facets. Such an example is given by the *permutahedron*, which is the convex hull of the vectors in $\mathbb{R}^n$ whose components form a permutation of the numbers $1, \dots, n$. The permutahedron has $2^n - 2$ facets, but it is the projection of a polyhedron Q in an $\frac{n(n-1)}{2}$-dimensional space that has only $n(n-1)$ facets (Q is the image of a cube under an affine transformation). (See [74] for the details.)

Therefore, among all the possible extended formulations of a polyhedron P, one can hope to find a description of P that requires a small number of facet-defining inequalities. However, Yannakakis [71] proved a very interesting (and perhaps surprising) theorem that gives a lower bound on the size of any extended formulation of a fixed polyhedron. Though such a bound cannot be easily used to predict the minimum size of an extended formulation of a given set, as an *a priori* knowledge of the facets and vertices is required, the theoretical relevance of this result is remarkable.

1.4.1 The role of extended formulations in mixed-integer programming

As discussed in Section 1.3 (see Theorem 1.9), a mixed-integer program reduces to a linear program once a linear inequality description of the convex hull of the feasible region is known. However such a convex hull may have a huge number of facets and it may be very hard to find them. We point out here how extended formulations can be useful in this context.

Let $X \subseteq \mathbb{R}^n$ be a mixed-integer set and suppose that we want to solve the problem $\min\{cx : x \in X\}$, or equivalently $\min\{cx : x \in \mathrm{conv}(X)\}$. Assume that we know an extended formulation of $\mathrm{conv}(X)$ and let $Q \subseteq \mathbb{R}^{n+p}$ be the polyhedron defined by such a formulation. It is immediate to see that then problem $\min\{cx : x \in \mathrm{conv}(X)\}$ is equivalent to problem $\min\{cx : (x, w) \in Q\}$.

This shows that if one knows an extended formulation of the convex hull of the feasible region of a mixed-integer program, then the problem can be equivalently solved in the extended space by means of linear programming. If, in addition, the size of such an extended formulation is polynomial in the size of the original description of X, this allows one to solve the mixed-integer program in polynomial time.

We say the an extended formulation of a mixed-integer set is *compact* if its size is polynomial in the size of the original description of the set. The above discussion can then be summarized in the following result:

Theorem 1.15 *If a mixed-integer set X admits an extended formulation which is compact, then linear optimization over X can be carried out in polynomial time by means of linear programming.*

1.4.2 Projections

When an extended formulation of a polyhedron P is available, in order to find a linear inequality description of P in its original space one has to calculate the projection of Q (the polyhedron defined by the extended formulation) onto the space where P is defined. We conclude this section by briefly discussing two possible ways of computing the projection of a polyhedron.

A first approach is *Fourier-Motzkin elimination* [25, 22, 48] (see e.g. [74]). This technique consists in eliminating one variable at a time.

Theorem 1.16 *Let $Q \in \mathbb{R}^{n+1}$ be a polyhedron in the variables $(x_1, \ldots, x_n, y)$. Assume without loss of generality that Q is described by a system of linear inequalities of the form $a^j x + \beta^j y \geq d^j$ for $j \in J$, where $\beta^j \in \{0, \pm 1\}$ for all $j \in J$. Then a linear inequality description of the* *polyhedron* $\mathrm{proj}_x(Q)$ *in the* x*-space is given by the inequalities*

$$\begin{array}{ll} a^j x \geq d^j & \text{for } j \in J \text{ such that } \beta^j = 0, \\ (a^j + a^k)x \geq d^j + d^k & \text{for } j,k \in J \text{ such that } \beta^j = 1 \text{ and } \beta^k = -1. \end{array}$$

If p variables have to be eliminated, p repetitions of the above procedure are needed. Note that at each iteration the number of inequalities may be squared, thus the elimination of p variables may result in a system with an exponential number of inequalities. This is coherent with what we observed above, namely that an extended formulation of a polyhedron may have less facet-defining inequalities than the polyhedron itself.

Note that the above theorem yields a proof of the fact that the projection of a polyhedron is a polyhedron. We also remark that Fourier-Motzkin elimination often produces a number of redundant inequalities.

A second approach, which allows one to eliminate all extra-variables together and will be used in Chapter 5, is now described. This result, which appears in Černikov [8], is based on Farkas' lemma (Theorem 1.7).

Theorem 1.17 *Let* Q *be a polyhedron in* $\mathbb{R}^{n+p}$ *defined by the linear system* $Ax+Dy \geq b$*. The projection of* Q *onto the space of the* x*-variables is the polyhedron defined by the inequalities* $u(Ax-b) \geq 0$ *for all vectors* u *(of suitable dimension) that are extreme rays of the polyhedral cone defined by*

$$uD = \mathbf{0}, \;\; u \geq \mathbf{0}. \tag{1.6}$$

If some inequalities of the system $Ax+Dy \geq b$ are replaced by equations, the nonnegativity bounds on the corresponding components of u must be removed. In this case cone (1.6) may be non-pointed and "extreme rays" should be replaced with "rays" in the statement of the theorem.

Note that even if the system defining Q has few constraints, the number of inequalities describing the projection can be huge, as one has to write an inequality for each extreme ray of cone (1.6). Similarly to Fourier-Motzkin elimination, this method can produce redundant inequalities.

In [6] the above result was applied for the first time to compute a linear inequality description of a combinatorial optimization problem by projecting an extended formulation.

1.5 Some well-known types of extended formulations

It is not possible to give a systematic presentation of all the techniques that have been successfully used to construct extended formulations in the past years, as such formulations usually exploit the peculiarities of the set under consideration. Nonetheless some of these approaches apply to a wide class of problems and have been used by several authors. In this section we survey some of the most relevant techniques that can be used to construct extended formulations of mixed-integer sets.

1.5.1 Hierarchies of formulations

We consider here mixed 0-1 programs, i.e. mixed-integer programs in which every integer variable must take a value in $\{0, 1\}$. We also assume that all continuous variables are nonnegative. Mixed 0-1 programs arise in many important combinatorial optimization problems, see e.g. [38, 49, 59].

Let X be a mixed 0-1 set, which we write in the form

$$Ax \geq b, \tag{1.7}$$

$$x \geq \mathbf{0}, \tag{1.8}$$

$$x_i \in \{0, 1\}, \quad i \in I, \tag{1.9}$$

where al entries of A and b are rational numbers. Without loss of generality we assume that the linear system $Ax \geq b$ include (or imply) inequalities $x_i \leq 1$ for $i \in I$.

Let P be the convex hull of X and P_0 be the linear relaxation of X. Several authors developed hierarchies of approximate formulations of P, i.e. sequences of polyhedra $P_1, \ldots, P_{|I|}$ such that

$$P_0 \supseteq P_1 \supseteq \cdots \supseteq P_{|I|} = P. \tag{1.10}$$

In the hierarchies that we consider here, each of the polyhedra P_t for $1 \leq t \leq |I|$ is defined implicitly as the projection of a polyhedron Q_t, which is explicitly given in a higher dimensional space. Thus we are provided with a sequence of approximate extended formulations of P, where the last formulation of the sequence is an exact extended formulation of P. As one can expect, in general such an exact formulation is non-compact.

We describe below three of the main hierarchies of formulations that one can find in the literature. The approaches that we describe are also called *lift-and-project* techniques, as the description of the set is first lifted (and strengthened) in a higher dimensional space and then projected onto the original space.

For the pure integer case, a presentation of these hierarchies in a unitary setting as well as an interesting comparison of the various relaxations can be found in [40].

The Sherali-Adams hierarchy

Sherali and Adams [60, 61] proposed the hierarchy of relaxations (1.10) that we now describe. For each fixed index $1 \leq t \leq |I|$, the polyhedra Q_t and P_t are constructed as follows.

1. Let S be the set of all polynomials of the form

$$\prod_{i \in J_1} x_i \prod_{i \in J_2} (1 - x_i),$$

   where J_1, J_2 are disjoint subsets of I satisfying $|J_1| + |J_2| = t$. Construct the nonlinear system consisting of all inequalities obtained by multiplying an inequality of the system $Ax \geq b$ by a polynomial in S.

2. Linearize the resulting system by performing the following two operations:

(a) for $i \in |I|$, substitute x_i for x_i^2 in all the inequalities of the system;

(b) for each monomial $\prod_{i\in J} x_i$, where $J \subseteq \{1,\dots,n\}$ and $|J| \geq 2$, introduce a new variable y_J and substitute y_J for $\prod_{i\in J} x_i$ throughout.

Let Q_t be the polyhedron defined by the resulting linear system of inequalities and let P_t be the projection of Q_t onto the x-space of variables.

Note that Steps 1 and 2 (a) give rise to inequality that are valid for X, as any point in X satisfies $x_i \in \{0,1\}$ for all $i \in I$.

Sherali and Adams [60, 61] proved that (1.10) holds for the polyhedra thus constructed. It is clear that the exact extended formulation $Q_{|I|}$ consists of an exponential number of variables and constraints.

The Sherali-Adams relaxation can be defined for a more general class of sets, namely mixed 0-1 polynomial sets that are linear in the continuous variables [61]. These sets have the form (1.7)–(1.9), except that the linear system $Ax \geq b$ is replaced by a system of inequalities involving polynomials in which the continuous variables appear with degree at most one. The procedure is similar to that described above and produces two sequences of polyhedra $Q_1,\dots,Q_{|I|}$ and $P_1,\dots,P_{|I|}$, where for each t the polyhedron P_t is the projection of Q_t onto the original space. Condition (1.10) is again satisfied, except for the inclusion $P_0 \supseteq P_1$ which might be violated. Note that linear optimization over a mixed 0-1 polynomial set of this type is converted into linear programming over $Q_{|I|}$.

A generalization of the procedure presented above was described in [62], while an extension to a more general class of sets was studied recently in [1].

The Lovász-Schrijver hierarchy

Lovász and Schrijver [41] proposed two hierarchies of formulations of P (in fact their original construction is for pure 0-1 problems only). The first hierarchy can be defined iteratively as follows: for $1 \leq r \leq |I|$, the polyhedra Q_r and P_r are obtained by applying the Sherali-Adams procedure with $t = 1$ to the linear system defining P_{r-1}. That is, the inequalities describing P_{r-1} have to be multiplied only by x_i and $1 - x_i$ for each $i \in I$ and then linearized.

It can be shown that (1.10) holds for the polyhedra thus constructed. Note in particular that $P_{|I|} = P$ even though the above construction uses only a partial version of the Sherali-Adams procedure.

The definition of the polyhedra $P_1,\dots,P_{|I|}$ given above is different from (though equivalent to) that appearing in [41]. The original equivalent construction of P_t is given below.

1. Define the cone $\tilde{P}_{t-1} := \left\{ \lambda \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in P_{t-1},\ \lambda \geq 0 \right\} \subseteq \mathbb{R}^{n+1}$. The additional coordinate is indexed by 0.

2. Let M_{t-1} be the set of symmetric $(|I|+1) \times (|I|+1)$ matrices $Y = (y_{ij} : i,j \in I \cup \{0\})$ such that

   (a) $Y_{ii} = Y_{0i}$ for $i \in I$,

(b) $Y_0, Y_0 - Y_i \in \tilde{P}_{t-1}$ for $i \in I$, where Y_i denotes the column of Y corresponding to index i.

3. Define $P_t := \left\{ x \in \mathbb{R}^n : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y_0 \text{ for some } Y \in M_{t-1} \right\}$.

The relaxation that is commonly referred to as the Lovász-Schrijver relaxation is obtained as above, except that Step 3 is replaced by the following:

3'. Define $P_t^+ := \left\{ x \in \mathbb{R}^n : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y_0 \text{ for some } Y \in M_{t-1}^+ \right\}$, where M_{t-1}^+ consists of the matrices in M_{t-1} that are positive semidefinite.

The convex sets P_t^+ satisfy (1.10). Furthermore it is clear that $P_t^+ \subseteq P_t$ for $1 \leq t \leq |I|$. Note however that P_t^+ is not a polyhedron: it is the feasible region of a *semidefinite program*. The interest in a relaxation of this type comes from the fact that semidefinite programs can be solved efficiently through interior point algorithm (see e.g. [66] for a survey on semidefinite programming).

Another hierarchy of semidefinite relaxations was given by Lasserre [39].

The Balas-Ceria-Cornuéjols hierarchy

Balas, Ceria and Cornuéjols [5] proposed the following lift-and-project procedure:

1. Pick an index $i_1 \in I$.

2. Construct the nonlinear system consisting of all inequalities obtained by multiplying an inequality of the system $Ax \geq b$ by one of x_{i_1} and $1 - x_{i_1}$.

3. Linearize the resulting system by performing the following two operations:

   (a) substitute x_{i_1} for $x_{i_1}^2$ in all the inequalities of the system;

   (b) for each $i \neq i_1$, introduce a new variable y_i and substitute y_i for $x_{i_1} x_i$ throughout.

   Let Q_1 be the polyhedron defined by the resulting linear system of inequalities and let P_1 be the projection of Q_1 onto the x-space of variables.

The polyhedra Q_2 and P_2 are constructed by choosing a different index $i_2 \in I \setminus \{i_1\}$ and performing the above operations on the linear system defining P_1. By iterating this construction, one defines the polyhedra Q_t and P_t for $1 \leq t \leq |I|$.

Results of Balas, Ceria and Cornuéjols [5] and Balas [3] show that

$$\begin{aligned} P_t &= \operatorname{conv}\left(\{x \in P_{t-1} : x_{i_t} = 0\} \cup \{x \in P_{t-1} : x_{i_t} = 1\}\right) \\ &= \operatorname{conv}\left(\{x \in P_0 : x_{i_r} \in \{0,1\} \text{ for } 1 \leq r \leq t\}\right), \end{aligned}$$

which implies all the inclusions and the equation in (1.10). In other words, at each iteration the lift-and-project procedure computes the convex hull of the current relaxation, where each time a single variable $x_i \in I$ is treated as a binary variable. Such a sequential convexification leads to the convex hull of the original set, i.e. $P_{|I|} = P$.

We remark that though this procedure requires much less effort than the Sherali-Adams and Lovász-Schrijver relaxations, still the $|I|$-th step yields a description of the convex hull in the original set. However, the intermediate relaxations $P_1, \dots, P_{|I|-1}$ are not as strong as those arising from the Sherali-Adams and Lovász-Schrijver procedures.

In [5] it is also shown how lift-and-project can be used to generate cutting planes.

1.5.2 Extended formulations based on Minkowski-Weyl theorem

We remarked in Section 1.1.2 that the formulation of a polyhedron given by Theorem 1.2 uses additional variables. Thus that theorem yields an extended formulation of a polyhedron.

In general an extended formulation of this type can hardly be explicitly given for the convex hull of a mixed-integer set, as it is usually difficult to characterize the extreme points and extreme rays of such a polyhedron (assuming it is pointed). Furthermore, the number of extreme points and extreme rays of the convex hull of a mixed-integer set is often huge, even if the original description of the set is small.

1.5.3 Extended formulations based on the properties of the extreme points

A refinement of the approach described in Section 1.5.2 is sometimes possible: the key idea is that some basic properties of the vertices, rather than their complete enumeration, may suffice to describe the convex hull of a mixed-integer set. This idea, which already appears in [53], will be exploited in the next chapters.

We demonstrate this technique by showing how Miller and Wolsey [45] used this approach to construct an extended formulation of the convex hull of the following mixed-integer set:

$$s + z_i \geq b_i, \quad 1 \leq i \leq n, \tag{1.11}$$

$$s \geq 0 \tag{1.12}$$

$$z_i \text{ integer}, \quad 1 \leq i \leq n, \tag{1.13}$$

where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$. The above set, which is now called *mixing set*, has important applications in production planning problems (in particular lot-sizing [55]). We will be analyzing it again in Sections 4.2 and 5.2.3.

The construction of an extended formulation of (1.11)–(1.13) can be divided into the three main steps below. We do not go into details or give any proofs, as we only want to convey the main idea of the technique. Furthermore, since an extension of this approach is described in Chapter 2, rigorous proofs can be found there.

1. First one observes that in every extreme point of the convex hull of (1.11)–(1.13), the fractional part of s is one of the values $f_0, \dots, f_n$, where for $1 \leq i \leq n$, $f_i := b_i - \lfloor b_i \rfloor$ is the fractional part of b_i, and $f_0 := 0$.

2. Then one adds the following constraints to the original formulation (1.11)–(1.13):

$$s = \mu + \textstyle\sum_{i=0}^{n} f_i \delta_i, \tag{1.14}$$
$$\textstyle\sum_{i=0}^{n} \delta_i = 1, \ \delta_1, \dots, \delta_n \geq 0, \tag{1.15}$$
$$\mu, \delta_1, \dots, \delta_n \text{ integer}. \tag{1.16}$$

   The above conditions force variable s to take a fractional part in the set of values $\{f_0, \dots, f_n\}$. One can show that adding constraints (1.14)–(1.16) does not change the convex hull of feasible solutions.

3. The set of constraints (1.11)–(1.13) and (1.14)–(1.16) is then tightened and an equivalent description is obtained that has the following form:

$$s = \mu + \textstyle\sum_{i=0}^{n} f_i \delta_i, \tag{1.17}$$
$$A\mu + B\delta + Cz \geq d, \tag{1.18}$$
$$\mu, \delta_1, \dots, \delta_n, z_1, \dots, z_n \text{ integer}, \tag{1.19}$$

   where $[A \mid B \mid C]$ is a totally unimodular matrix and d is an integral vector. Since variable s does not appear in any of inequalities (1.18), by Theorem 1.13 the integrality conditions (1.19) can be removed without affecting the convex hull of feasible solutions. The resulting linear system is an extended formulation of the mixing set (1.11)–(1.13).

Step 3 suggests that such an approach can only be used for some particular mixed-integer sets, as one needs to obtain a linear system with totally unimodular matrix. The idea of exploiting the total unimodularity of a pure integer reformulation of a mixed-integer set appears in [53].

The general idea underlying the above technique —modeling the continuous variable according to the possible fractional parts taken at the vertices— can be extended to mixed-integer sets with more than one continuous variable. Such an extension was successfully used by Miller and Wolsey [45, 46], Van Vyve [63, 65] and Conforti, Di Summa and Wolsey [12] in tackling specific mixed-integer sets that appear in lot-sizing problems.

In Chapter 2 we present a modeling technique that generalizes that described here and can be used to formulate a quite large family of mixed-integer sets, which includes as special cases several sets studied by the authors cited above.

We remark that the technique sketched above is just one of the possible ways of exploiting the properties of the vertices (see e.g. [53, 63, 65]).

1.5.4 Extended formulations based on the union of polyhedra

Theorem 1.3 yields an extended formulation for the convex hull of several polyhedra $P_1, \dots, P_k$ in $\mathbb{R}^n$, provided that external descriptions of these polyhedra are available. We remark that such an extended formulation is compact, while the description of $\mathrm{conv}(P_1 \cup \dots \cup P_k)$ in its original space $\mathbb{R}^n$ may have an exponential number of facet-defining inequalities (such an example is given in [18]).

Balas' result was recently applied by Conforti and Wolsey in [16], where a technique is introduced and used to find extended formulations of some mixed-integer sets arising in lot-sizing problems. The same idea, which we present below, had been also used by Atamtürk [2] to formulate a simple mixed-integer set that has application in robust optimization.

To summarize the approach, we use the following notation: given a mixed-integer set X, let V be the set of vertices of $\text{conv}(X)$ and let R be the set of its extreme rays (we assume that $\text{conv}(X)$ is a pointed polyhedron). The technique proposed in [16] is as follows (we present it in a simplified version):

1. First the set of vertices V is partitioned into subsets $V_1, \dots, V_k$ according to some criterion (usually the fractional part of one or some of the continuous variables).

2. For each $1 \le i \le k$, let P_i be the polyhedron generated by the points in V_i and the rays in R. Note that $\text{conv}(X) = \text{conv}(P_1 \cup \cdots \cup P_k)$, as all these polyhedra have the same recession cone. For $1 \le i \le k$, an extended formulation Q_i of P_i is constructed in some way: this is usually done by (i) introducing new variables to model the common property of the vertices in V_i and (ii) observing that the resulting set belongs to a class of mixed-integer sets for which an extended formulation is known.

3. Balas' result is then applied either to the polyhedra $P_1, \dots, P_k$ (which can be determined by computing the projection of $Q_1, \dots, Q_k$), or to their extended formulations $Q_1, \dots, Q_k$. In both cases an extended formulation of $\text{conv}(X)$ is found.

The above approach will be used in Section 8.3 to tackle a mixed-integer set which has application both in deterministic and stochastic lot-sizing problems with backlogging.

1.5.5 Extended formulations more generally

The definition of extended formulation of a polyhedron that we gave at the beginning of Section 1.4 can be stated in a different way, as the following result shows:

Proposition 1.18 *Let P be a polyhedron in the variables $x \in \mathbb{R}^n$ and Q a polyhedron in the variables $(x, w) \in \mathbb{R}^{n+p}$. The following conditions are equivalent:*

(i) *P is the projection of Q onto the x-space of variables;*

(ii) *for every vector $c \in \mathbb{R}^n$, $\bar{x}$ is an optimal solution of the linear program* $\min\{cx : x \in P\}$ *if and only if there exists $\bar{w}$ such that $(\bar{x}, \bar{w})$ is an optimal solution of the linear program* $\min\{cx : (x, w) \in Q\}$.

Thus condition (ii) could be taken as definition of extended formulation. We now show that such a definition is sometimes too restrictive, in the sense that a softer version may be sufficient to transform a mixed-integer program into a linear program on a different space of variables.

Specific objective functions

In many cases the mixed-integer program under consideration is the model of a real-world problem for which not all possible objective functions are meaningful. For instance, when the objective function cx represents a cost, one will probably be interested only in vectors c that have nonnegative components.

For a fixed mixed-integer set X, let F be the set of vectors $c \in \mathbb{R}^n$ that correspond to "interesting" objective functions, i.e. objective functions that can really occur in the problem that is modeled by X. Define $P := \text{conv}(X)$ and let Q be a polyhedron in the variables $(x, w) \in \mathbb{R}^{n+p}$ that satisfies the following weak version of condition (ii) of Proposition 1.18:

(ii') For every vector $c \in F$, $\bar{x}$ is an optimal solution of the linear program $\min\{cx : x \in P\}$ if and only if there exists $\bar{w}$ such that $(\bar{x}, \bar{w})$ is an optimal solution of the linear program $\min\{cx : (x, w) \in Q\}$.

Such a polyhedron Q is not an extended formulation of P according to the definition given in Section 1.4, however it is sufficient to convert the mixed-integer program $\min\{cx : x \in X\}$ into the linear program $\min\{cx : (x, w) \in Q\}$ for all "interesting" objective functions.

To demonstrate that such a weaker version of the concept of extended formulation can be useful, we consider lot-sizing problems. In a lot-sizing problem several costs need to be considered: for each period i, one usually has a per unit production cost p_i, a fixed cost q_i that one must pay if production takes place in period i, a per unit holding cost h_i for storing the excess of production at the end of period i and a per unit backlogging (recovery) cost r_i.

Several kinds of lot-sizing problems (and relaxations of them) were studied and successfully formulated without any assumptions on the objective function (i.e. on the costs), see e.g. [12, 13, 16, 30, 45, 46, 64, 65], but many others do not seem to be easily tractable in the general case. However it turns out that in practice many instances satisfies the following special condition: for $2 \le i \le N$ (where N is the total number of periods), $p_{i-1} + h_{i-1} \ge p_i$ and $p_i + r_{i-1} \ge p_{i-1}$. A problem satisfying such a property is said to have *Wagner-Whitin costs*.

A number of lot-sizing problems with Wagner-Whitin costs were studied in the last years, see e.g. [45, 53, 63, 65]. Under Wagner-Whitin hypotheses, the optimal solutions satisfy some special properties that can be exploited to construct compact extended formulation in the weaker sense discussed above.

Linear inequality formulations based on dynamic programming

(We use here some basic concepts about dynamic programming, shortest path problems on digraphs and linear programming duality, see e.g. [7, 38, 58] respectively. Our presentation is mostly based on [68].)

A number of problems that can be solved through dynamic programming can be formalized as follows: states are labeled $0, \dots, N$ and the recursive function has the form

$$F(0) = 0, \;\; F(j) = \min_{0 \le i < j} \{F(i) + c(i, j)\} \text{ for } 1 \le j \le n, \tag{1.20}$$

where $c(i, j)$ is the nonnegative cost of the transition from state i to state j. The application of the recursion yields the optimal value $F(n)$ along with an optimal solution that is determined as follows: if $0 = j_0 < j_1 < \cdots < j_k = N$ is a sequence of indices such that $F(j_\ell) = F(j_{\ell-1}) + c(j_{\ell-1}, j_\ell)$ for $1 \le \ell \le k$, then the optimal solution consists of the following decisions: for each $1 \le \ell \le k$, go from state $j_{\ell-1}$ to state j_ℓ.

Let $\mathcal{D} = (V, A)$ be the directed graph with node set $V := \{0, \dots, N\}$ and arc set $A := \{(i, j) : 0 \le i < j \le N\}$. Note that $\mathcal{D}$ contains no cycles. If we assign weight $c(i, j)$ to arc (i, j), then the dynamic programming recursion amounts to finding a shortest path in $\mathcal{D}$ connecting nodes 0 and N. The well-known linear programming formulation of such a problem is

$$\begin{aligned}
\min \quad & \sum_{0 \le i < j \le N} c(i, j) w_{ij} \\
\text{subject to} \quad & \sum_{j > 0} w_{0j} = 1, \\
& \sum_{j > k} w_{jk} - \sum_{i < k} w_{ik} = 0, \quad 1 \le k \le N - 1, \\
& \sum_{i < N} w_{iN} = 1, \\
& w_{ij} \ge 0, \quad 0 \le i < j \le N.
\end{aligned}$$

The above problem has an optimal solution with $w_{ij} \in \{0, 1\}$ for all $0 \le i < j \le N$. Arcs (i, j) corresponding to variables that take value 1 form an optimal path. By interpreting each w_{ij} as a decision variable corresponding to the transition from state i to state j, such a path yields an optimal solution of the original problem.

This shows that the above linear program is a linear formulation of the original problem, in the sense that solving it yields the optimal solution of the original problem. This property is similar to condition (ii) of Proposition 1.18, in the sense that a given problem is converted into a linear program on a different space.

The same linear program can also be obtained by using linear programming duality. Specifically, observe that the following linear program is the equivalent of recursion (1.20):

$$\begin{aligned}
\max \quad & F(n) \\
\text{subject to} \quad & F(j) - F(i) \le c_{ij}, \quad 0 \le i < j \le N, \\
& F(0) = 0.
\end{aligned}$$

By interpreting $F(0), \dots, F(n)$ as variables, the dual of the above linear program is essentially the linear programming formulation of the shortest path problem seen above.

Clearly such a shortest path formulation can be given only for problems that admit a dynamic programming algorithm with a recursion of type (1.20). However Martin, Rardin and Campbell [43] showed that this approach can be generalized to a wider class of problem that can be solved by discrete dynamic programming: given a dynamic programming algorithm, they formulate the original instance as a linear program arising from a problem on a hypergraph.

General affine transformations

The definition of extended formulation given in Section 1.4 is based on the notion of projection. Since a projection is a particular type of full-rank affine transformation, such a definition can be generalized as we now describe.

Let P be a polyhedron in the variables $x \in \mathbb{R}^n$ and Q a polyhedron in the variables $y \in \mathbb{R}^m$, where $m \geq n$. Let T be a full-rank $n \times m$ matrix and let t be a vector in $\mathbb{R}^n$. The mapping g defined by $g(y) := Ty + t$ for $y \in \mathbb{R}^m$ is a full-rank affine transformation of $\mathbb{R}^m$ into $\mathbb{R}^n$.

Assume that $g(Q) = P$. Then one can easily check that the following analogue of condition (ii) of Proposition 1.18 holds:

(ii") For every $c \in \mathbb{R}^n$, $\bar{x}$ is an optimal solution of the linear program $\min\{cx : x \in P\}$ if and only if $\bar{x} = T\bar{y} + t$, where $\bar{y} \in Q$ is an optimal solution of the linear program $\min\{cTy + ct : y \in Q\}$.

Note that if $t = \mathbf{0}$ and $T = [I_n \mid \mathbb{O}]$ (where I_n is the $n \times n$ identity matrix), we reobtain condition (ii) of Proposition 1.18 and Q is an extended formulation of P according to our definition.

This more general kind of extended formulation was studied by Padberg and Sung [50], who proved a generalization of Theorem 1.17 that we now describe. Following [50], we assume without loss of generality that the columns of T are ordered so that $T = [T_1 \mid T_2]$, where T_1 is a non-singular $n \times n$ matrix.

Theorem 1.19 *Let Q be a polyhedron in $\mathbb{R}^m$ defined by the linear system*

$$Ay \geq b, \ \ Cy = d.$$

Partition $A = [A_1 \mid A_2]$ and $C = [C_1 \mid C_2]$, where A_1, C_1 are the column submatrices formed by the first n columns of A, C respectively. The polyhedron $g(Q)$ is defined by the inequalities

$$(uA_1 + vC_1)T_1^{-1}(x - t) \geq ub + vd$$

for all vectors (u, v) (of suitable dimension) belonging the following polyhedral cone:

$$u(A_2 - A_1T_1^{-1}T_2) + v(C_2 - C_1T_1^{-1}T_2) = \mathbf{0}, \ \ u \geq \mathbf{0}.$$

If the above is a polyhedral cone with apex, then its extreme rays are sufficient.

If $t = \mathbf{0}$ and $T = [I_n \mid \mathbb{O}]$, the above statement coincides with Theorem 1.17. Padberg and Sung [50] used this result to compare four approximate extended formulations of the traveling salesman problem, each defined on a different space of variables.

1.6 Outline of the thesis

The main subject of this work is the study of a class of mixed-integer sets whose constraint matrices are totally unimodular. A technique is presented that allows one to construct extended formulations for such sets, and the description in the original space is also considered for some special cases. Furthermore, possible extensions to other sets are considered.

In Chapter 2 we study mixed-integer sets of the type

$$MIX^{TU} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}, \tag{1.21}$$

where A is a totally unimodular matrix, b is a column vector and I is a nonempty subset of $\{1, \dots, n\}$. By a result of Eisenbrand [23, 11], the problem of checking nonemptiness of a set MIX^{TU} is $\mathcal{NP}$-complete, even if A is a totally unimodular matrix with at most two nonzero entries per column and b is a half-integral vector (i.e. $2b$ is integral). This, together with the equivalence between separation and optimization (Theorem 1.6), indicates that finding an explicit inequality description of the polyhedron $\mathrm{conv}(MIX^{TU})$ will most likely be an elusive task.

We then focus on sets of the type MIX^{TU} for which the matrix A contains at most two nonzero entries per *row* (a set of this type is denoted by MIX^{2TU}), and sets of the type MIX^{TU} for which A is the transpose of a flow network matrix (denoted MIX^{DN}). We provide an extended formulation for the polyhedron $\mathrm{conv}(MIX^{DN})$, and this will also yield an extended formulation for $\mathrm{conv}(MIX^{2TU})$. We summarize below the approach used to find an extended formulation of $\mathrm{conv}(MIX^{DN})$, which is based on a general idea that was also adopted by Miller and Wolsey [45, 46] and Van Vyve [63, 65] to tackle some specific mixed-integer sets arising from lot-sizing problems.

First we study a mixed-integer set $X^{\mathcal{F}}$, which is the set of points that satisfy the system $Ax \geq b$ (which defines MIX^{DN}), where all variables are required to take a fractional part belonging to a given list $\mathcal{F}$. We introduce additional variables to model the conditions defining $X^{\mathcal{F}}$ and obtain a pure integer description of this set. The constraints are then strengthened and an equivalent pure integer description is obtained, where the constraint matrix is now totally unimodular. This will provide an extended formulation of $\mathrm{conv}(X^{\mathcal{F}})$.

Next we study the case in which the list $\mathcal{F}$ is *complete*: that is, it contains all possible fractional parts that the variables take over the set of vertices of $\mathrm{conv}(MIX^{DN})$. We prove that under this assumption the above result yields an extended formulation of $\mathrm{conv}(MIX^{DN})$. We show that a complete list for a set of the type MIX^{DN} can always be exhibited, thus an extended formulation of our type can be constructed in all cases. We also show that if there is a complete list $\mathcal{F}$ that contains a polynomial number of elements, then the extended formulation is compact. This proves that linear optimization over sets of the type MIX^{DN} (or MIX^{2TU}) that have this property can be carried out efficiently through linear programming. This is in contrast to the $\mathcal{NP}$-completeness result mentioned above, which holds when the matrix A in (1.21) has at most two nonzero entries per *column*.

In Chapter 3 we discuss the size of an extended formulation of the type introduced in Chapter 2.

On the negative side, we show that there exist mixed-integer sets of the type MIX^{2TU} that do not admit a complete list of fractional parts containing only a polynomial number of elements. This implies that for such sets, no extended formulation of our type is compact.

On the other hand, we give some sufficient conditions ensuring that a mixed-integer set MIX^{2TU} admits a complete list of polynomial length, thus proving that under these conditions the extended formulation of Chapter 2 is polynomial in the original description of the set. The list of fractional parts is explicitly given through a construction based on a graph associated with the set.

In Chapter 4 we show that several mixed-integer sets that have been studied in the literature can be transformed into sets of the type MIX^{2TU} and thus admit an extended formulation of the type introduced in Chapter 2. For many of these sets, one of the conditions ensuring the existence of a complete list of fractional parts with a polynomial number of elements is satisfied, and such a list can be explicitly given. Therefore the extended formulation is compact for such sets.

We will see that most of the mixed-integer sets considered in this chapter have application in real-word problems, such as production planning. Our results provide a unified framework for the extended formulations of these sets found in the last years.

In Chapter 5 we consider the problem of carrying out explicitly the projection of an extended formulation of a mixed-integer set of the type MIX^{2TU}. When this can be done, we obtain a linear inequality description of the polyhedron $\text{conv}(MIX^{2TU})$ in its original space.

Since computing the projection of our extended formulation seems to be an extremely hard task in general, we only consider two special cases for which the projection can be calculated: the first case is a general set of the type MIX^{2TU} having a single continuous variable, while the second set studied is a mixed-integer set arising from some lot-sizing problems.

We will see that the problem of computing the projection of an extended formulation of the type given in Chapter 2 amounts to solving a family of circulation problems on a network depending on continuous parameters.

Chapter 6 is entirely devoted to mixed-integer sets of the type MIX^{2TU} having a single integer variable. We give a linear inequality description (in the original space) of the convex hull of an arbitrary set in this class. In contrast to the “opposite” case of a single continuous variable considered in Chapter 5, such a description is obtained without constructing or projecting any extended formulation of the set. A technique appearing in [24] will be used.

We will point out that all the inequalities of the formulation can be derived as simple MIR-inequalities, while the Chvátal-Gomory procedure is not sufficient to generate all of them.

In Chapter 7 we consider two examples of a mixed-integer set whose constraint matrix has a simple structure but is not totally unimodular (in fact, it is not even a $0, \pm 1$-matrix). We show how the approach described in Chapter 2 can be extended and how this yields extended formulations for the two sets that are analyzed.

The coefficients of the first set form a sequence of divisible number, while the constraints of the second set contain only two distinct (but arbitrary) coefficients on the integer variables. For the former set the size of the extended formulation is polynomial in the size of the original description of the set, while for the latter we can only obtain a pseudo-polynomial description.

We will also point out that in both cases the success in finding such formulations relies upon the very special properties that each integer variable appears in a single inequality of the original description of the set.

In Chapter 8 we present a different approach to construct formulations of mixed-integer sets in the original space or in an extended space. In contrast to the technique of Chapter 2 and its extension described in Chapter 7, no explicit enumeration of fractional parts or other numbers is required (except possibly in the final phase of the process). We adopt this technique to formulate two specific sets, but we cannot determine a class of mixed-integer sets for which this approach can be used.

The idea can be summarized as follows. A given mixed-integer set X is written as $X = Z \cap P$ for some mixed-integer set Z and some polyhedron P that is described by a small number of inequalities. Then one proves that for a particular choice of Z and P, $\mathrm{conv}(X) = \mathrm{conv}(Z) \cap P$. Next the set Z is shown to be equivalent to a mixed-integer set for which a formulation is known either in the original space or in an extended space. This can be used to derive a formulation of X.

Finally in Chapter 9 some open problems in this field are discussed.

Note The results presented in Chapters 2–4 are joint work with Michele Conforti, Friedrich Eisenbrand and Laurence A. Wolsey. The results of Chapter 8 and partly of Chapters 5 and 7 are joint work with Michele Conforti and Laurence A. Wolsey.

Chapter 2

Extended formulations of dual network sets

In this chapter we study mixed-integer sets of the type

$$MIX^{TU} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}, \tag{2.1}$$

where A is a totally unimodular matrix, b is a column vector and I is a nonempty subset of $\{1, \dots, n\}$.

We point out in Section 2.1 that the problem of checking nonemptiness of a set MIX^{TU} is $\mathcal{NP}$-complete, even if A is a totally unimodular matrix with at most two nonzero entries per column and b is a half-integral vector (i.e. $2b$ is integral). This, together with the equivalence between separation and optimization (Theorem 1.6), indicates that finding an explicit inequality description of the polyhedron $\operatorname{conv}(MIX^{TU})$ will most likely be an elusive task.

In Section 2.2 we introduce two families of matrices that are studied in this chapter: one is the class of *dual network matrices*, i.e. the transposes of matrices of circulation problems on a network; the other consists of the totally unimodular matrices with at most two nonzero entries per row. We recall some well-known results about these matrices and in particular we observe that the matrices of the second class can be easily "transformed" into matrices of the first class.

Let MIX^{DN} be a mixed-integer set of the type MIX^{TU} defined above, with the additional restriction that A is a dual network matrix. Similarly, let MIX^{2TU} be a mixed-integer set of the type MIX^{TU} where A has at most two nonzero entries per row. In Sections 2.3–2.4 we provide an extended formulation for the polyhedron $\operatorname{conv}(MIX^{DN})$. This, together with the observations made in Section 2.2, gives an extended formulation of $\operatorname{conv}(MIX^{2TU})$.

The technique that we present is based on a general idea that was also used by Miller and Wolsey [45, 46] and Van Vyve [63, 65] to tackle some specific mixed-integer sets arising from lot-sizing problems. Their common approach consisted in modeling the continuous variables of the problem by introducing integer variables, so that a pure integer description of the set was derived. A linear inequality description of this pure integer formulation was then obtained (see also Section 1.5.3). In this last step total unimodularity usually plays a central role. The idea of constructing compact extended formulations by exploiting the total unimodularity of a pure integer reformulation of the set appears in a paper by Pochet and Wolsey [53].

The approach used here to find an extended formulation of $\operatorname{conv}(MIX^{DN})$ is now summarized. In Section 2.3 we study a mixed-integer set $X^{\mathcal{F}}$, which is the set of points that satisfy the system $Ax \geq b$ (which defines MIX^{DN}), where all variables are required to take a fractional part belonging to a given list $\mathcal{F}$. We introduce additional variables to model the conditions defining $X^{\mathcal{F}}$ and obtain a pure integer description of this set. The constraints are then strengthened and an equivalent pure integer description is obtained, where the constraint matrix is now totally unimodular. This will provide an extended formulation of $\operatorname{conv}(X^{\mathcal{F}})$.

In Section 2.4 we study the case in which the list $\mathcal{F}$ is *complete*: that is, it contains all possible fractional parts that the variables take over the set of vertices of $\operatorname{conv}(MIX^{DN})$. We prove that under this assumption the result of Section 2.3 yields an extended formulation of $\operatorname{conv}(MIX^{DN})$. We show that a complete list for a set of the type MIX^{DN} can always be exhibited, thus an extended formulation of our type can be constructed in all cases. We also show that if there is a complete list $\mathcal{F}$ that contains a polynomial number of elements, then the extended formulation is compact. This proves that linear optimization over sets of the type MIX^{DN} (or MIX^{2TU}) that have this property can be carried out in polynomial time through linear programming. This is in contrast to the $\mathcal{NP}$-completeness result mentioned above, which holds when the matrix A in (2.1) has at most two nonzero entries per *column*.

Finally in Section 2.5 we discuss a variant of the above approach which allows one to reduce the size of the extended formulation. Such a variant consists in using a different list of fractional parts $\mathcal{F}_i$ for each variable x_i rather than a single list $\mathcal{F}$ for all variables of the set. This reduces the number of variables and constraints of the extended formulation.

The results of this chapter are joint work with Michele Conforti, Friedrich Eisenbrand and Laurence A. Wolsey and are also summarized in [11].

2.1 Complexity

As recalled in Section 1.3.2, a linear system with totally unimodular matrix and integral right-hand side defines an integral polyhedron, i.e. a polyhedron which is the convex hull of its integral points. Thus optimization of a linear function over pure integer sets defined by systems of this type can be carried out in polynomial time by means of linear programming. It is then natural to wonder whether a similar result also holds in the mixed-integer case.

A result due to Eisenbrand [23] (which also appears in [11]) shows that the answer to the above question is negative (unless $\mathcal{P} = \mathcal{NP}$) even under some more restrictive assumptions.

Theorem 2.1 *[23, 11] The problem of deciding whether a mixed-integer set with totally unimodular constraint matrix contains a feasible point is $\mathcal{NP}$-complete, even if the constraint matrix has at most two nonzero entries per column and all components of the right-hand side vector are half-integer. In particular, it follows that linear optimization over such sets is an $\mathcal{NP}$-hard problem.*

The proof of the above theorem is via reduction to CNF-SAT.

Consider a mixed-integer set of the type

$$MIX^{TU} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\},$$

where A is a totally unimodular matrix and I is a nonempty subset of $\{1,\dots,n\}$. Let cx be an objective function to be minimized over MIX^{TU} and assume that we know the fractional parts $f_1,\dots,f_n$ of the components of an optimal solution. For $1 \leq i \leq n$, we introduce an integer variable μ_i that represents the integer part of x_i and we consider the mixed-integer set

$$\begin{aligned} x_i &= \mu_i + f_i, \quad 1 \leq i \leq n, \\ Ax &\geq b, \\ \mu_i &\text{ integer}, \quad 1 \leq i \leq n. \end{aligned}$$

The above constraints define a subset of MIX^{TU} (as $f_i = 0$ for $i \in I$) which contains an optimal solution of the minimization problem. Thus optimizing cx over this set is the same as optimizing over the original set. Furthermore, since we know that $f_1,\dots,f_n$ are the fractional parts of an optimal solution, we can equivalently minimize the function $c\mu$.

System $Ax \geq b$ can now be rewritten as $A\mu \geq b - Af$, which can be tightened to $A\mu \geq \lceil b - Af \rceil$, where $\lceil b - Af \rceil$ indicates the vector whose components are $\lceil b_j - (Af)_j \rceil$. We then obtain the system

$$\begin{aligned} x_i &= \mu_i + f_i, \qquad 1 \leq i \leq n, \\ A\mu &\geq \lceil b - Af \rceil, \\ \mu_i &\text{ integer}, \qquad 1 \leq i \leq n. \end{aligned}$$

Note that each variable x_i only appears in one equation, which determines its value. Since A is a totally unimodular matrix and the right-hand side $\lceil b - Af \rceil$ is an integral vector, by Theorem 1.13 we can drop the integrality constraints from the above system. The original minimization problem can now be solved by means of linear programming.

Together with Theorem 2.1, the above discussion shows that given an optimization problem of the form $\min\{cx : x \in MIX^{TU}\}$, finding the fractional parts of the components of any optimal solution is an $\mathcal{NP}$-hard problem (even if the constraint matrix contains at most two nonzero entries per column and all components of the right-hand side vector are half-integer).

2.2 Dual network matrices

We recall here some basic facts about the matrices that are the object of this study.

Given a network $\mathcal{N} = (V, A)$ with node set V and arc set A, the node-arc incidence matrix of $\mathcal{N}$ is the matrix $M = (m_{v,a} : v \in V, a \in A)$ defined by

$$m_{v,a} := \begin{cases} +1 & \text{if } v \text{ is the head of } a, \\ -1 & \text{if } v \text{ is the tail of } a, \\ 0 & \text{otherwise.} \end{cases}$$

Such a matrix has exactly two nonzero entries per column (one $+1$ and one -1). If we allow arcs having only one endpoint in the network (the other endpoint being a dummy node), matrix M has *at most* two nonzero entries per column, and each column with two nonzero entries contains one $+1$ and one -1. The matrices of this type are the constraint matrices of circulation problems on networks (this will be discussed in Section 5.1).

We say that a $0, \pm 1$-matrix A with at most two nonzero entries per row is a *dual network matrix* if each row of A having two nonzero entries contains one $+1$ and one -1. Thus dual network matrices are the transposes of the constraint matrices of circulation problems on networks.

In this chapter we study mixed-integer sets whose constraint matrix is totally unimodular and contains at most two nonzero entries per row. A matrix of this type can be converted into a dual network matrix by changing the sign of some of its columns. To see this, we first recall the following characterization, which is due to Heller and Tompkins [32], see e.g. Theorem 2.8 in [49].

Theorem 2.2 *Let A be a $0, \pm 1$-matrix with at most two nonzero entries per row, where $\{a_j : j \in N\}$ is the set of columns of A. Then A is totally unimodular if and only if the set N can be partitioned into two classes R, B such that all entries of the vector $\sum_{j \in R} a_j - \sum_{j \in B} a_j$ are $0, \pm 1$.*

This is a particular case of the characterization of totally unimodular matrices given by Ghouila-Houri [26] (see also Theorem 1.14). The condition in the above theorem can be stated this way: in every row of A with two nonzero elements, the nonzero entries have the same sign if and only if they belong to columns in distinct classes.

Corollary 2.3 *Every dual network matrix is totally unimodular.*

Proof. Just choose $R := N$ and $B := \varnothing$. □

Another well-known consequence of Theorem 2.2 is the following:

Corollary 2.4 *Let A be a $0, \pm 1$-matrix with at most two nonzero entries per row, where $\{a_j : j \in N\}$ is the set of columns of A. Then A is totally unimodular if and only if N contains a subset R such that the matrix A^R, obtained by multiplying by -1 the columns a_j for $j \in R$, is a dual network matrix.*

Proof. If A is a totally unimodular matrix with at most two nonzero entries per row, take a partition (R, B) of N satisfying the condition of Theorem 2.2. It is easily checked that then A^R is a dual network matrix.

For the converse, observe that if there is a subset $R \subseteq N$ such that A^R is a dual network matrix, then the partition (R, B), where $B := N \setminus R$, satisfies the condition of Theorem 2.2 and thus A is totally unimodular. □

2.3 Dual network systems and lists of fractional parts

The goal of this chapter is to provide an extended formulation for a set of the type MIX^{2TU}, i.e. a mixed-integer set whose constraint matrix is totally unimodular and contains at most two nonzero entries per row. To achieve this result, we first study subsets of $\mathbb{R}^n$ that are defined by a linear system with dual network matrix, with the additional restriction that all variables can only take a fractional part belonging to a given list.

Given a real number α, we write $f(\alpha)$ to denote the fractional part of α. Also, throughout this dissertation *fractional part* stands for any real number in the interval $[0, 1)$.

Define $N := \{1, \ldots, n\}$ and consider a general linear system with dual network matrix in the variables $x_1, \ldots, x_n$:

$$x_i - x_j \geq l_{ij}, \quad (i,j) \in N^e, \tag{2.2}$$
$$x_i \geq l_i, \quad i \in N^l, \tag{2.3}$$
$$x_i \leq u_i, \quad i \in N^u, \tag{2.4}$$

where $N^e \subseteq N \times N$ and $N^l, N^u \subseteq N$. The set N^e does not contain any pair of the type (i,i) for $i \in N$. The values l_{ij}, l_i, u_i are arbitrary real numbers. We remark that the above system may also include constraints of the type $x_i - x_j \leq u_{ij}$, as this inequality is equivalent to $x_j - x_i \geq l_{ij}$ for $l_{ij} := -u_{ij}$.

Suppose we are given a list of fractional parts $\mathcal{F} = \{f_1, \ldots, f_k\}$, with $f_1 > \cdots > f_k$, and let $K := \{1, \ldots, k\}$ be its set of indices. Let $X^{\mathcal{F}}$ be the set of points x satisfying inequalities (2.2)–(2.4) along with the additional condition that all variables take a fractional part in $\mathcal{F}$:

$$X^{\mathcal{F}} := \{x \in \mathbb{R}^n : x \text{ satisfies (2.2)–(2.4)},\ f(x_i) \in \mathcal{F} \text{ for } i \in N\}.$$

That is, $X^{\mathcal{F}}$ is the set of points $x \in \mathbb{R}^n$ such that there exist μ^i, δ^i_ℓ, for $i \in N$ and $\ell \in K$, satisfying the following constraints:

$$x_i = \mu^i + \textstyle\sum_{\ell=1}^k f_\ell \delta^i_\ell, \quad i \in N, \tag{2.5}$$
$$\textstyle\sum_{\ell=1}^k \delta^i_\ell = 1,\ \delta^i_\ell \geq 0, \quad i \in N, \ell \in K, \tag{2.6}$$
$$x_i - x_j \geq l_{ij}, \quad (i,j) \in N^e, \tag{2.7}$$
$$x_i \geq l_i, \quad i \in N^l, \tag{2.8}$$
$$x_i \leq u_i, \quad i \in N^u, \tag{2.9}$$
$$\mu^i, \delta^i_\ell \text{ integer}, \quad i \in N, \ell \in K. \tag{2.10}$$

In other words, $X^{\mathcal{F}}$ is the projection of the mixed-integer set (2.5)–(2.10) onto the x-space of variables. In the remainder of this section we give an extended formulation of the polyhedron $\operatorname{conv}(X^{\mathcal{F}})$.

Consider the following transformation:

$$\mu^i_0 := \mu^i, \ \ \mu^i_\ell := \mu^i + \sum_{j=1}^{\ell} \delta^i_j \ \text{ for } i \in N \text{ and } \ell \in K. \tag{2.11}$$

Since the above is a unimodular transformation (see e.g. [38]), we can equivalently study the transformed of (2.5)–(2.10) under (2.11).

Define $f_0 := 1$ and $f_{k+1} := 0$. For fixed $i \in N$, under transformation (2.11) an equation in (2.5) becomes

$$x_i = \sum_{\ell=0}^{k} (f_\ell - f_{\ell+1})\mu_\ell^i \tag{2.12}$$

and the $k+1$ constraints in (2.6) become

$$\mu_k^i - \mu_0^i = 1, \;\; \mu_\ell^i - \mu_{\ell-1}^i \geq 0 \text{ for } \ell \in K. \tag{2.13}$$

In the following we strengthen constraints (2.7)–(2.9). Consider first an inequality of the type $x_i \leq l_i$ with $i \in N^l$. Let $\ell(l_i)$ be the highest index $\ell \in \{0, \ldots, k\}$ such that $f_\ell \geq f(l_i)$.

Lemma 2.5 *Assume that x_i, δ_ℓ^i and μ_ℓ^i for $\ell \in K$ satisfy (2.5), (2.6), (2.10) and (2.11). Then $x_i \geq l_i$ if and only if*

$$\mu_{\ell(l_i)}^i \geq \lfloor l_i \rfloor + 1. \tag{2.14}$$

Proof. The result can be checked directly. We show here that inequality (2.14) can be obtained through the Chvátal-Gomory procedure (see Theorem 1.10).

By equation (2.5), inequality $x_i \geq l_i$ is equivalent to $\mu^i + \sum_{\ell=1}^{k} f_\ell \delta_\ell^i \geq l_i$. For $\varepsilon > 0$ small enough, combining such inequality with equation

$$-(f(l_i) - \varepsilon) \sum_{\ell=1}^{k} \delta_\ell^i = -(f(l_i) - \varepsilon)$$

(which holds by (2.6)) and with the nonnegativity of the δ_ℓ^i, and then applying Chvátal-Gomory rounding, gives inequality $\mu^i + \sum_{\ell \leq \ell(l_i)} \delta_\ell^i \geq \lfloor l_i \rfloor + 1$, which is equivalent to (2.14). □

For $i \in N^u$, let $\ell'(u_i)$ be the highest index $\ell \in \{0, \ldots, k\}$ such that $f_\ell > f(u_i)$.

Lemma 2.6 *Assume that x_i, δ_ℓ^i and μ_ℓ^i for $\ell \in K$ satisfy (2.5), (2.6), (2.10) and (2.11). Then $x_i \leq u_i$ if and only if*

$$\mu_{\ell'(u_i)}^i \leq \lfloor u_i \rfloor. \tag{2.15}$$

Proof. The proof is similar to that of Lemma 2.5, with $\varepsilon = 0$. □

We now consider an inequality of the type $x_i - x_j \geq l_{ij}$ for $(i,j) \in N^e$. Define k_{ij} to be the highest index $\ell \in \{0, \ldots, k\}$ such that $f_\ell + f(l_{ij}) \geq 1$. Given an index $t \in K$, define t'_{ij} to be the highest index $\ell \in \{0, \ldots, k\}$ such that $f_\ell \geq f(f_t + f(l_{ij}))$.

Lemma 2.7 *Assume that x_i, x_j, δ_ℓ^i, δ_ℓ^j, μ_ℓ^i, μ_ℓ^j for $\ell \in K$ satisfy (2.5), (2.6), (2.10) and (2.11). Then $x_i - x_j \geq l_{ij}$ if and only if the following inequalities are satisfied:*

$$\mu_{t'_{ij}}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor + 1, \quad 1 \leq t \leq k_{ij}, \tag{2.16}$$

$$\mu_{t'_{ij}}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor, \qquad k_{ij} < t \leq k. \tag{2.17}$$

Proof. Substituting for x_j using equation (2.5), inequality $x_i - x_j \geq l_{ij}$ becomes

$$x_i \geq \mu^j + \sum_{\ell=1}^{k} f_\ell \delta_\ell^j + \lfloor l_{ij} \rfloor + f(l_{ij}). \tag{2.18}$$

First we show that inequality (2.17) is valid for $t > k_{ij}$. As $\sum_{\ell=1}^{k} f_\ell \delta_\ell^j \geq \sum_{\ell \leq t} f_\ell \delta_\ell^j \geq f_t \sum_{\ell \leq t} \delta_\ell^j$, we obtain from (2.18) the following valid inequality:

$$x_i \geq \mu^j + f_t \sum_{\ell \leq t} \delta_\ell^j + \lfloor l_{ij} \rfloor + f(l_{ij}).$$

Adding the valid inequality $(1 - f_t) \geq (1 - f_t) \sum_{\ell \leq t} \delta_\ell^j$ and isolating x_i gives

$$x_i \geq \mu^j + \sum_{\ell \leq t} \delta_\ell^j + \lfloor l_{ij} \rfloor + f(l_{ij}) - 1 + f_t. \tag{2.19}$$

Let β be the right-hand side of the above inequality. We now strengthen inequality $x_i \geq \beta$ by using Lemma 2.5. For this purpose, we observe that condition $t > k_{ij}$ implies $f_t + f(l_{ij}) < 1$, so $\lfloor \beta \rfloor = \mu^j + \sum_{\ell \leq t} \delta_\ell^j + \lfloor l_{ij} \rfloor - 1 = \mu_t^j + \lfloor l_{ij} \rfloor - 1$. Also $f(\beta) = f(f_t + f(l_{ij}))$, thus Lemma 2.5 yields the valid inequality $\mu_{t'_{ij}}^i \geq \mu_t^j + \lfloor l_{ij} \rfloor$ and the validity of (2.17) is proven.

The argument when $t \leq k_{ij}$ is the same, except that $f_t + f(l_{ij}) \geq 1$.

To establish the converse, let $t \in K$ be the index such that $\delta_t^j = 1$. Then $\mu_t^j = \mu_0^j + 1$, $\mu_{t-1}^j = \mu_0^j$ and $x_j = \mu_0^j + f_t$. Inequality $\mu_{t'_{ij}}^i \geq \mu_t^j + \lfloor l_{ij} \rfloor$ implies that either $\mu_0^i \geq \mu_0^j + 1 + \lfloor l_{ij} \rfloor$, or $\mu_0^i = \mu_0^j + \lfloor l_{ij} \rfloor$ and $\sum_{\ell \leq t'_{ij}} \delta_\ell^i = 1$. In both cases, this implies that $x_i \geq \mu_0^j + \lfloor l_{ij} \rfloor + f_{t'_{ij}}$. Now, assuming $t > k_{ij}$,

$$\begin{aligned} x_i - x_j &\geq \mu_0^j + \lfloor l_{ij} \rfloor + f_{t'_{ij}} - \mu_0^j - f_t \\ &= \lfloor l_{ij} \rfloor + f_{t'_{ij}} - f_t \\ &\geq \lfloor l_{ij} \rfloor + f(l_{ij}), \end{aligned}$$

as $f_{t'_{ij}} \geq f(f_t + f(l_{ij}))$ and $f_t + f(l_{ij}) < 1$. Again the other case with $t \leq k_{ij}$ is similar. □

We can now give an extended formulation of $\text{conv}(X^{\mathcal{F}})$. For this purpose, let $Q^{\mathcal{F}}$ be the polyhedron in the space of the variables $(x_i, \mu_\ell^i : i \in N, \ell \in K \cup \{0\})$ defined by the inequalities (2.12), (2.13), (2.14), (2.15) and (2.16)–(2.17):

$$\begin{aligned}
x_i = \textstyle\sum_{\ell=0}^{k} (f_\ell - f_{\ell+1}) \mu_\ell^i, &\quad i \in N, && (2.20)\\
\mu_k^i - \mu_0^i = 1,\ \mu_\ell^i - \mu_{\ell-1}^i \geq 0, &\quad i \in N,\ \ell \in K, && (2.21)\\
\mu_{\ell(l_i)}^i \geq \lfloor l_i \rfloor + 1, &\quad i \in N^l, && (2.22)\\
\mu_{\ell'(u_i)}^i \leq \lfloor u_i \rfloor, &\quad i \in N^u, && (2.23)\\
\mu_{t'_{ij}}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor + 1, &\quad i \in N^e,\ 1 \leq t \leq k_{ij}, && (2.24)\\
\mu_{t'_{ij}}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor, &\quad i \in N^e,\ k_{ij} < t \leq k. && (2.25)
\end{aligned}$$

Theorem 2.8 *The polyhedron* $\operatorname{conv}\left(X^{\mathcal{F}}\right)$ *is the projection of the polyhedron* $Q^{\mathcal{F}}$ *onto the space of the* x*-variables.*

Proof. Recall that $X^{\mathcal{F}}$ is the projection onto the x-space of the mixed-integer set (2.5)–(2.10), which, as the above discussion shows, is equivalent to the mixed-integer set

$$\left\{(x, \mu) \in Q^{\mathcal{F}} : \mu \text{ is integral}\right\}. \tag{2.26}$$

Therefore $\operatorname{conv}\left(X^{\mathcal{F}}\right)$ is the projection of the convex hull of (2.26) onto the x-space of variables. We then have to show that such a convex hull is given by inequalities (2.20)–(2.25).

Since, for $i \in N$, variable x_i is determined by the corresponding equation (2.20) (and this variable does not appear in any other constraints), we only need to show that the polyhedron defined by inequalities (2.21)–(2.25) is integral.

Let A_μ be the constraint matrix of the above system. By construction, A_μ is a dual network matrix. Since dual network matrices are totally unimodular (see Theorem 2.3) and the right-hand sides of the above inequalities are all integer, the statement follows from Theorem 1.13. □

2.4 Complete lists of fractional parts

We use the results of the previous section to construct an extended formulation of a set of the type MIX^{2TU}, i.e. a mixed-integer set whose constraint matrix is totally unimodular and contains at most two nonzero entries per row. For this purpose, we now introduce the concept of compete list of fractional parts for an arbitrary mixed-integer set.

Let $X := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$ be a mixed-integer set, where $(A \mid b)$ is an arbitrary matrix and I is a nonempty subset of the set of column indices of A. A list $\mathcal{F} = \{f_1, \ldots, f_k\}$ of fractional parts is *complete* for X if the following property is satisfied:

$$\begin{gathered}\textit{Every minimal face of } \operatorname{conv}(X) \textit{ contains a point } \bar{x} \textit{ such that} \\ f(\bar{x}_i) \in \mathcal{F} \textit{ for each } i \in N, \textit{ and } f(\bar{x}_i) = 0 \textit{ for each } i \in I.\end{gathered} \tag{2.27}$$

In our applications (Chapters 4–5), minimal faces are vertices and the above condition becomes:

$$\textit{If } \bar{x} \textit{ is a vertex of } \operatorname{conv}(X), \textit{ then } f(\bar{x}_i) \in \mathcal{F} \textit{ for each } i \in N,$$

as every vertex $\bar{x}$ of $\operatorname{conv}(X)$ certainly satisfies $f(\bar{x}_i) = 0$ for all $i \in I$. However, for the sake of generality we do not assume here that minimal faces are vertices.

We now consider a mixed-integer set

$$MIX^{DN} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\},$$

where A is a dual network matrix. That is, the system $Ax \geq b$ consists of inequalities of type (2.2)–(2.4). We sometimes call a set of this type a *dual network set*.

We assume that we are given a list of fractional parts $\mathcal{F} = \{f_1, \dots, f_k\}$, with $f_1 > \cdots > f_k$, which is complete for MIX^{DN}. Note that since I is nonempty, $\mathcal{F}$ must include the value 0, thus $f_k = 0$.

We first give an extended formulation of the polyhedron $\text{conv}(MIX^{DN})$ and then show how this easily leads to an extended formulation for the convex hull of a more general set MIX^{2TU}.

In order to obtain an extended formulation of $\text{conv}(MIX^{DN})$, we consider the following mixed-integer set:

$$
\begin{aligned}
x_i = \mu^i + \textstyle\sum_{\ell=1}^k f_\ell \delta_\ell^i, \quad & i \in N, && (2.28)\\
\textstyle\sum_{\ell=1}^k \delta_\ell^i = 1,\ \delta_\ell^i \geq 0, \quad & i \in N, \ell \in K, && (2.29)\\
\delta_k^i = 1, \quad & i \in I, && (2.30)\\
x_i - x_j \geq l_{ij}, \quad & (i,j) \in N^e, && (2.31)\\
x_i \geq l_i, \quad & i \in N^l, && (2.32)\\
x_i \leq u_i, \quad & i \in N^u, && (2.33)\\
\mu^i, \delta_\ell^i \text{ integer}, \quad & i \in N, \ell \in K, && (2.34)
\end{aligned}
$$

where inequalities (2.31)–(2.33) constitute the system $Ax \geq b$.

Let $MIX^{\mathcal{F}}$ be the set of points $x \in \mathbb{R}^n$ such that there exist μ^i, δ_ℓ^i, for $i \in N$ and $\ell \in K$, satisfying constraints (2.28)–(2.34). Note that equations (2.30) force variables x_i for $i \in I$ to be integer valued in $MIX^{\mathcal{F}}$.

Lemma 2.9 $\text{conv}(MIX^{DN}) = \text{conv}(MIX^{\mathcal{F}})$.

Proof. If $\bar{x} \in MIX^{\mathcal{F}}$ then $\bar{x}$ satisfies the system $Ax \geq b$ (i.e. inequalities (2.31)–(2.33)). Furthermore equations (2.30) force x_i for $i \in I$ to take an integer value. So $\bar{x} \in MIX^{DN}$. This shows that $MIX^{\mathcal{F}} \subseteq MIX^{DN}$ and therefore $\text{conv}(MIX^{\mathcal{F}}) \subseteq \text{conv}(MIX^{DN})$.

To prove the reverse inclusion, we show that all rays and minimal faces of $\text{conv}(MIX^{DN})$ belong to $\text{conv}(MIX^{\mathcal{F}})$. Recall that since the constraint matrix of the system $Ax \geq b$ is rational, the extreme rays of $\text{conv}(MIX^{DN})$ and $\text{conv}(MIX^{\mathcal{F}})$ coincide with those of their linear relaxations (see Theorem 1.8). Now, if $\bar{x}$ is a ray of $\text{conv}(MIX^{DN})$, the vector defined by

$$x_i := \bar{x}_i,\ \ \mu_i := \bar{x}_i,\ \ \delta_\ell^i := 0 \quad \text{for } i \in N \text{ and } \ell \in K$$

is a ray of the polyhedron that is the convex hull of (2.28)–(2.34). This implies that $\bar{x}$ is a ray of $\text{conv}(MIX^{\mathcal{F}})$.

Since the list $\mathcal{F}$ is complete, every minimal face F of $\text{conv}(MIX^{DN})$ contains a point $\bar{x} \in MIX^{\mathcal{F}}$. Furthermore F is an affine subspace which can be expressed as $\{x \in \mathbb{R}^n : x = \bar{x} + \sum_{t=1}^h \lambda_t r_t,\ \lambda_t \in \mathbb{R}\}$ for some subset of rays $r_1, \dots, r_h$ of $\text{conv}(MIX^{DN})$. Since $\bar{x} \in MIX^{\mathcal{F}}$ and $r_1, \dots, r_h$ are all rays of $\text{conv}(MIX^{\mathcal{F}})$, then $F \subseteq \text{conv}(MIX^{\mathcal{F}})$. □

As shown in Section 2.3, by applying the unimodular transformation (2.11) inequalities (2.28)–(2.29) become inequalities (2.20)–(2.21), while (2.31)–(2.33) become (2.22)–(2.25).

Let Q be the polyhedron in the space of the variables $\big(x_i, \mu^i_\ell : i \in N, \ell \in K \cup \{0\}\big)$ defined by inequalities (2.20)–(2.25), which correspond to inequalities (2.28), (2.29), (2.31), (2.32), (2.33) under transformation (2.11), and let Q^I be the face of Q defined by equations

$$\mu^i_k - \mu^i_{k-1} = 1, \;\; i \in I, \tag{2.35}$$

which are equivalent to equations (2.30) under transformation (2.11). More explicitly, Q_I is the polyhedron defined by the following linear system:

$$\begin{aligned}
x_i = \textstyle\sum_{\ell=0}^{k}(f_\ell - f_{\ell+1})\mu^i_\ell, &\qquad i \in N, &(2.36)\\
\mu^i_k - \mu^i_0 = 1,\; \mu^i_\ell - \mu^i_{\ell-1} \geq 0, &\qquad i \in N,\, \ell \in K, &(2.37)\\
\mu^i_k - \mu^i_{k-1} = 1, &\qquad i \in I, &(2.38)\\
\mu^i_{\ell(l_i)} \geq \lfloor l_i \rfloor + 1, &\qquad i \in N^l, &(2.39)\\
\mu^i_{\ell'(u_i)} \leq \lfloor u_i \rfloor, &\qquad i \in N^u, &(2.40)\\
\mu^i_{t'_{ij}} - \mu^j_t \geq \lfloor l_{ij} \rfloor + 1, &\qquad i \in N^e,\, 1 \leq t \leq k_{ij}, &(2.41)\\
\mu^i_{t'_{ij}} - \mu^j_t \geq \lfloor l_{ij} \rfloor, &\qquad i \in N^e,\, k_{ij} < t \leq k. &(2.42)
\end{aligned}$$

Theorem 2.10 *The polyhedron* $\mathrm{conv}\big(MIX^{DN}\big)$ *is the projection of the face* Q^I *of* Q *onto the space of the* x*-variables. In other words, the linear system* (2.36)–(2.42) *is an extended formulation of* $\mathrm{conv}\big(MIX^{DN}\big)$.

Proof. Theorem 2.8 shows that every minimal face of Q contains a vector $(\bar{x}, \bar{\mu})$ with integral $\bar{\mu}$. So the same holds for Q^I, which is a face of Q. By applying the transformation that is the inverse of (2.11), this shows that every minimal face of the polyhedron defined by (2.28)–(2.33) contains a point $(\bar{x}, \bar{\mu}, \bar{\delta})$ where $(\bar{\mu}, \bar{\delta})$ is integral. So the projection of this polyhedron onto the x-space coincides with $\mathrm{conv}\big(MIX^{\mathcal{F}}\big)$ and by Lemma 2.9 we are done. □

We now consider a more general mixed-integer set of the type $MIX^{2TU} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$, where A is a totally unimodular matrix with at most two nonzero entries per row. By Corollary 2.4, A can be transformed into a dual network matrix by changing the sign of some of its columns. Then MIX^{2TU} is transformed into a set of the type MIX^{DN}. Note that if $\mathcal{F} = \{f_1, \dots, f_k\}$ is a list of fractional parts which is complete for MIX^{2TU}, then the list $\mathcal{F}' := \{f_\ell, 1 - f_\ell : 1 \leq \ell \leq k-1\} \cup \{0\}$ is complete for the transformed set MIX^{DN}. This shows that an extended formulation of MIX^{2TU} can be easily obtained from the extended formulation of the corresponding set MIX^{DN}. We also remark that the list $\mathcal{F}'$ contains at most the double of the number of elements in $\mathcal{F}$.

2.4.1 An explicit complete list of fractional parts

Clearly an extended formulation of the type (2.36)–(2.42) can be derived only if a complete list of fractional parts is known for the set. However, the following result holds:

Lemma 2.11 *Let* $X := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$ *be a mixed-integer set, where* A *is an* $m \times n$ *totally unimodular matrix,* $b \in \mathbb{R}^m$ *and* $I \subseteq \{1, \ldots, n\}$*. Then every minimal face of* $\operatorname{conv}(X)$ *contains a point* $\bar{x} \in X$ *such that*

$$f(\bar{x}_i) = f\left(\sum_{j=1}^m \sigma_{ij} b_j\right) \quad \text{for } i \notin I, \tag{2.43}$$

where $\sigma_{ij} \in \{0, \pm 1\}$ *for all* $i \notin I$ *and* $1 \leq j \leq m$*.*

Proof. Let F be a minimal face of $\operatorname{conv}(X)$ and pick any point $\hat{x} \in F \cap X$. Define the nonempty polyhedron

$$P := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i = \hat{x}_i \text{ for } i \in I\}.$$

Let G be a minimal face of P. Then G is an affine variety in $\mathbb{R}^n$. Let d denote the dimension of G. Since the equations $\bar{x}_i = \hat{x}_i$ for $i \in I$ are linearly independent, then $d \leq n - |I|$ and there exists a subsystem $A'x \geq b'$ of $Ax \geq b$ with $n - |I| - d$ rows such that the $n - d$ equations

$$A'x = b',\ \ x_i = \hat{x}_i \text{ for } i \in I$$

are linearly independent and define G.

By standard linear algebra, there is a subset $J \subseteq \{1, \ldots, n\} \setminus I$, with $|J| = d$, such that the n equations of the system

$$A'x = b',\ \ x_i = \hat{x}_i \text{ for } i \in I,\ \ x_i = 0 \text{ for } i \in J \tag{2.44}$$

are linearly independent.

Let $\bar{x}$ be the unique solution to system (2.44). Since $\bar{x} \in G$ and $\bar{x}_i = \hat{x}_i \in \mathbb{Z}$ for $i \in I$, then $\bar{x}$ belongs to X. We now prove that $\bar{x}$ satisfies conditions (2.43).

Since A is a totally unimodular, the constraint matrix of system (2.44) is totally unimodular as well. Equation (2.43) then follows from the observation that the inverse of a nonsingular totally unimodular matrix is a $0, \pm 1$-matrix. □

Lemma 2.11 is useful for at least two reasons. First, it provides an explicit (though long) list of fractional parts which is guaranteed to be complete for the set, thus showing that an extended formulation of the type (2.36)–(2.42) can be explicitly given for the convex hull of an arbitrary set MIX^{DN}. We will show in Chapter 3 that such a huge list can sometimes be shortened.

To illustrate the second reason why the above lemma is useful, observe that the size of formulation (2.36)–(2.42) depends not only on the number of variables and constraints of the original system $Ax \geq b$, but on the size of the list $\mathcal{F}$ too. The size of $\mathcal{F}$ in turn depends on two elements: the number k of fractional parts that it contains and the size of such fractional parts. However Lemma 2.11 shows that one can assume without loss of generality that the fractional parts of a complete list $\mathcal{F}$ are all of the form $f\big(\sum_{j=1}^m \sigma_j b_j\big)$ for $\sigma_j \in \{0, \pm 1\}$, where m is the dimension of b. Observe that the size of a number of this type is bounded by a polynomial function of the size of vector b (assuming that b has rational components). Thus from now on, when considering the size of a list of fractional parts, we will only take into account its length (i.e. cardinality) k.

We remark that the latter consideration implies the following immediate consequence of Theorem 2.10:

Corollary 2.12 *If a mixed-integer set of the type MIX^{2TU} (with rational right-hand side) admits a complete list of fractional parts $\mathcal{F}$ whose length k is polynomial in the size of its description (given by the system $Ax \geq b$), the extended formulation (2.36)–(2.42) of the corresponding set $\text{conv}(MIX^{DN})$ is compact: it uses $\mathcal{O}(nk)$ variables and $\mathcal{O}((n + |N^e|)k)$ constraints. Therefore the problem of optimizing a linear function over sets of the type MIX^{2TU} with this property can be solved in polynomial time.*

2.4.2 A different approach?

As observed above, a list including all values of the form $f\left(\sum_{j=1}^m \sigma_j b_j\right)$ for $\sigma_j \in \{0, \pm 1\}$ is always complete. Unfortunately such a list has (in general) an exponential number of elements. We will see in Chapter 3 that in fact there exist mixed-integer sets with dual network constraint matrix that do not admit a complete list of compact size.

In order to obtain a compact extended formulation of a set MIX^{2TU} even if there is no complete list for the set having compact size, one could try to modify the approach described in the previous sections by modeling the variables of the problem in a differen way, e.g.

$$x_i = \mu^i + \textstyle\sum_{\ell=1}^m f(b_\ell)\delta_\ell^i, \quad i \in N, \tag{2.45}$$

$$\mu^i \text{ integer}, \; \delta_\ell^i \in \{0, \pm 1\}, \quad i \in N, \, 1 \leq \ell \leq m. \tag{2.46}$$

By the above observation, every minimal face of $\text{conv}\left(MIX^{2TU}\right)$ contains a point x that satisfies the above conditions for some μ^i, δ_ℓ^i. Note that for each $i \in N$, only $m + 1$ additional variables are used.

Unfortunately tightening the inequalities defining MIX^{2TU} under the above conditions seems to be hard. To demonstrate this, assume that some variable x is defined by the conditions

$$x = \mu + 0.9\delta_1 + 0.5\delta_2 + 0.3\delta_3, \tag{2.47}$$

$$\mu \text{ integer}, \; \delta_1, \delta_2, \delta_3 \in \{0, \pm 1\}. \tag{2.48}$$

Suppose that one of the constraints describing MIX^{2TU} is inequality $x \geq 0$. It can be checked (we did so by using PORTA [9]) that a linear inequality description of the set of points (x, μ, δ) satisfying (2.47)–(2.48) and $x \geq 0$ is given by the following constraints:

$$\begin{aligned}
\mu + \delta_1 \qquad\qquad &\geq 0, \\
\mu + \delta_1 + \delta_2 \qquad &\geq 0, \\
2\mu + 2\delta_1 + \delta_2 + \delta_3 &\geq 0, \\
4\mu + 3\delta_1 + 2\delta_2 + \delta_3 &\geq 0, \\
-1 \leq \delta_1, \delta_2, \delta_3 &\leq 1.
\end{aligned}$$

When considering the systems originating from similar examples, we could not see any particular structure that could lead us to characterize the convex hull of the integral points. This is not surprising: for instance, modeling $x \geq 0$ under conditions (2.47)–(2.48) amounts to finding the convex hull of the following integer knapsack set:

$$10\mu + 9\delta_1 + 5\delta_2 + 3\delta_3 \geq 0,$$
$$\mu \text{ integer}, \ \delta_1, \delta_2, \delta_3 \in \{0, \pm 1\}.$$

It is well-known that problems of this type are hard. Furthermore, if two or more constraints —instead of a single inequality— are considered, tightening each inequality separately does not give (in general) the convex hull of the mixed-integer set. This suggests that it is unlikely to find a straightforward modification of our approach that uses the modeling conditions (2.45)–(2.46).

Note that conditions $\delta_\ell^i \in \{0, \pm 1\}$ in (2.46) could be replaced with conditions $\delta_\ell^i \in \mathbb{Z}$. In this case the strengthening of a single inequality is easy: after transforming all coefficients into coprime integers by multiplying the inequality by a suitable number (provided that all coefficients are rational), it is sufficient to round up the right-hand side. However, when there are two or more constraints, tightening each inequality separately does not give (in general) the convex hull of the mixed-integer set.

Finding a compact extended formulation of the convex hull of a set MIX^{2TU} that does not admit a "short" list of fractional parts is an open problem.

2.5 Specific lists of fractional parts

We discuss here a simple variant of the results presented in Sections 2.3–2.4. Such a variant allows us to reduce the size of the extended formulation given by Theorem 2.10 and will be useful in Chapter 5, where for some special sets we compute explicitly the projection of the extended formulation onto the original space of variables.

2.5.1 A more compact extended formulation

In Section 2.3 we considered a system of inequalities of the form (2.2)–(2.4) and a list $\mathcal{F}$ of fractional parts, and we gave an extended formulation of the polyhedron which is the convex hull of the set of points x satisfying (2.2)–(2.4) along with the additional condition that $f(x_i) \in \mathcal{F}$ for all $i \in N$.

Now assume that instead of a single list $\mathcal{F}$, we are given a (possibly) different list of fractional part $\mathcal{F}_i$ for each $i \in N$. We assume $\mathcal{F}_i = \{f_1^i, \dots, f_{k_i}^i\}$, with $f_1^i > \cdots > f_{k_i}^i$, and set $K_i := \{1, \dots, k_i\}$. We define $X^{\mathcal{F}}$ as the set of points x satisfying the linear system (2.2)–(2.4) along with the additional condition that $f(x_i) \in \mathcal{F}_i$ for all $i \in N$. That is, $X^{\mathcal{F}}$ is the set of points $x \in \mathbb{R}^n$ such that there exist μ^i, δ_ℓ^i, for $i \in N$ and $\ell \in K_i$, satisfying the following constraints:

$$x_i = \mu^i + \textstyle\sum_{\ell=1}^{k_i} f_\ell^i \delta_\ell^i, \quad i \in N, \tag{2.49}$$
$$\textstyle\sum_{\ell=1}^{k_i} \delta_\ell^i = 1,\ \delta_\ell^i \geq 0, \quad i \in N, \ell \in K_i, \tag{2.50}$$
$$x_i - x_j \geq l_{ij}, \quad (i,j) \in N^e, \tag{2.51}$$
$$x_i \geq l_i, \quad i \in N^l, \tag{2.52}$$
$$x_i \leq u_i, \quad i \in N^u, \tag{2.53}$$
$$\mu^i, \delta_\ell^i \text{ integer}, \quad i \in N, \ell \in K_i. \tag{2.54}$$

Similarly to Section 2.3, $X^{\mathcal{F}}$ is the projection of the mixed-integer set (2.49)–(2.54) onto the x-space.

An extended formulation of $\operatorname{conv}(X^{\mathcal{F}})$ can be found as in Section 2.3, with just some slight changes. We summarize the construction of the extended formulation below; the details and the proofs are perfectly analogous to those of Section 2.3.

First of all, we define a unimodular transformation which is identical to transformation (2.11), except that now K has to be replaced with K_i:

$$\mu_0^i := \mu^i, \ \ \mu_\ell^i := \mu^i + \sum_{j=1}^{\ell} \delta_j^i \ \text{ for } i \in N \text{ and } \ell \in K_i. \tag{2.55}$$

Similarly, after setting $f_0^i := 1$ and $f_{k_i+1}^i := 0$ for all $i \in N$, constraints (2.49)–(2.50) transform into constraints that are almost identical to (2.12)–(2.13):

$$x_i = \textstyle\sum_{\ell=0}^{k_i} (f_\ell^i - f_{\ell+1}^i)\mu_\ell^i,$$
$$\mu_{k_i}^i - \mu_0^i = 1,\ \mu_\ell^i - \mu_{\ell-1}^i \geq 0, \quad \ell \in K_i.$$

For $i \in N^l$, inequality $x_i \geq l_i$ can be modeled as $\mu^i_{\ell_i(l_i)} \geq \lfloor l_i \rfloor + 1$, where $\ell_i(l_i)$ is the highest index $\ell \in \{0, \dots, k_i\}$ such that $f_\ell \geq f(l_i)$. For $i \in N^u$, inequality $x_i \leq u_i$ can be modeled as $\mu^i_{\ell'_i(u_i)} \leq \lfloor u_i \rfloor$, where $\ell'_i(u_i)$ is the highest index $\ell \in \{0, \dots, k_i\}$ such that $f_\ell > f(u_i)$.

Finally, to model inequality $x_i - x_j \geq l_{ij}$ for $(i,j) \in N^e$, we define k_{ij} to be the highest index $\ell \in \{0, \dots, k_j\}$ such that $f_\ell^j + f(l_{ij}) \geq 1$. Given an index $t \in K_j$, define t'_{ij} to be the highest index $\ell \in \{0, \dots, k_i\}$ such that $f_\ell^i \geq f\big(f_t^j + f(l_{ij})\big)$. Now a result almost identical to Lemma 2.7 (just replace k with k_i in (2.17)) can be proven exactly as in Section 2.3.

With a proof that is identical to that of Theorem 2.8 one can prove that an extended formulation of $\operatorname{conv}(X^{\mathcal{F}})$ is given by the following linear system:

$$x_i = \textstyle\sum_{\ell=0}^{k_i} (f_\ell^i - f_{\ell+1}^i)\mu_\ell^i, \quad i \in N, \tag{2.56}$$
$$\mu_k^i - \mu_0^i = 1,\ \mu_\ell^i - \mu_{\ell-1}^i \geq 0, \quad i \in N,\ \ell \in K_i, \tag{2.57}$$
$$\mu^i_{\ell_i(l_i)} \geq \lfloor l_i \rfloor + 1, \quad i \in N^l, \tag{2.58}$$
$$\mu^i_{\ell'_i(u_i)} \leq \lfloor u_i \rfloor, \quad i \in N^u, \tag{2.59}$$
$$\mu^i_{t'_{ij}} - \mu^j_t \geq \lfloor l_{ij} \rfloor + 1, \quad i \in N^e,\ 1 \leq t \leq k_{ij}, \tag{2.60}$$
$$\mu^i_{t'_{ij}} - \mu^j_t \geq \lfloor l_{ij} \rfloor, \quad i \in N^e,\ k_{ij} < t \leq k_j. \tag{2.61}$$

We now extend the definition of complete list given in Section 2.4. For $i \in N$, a list $\mathcal{F}_i = \{f_1^i, \dots, f_k^i\}$ of fractional parts is *complete* for X *with respect to variable* x_i if the following property is satisfied:

Every minimal face F of $\mathrm{conv}(X)$ *contains a point $\bar{x}$ such that*
$f(\bar{x}_i) \in \mathcal{F}_i$ for each $i \in N$, and $f(\bar{x}_i) = 0$ for each $i \in I$.

When $\mathrm{conv}(X)$ is a pointed polyhedron, the above definition reads as follows:

If $\bar{x}$ is a vertex of $\mathrm{conv}(X)$*, then $f(\bar{x}_i) \in \mathcal{F}_i$ for each $i \in N$.*

Let $MIX^{DN} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$ be a mixed-integer set with dual network constraint matrix A. If for each $i \in N$ we are given a list of fractional parts $\mathcal{F}_i = \{f_1^i, \dots, f_{k_i}^i\}$ which is complete for MIX^{DN} with respect to variable x_i, one can repeat the process of Section 2.4 and prove the following result (as usual, $f_1^i > \cdots > f_{k_i}^i$):

Theorem 2.13 *The following linear system is an extended formulation of the polyhedron* $\mathrm{conv}(MIX^{DN})$*:*

$$
\begin{aligned}
x_i = \textstyle\sum_{\ell=0}^{k_i}(f_\ell^i - f_{\ell+1}^i)\mu_\ell^i, &\qquad i \in N, &(2.62)\\
\mu_{k_i}^i - \mu_0^i = 1,\ \mu_\ell^i - \mu_{\ell-1}^i \geq 0, &\qquad i \in N,\ \ell \in K_i, &(2.63)\\
\mu_{k_i}^i - \mu_{k-1}^i = 1, &\qquad i \in I, &(2.64)\\
\mu_{\ell_i(l_i)}^i \geq \lfloor l_i \rfloor + 1, &\qquad i \in N^l, &(2.65)\\
\mu_{\ell_i'(u_i)}^i \leq \lfloor u_i \rfloor, &\qquad i \in N^u, &(2.66)\\
\mu_{t_{ij}'}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor + 1, &\qquad i \in N^e,\ 1 \leq t \leq k_{ij}, &(2.67)\\
\mu_{t_{ij}'}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor, &\qquad i \in N^e,\ k_{ij} < t \leq k_j. &(2.68)
\end{aligned}
$$

The extension to a set of the type MIX^{2TU} can be done as in Section 2.4.

Corollary 2.14 *Given a mixed-integer set of the type MIX^{2TU} (with rational right-hand side), let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be lists of fractional parts which are complete for MIX^{2TU} with respect to variables $x_1, \dots, x_n$ respectively. Define $\bar{k} := \max_{1 \leq i \leq n} |\mathcal{F}_i|$. Then the extended formulation (2.62)–(2.68) of the corresponding set MIX^{DN} uses $\mathcal{O}(n\bar{k})$ variables and $\mathcal{O}((n + |N^e|)\bar{k})$ constraints.*

Let us compare Corollaries 2.12 and 2.14. Let $\mathcal{F}$ be a list of fractional parts which is complete for MIX^{2TU} and whose length is minimum. Similarly, let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be lists of fractional parts which are complete for MIX^{2TU} with respect to variables $x_1, \dots, x_n$ respectively and whose lengths are minimum. It is clear that $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$, thus in this case $\bar{k} \leq k \leq n\bar{k}$. This implies that formulation (2.62)–(2.68) can be more compact than formulation (2.36)–(2.42). However inequalities $\bar{k} \leq k \leq n\bar{k}$ show that a mixed-integer set MIX^{2TU} admits a *compact* extended formulation of type (2.36)–(2.42) if and only if it admits a *compact* extended formulation of type (2.62)–(2.68). Therefore, when aiming at showing the existence of a compact extended formulation of a set of the type MIX^{2TU}, one can consider without loss of generality a single list $\mathcal{F}$ of fractional parts as in Section 2.4.

2.5.2 Inequalities involving integer variables

To conclude this section, we show more explicitly the form of inequalities (2.65)–(2.68) when i and/or j belong to I. This will be useful in Chapter 5.

If $i \in I$ (i.e. x_i is an integer variable), we can safely choose $\mathcal{F}_i := \{0\}$: such a list is certainly complete for MIX^{DN} with respect to variable x_i. So we now assume that $\mathcal{F}_i = \{0\}$ for all $i \in I$. We also observe that when x_i is an integer variable, we do not need to introduce variables μ^i_ℓ, as $\mu^i_0 = x_i$ and $\mu^i_1 = x_i + 1$. In other words variables x_i for $i \in I$ can be kept in the formulation without introducing any additional variables to model them.

Given an index $i \in I \cap N^l$, inequality $x_i \geq l_i$ can be trivially tightened to $x_i \geq \lceil l_i \rceil$. It is interesting to observe that this is equivalent to inequality (2.65), as we now prove.

Note that

$$\ell_i(l_i) = \begin{cases} 0 & \text{if } l_i \notin \mathbb{Z}, \\ 1 & \text{if } l_i \in \mathbb{Z}. \end{cases}$$

In the former case inequality (2.65) reads $\mu^i_0 \geq \lfloor l_i \rfloor + 1 = \lceil l_i \rceil$, as $l_i \notin \mathbb{Z}$; in the latter case inequality (2.65) reads $\mu^i_1 \geq \lfloor l_i \rfloor + 1$, which is equivalent to $\mu^i_0 \geq \lceil l_i \rceil$, as $\mu^i_1 = \mu^i_0 + 1$ and $\lfloor l_i \rfloor = \lceil l_i \rceil$. Thus in both cases inequality (2.65) is equivalent to $\mu^i_0 \geq \lceil l_i \rceil$, that is, $x_i \geq \lceil l_i \rceil$.

Given an index $i \in I \cap N^u$, inequality $x_i \leq u_i$ can be trivially tightened to $x_i \leq \lfloor u_i \rfloor$, that is, $\mu^i_0 \leq \lfloor u_i \rfloor$. This is equivalent to (2.66), as $\ell'_i(u_i) = 0$.

Now consider a pair $(i, j) \in N^e$ with $j \in I$. Since x_j is an integer variable, inequality $x_i - x_j \geq l_{ij}$ could be modeled as done for the inequalities of group (2.52), thus obtaining $\mu^i_{\ell_i(l_{ij})} - x_j \geq \lfloor l_{ij} \rfloor + 1$, or in other words, $\mu^i_{\ell_i(l_{ij})} - \mu^j_0 \geq \lfloor l_{ij} \rfloor + 1$. We now show that in fact the set of inequalities (2.67)–(2.68) reduces to this single inequality.

Note that $k_{ij} = 0$. For $t = 1$, it easily checked that $t'_{ij} = \ell_i(l_{ij})$. Thus constraints (2.67)–(2.68) reduce to the single inequality $\mu^i_{\ell_i(l_{ij})} - \mu^j_1 \geq \lfloor l_{ij} \rfloor$, that is, $\mu^i_{\ell_i(l_{ij})} - x_j \geq \lfloor l_{ij} \rfloor + 1$.

If $(i, j) \in N^e$ with $i \in I$, inequality $x_i - x_j \geq l_{ij}$ could be modeled as done for inequalities of group (2.53): after writing the inequality as $x_j - x_i \leq -l_{ij}$, we obtain $\mu^j_{\ell_j(-l_{ij})} - x_i \leq \lfloor -l_{ij} \rfloor$. However, in this case the set of inequalities (2.67)–(2.68) consists of k_j constraints, thus whenever $k_j > 1$ (i.e. x_j is a continuous variable) there are redundant inequalities in (2.67)–(2.68). We only mention that it is possible to swap to role of x_i and x_j in the tightening of $x_i - x_j \geq l_{ij}$, thus obtaining a set of k_i inequalities. In the case $(i, j) \in N^e$ with $i \in I$, such a set of inequalities reduce to a single constraint.

When $(i, j) \in N^e$ and both $i, j \in I$, the set of inequalities (2.67)–(2.68) reduces to the single (obvious) inequality $x_i - x_j \geq \lceil l_{ij} \rceil$.

The above observations are summarized below:

Observation 2.15 *If no variable is introduced to model the integer variables, then:*

(i) *If $i \in I \cap N^l$, inequality (2.65) reads $x_i \geq \lceil l_i \rceil$.*

(ii) *If $i \in I \cap N^u$, inequality (2.66) reads $x_i \leq \lfloor l_i \rfloor$.*

(iii) *If* $(i,j) \in N^e$ *with* $j \in I$*, the set of inequalities* (2.67)–(2.68) *reduces to the single inequality* $\mu^i_{\ell_i(l_{ij})} - x_j \geq \lfloor l_{ij} \rfloor + 1$.

(iv) *If* $(i,j) \in N^e$ *with* $i \in I$*, the set of inequalities* (2.67)–(2.68) *can be replaced with the single inequality* $x_i - \mu^j_{\ell_j(-l_{ij})} \geq \lceil l_{ij} \rceil$.

(v) *If* $(i,j) \in N^e$ *with* $j \in I$*, the set of inequalities* (2.67)–(2.68) *reduces to the single inequality* $x_i - x_j \geq \lceil l_{ij} \rceil$.

The simple observation in (v) implies the following result:

Proposition 2.16 *Let* MIX^{DN} *be a mixed-integer set with dual network constraint matrix and let* $Bx \geq d$ *be a linear system whose inequalities are all of the type* $x_i - x_j \geq d_{ij}$ *with* $i, j \in I$*, where* d *is an integral vector. Then*

$$\operatorname{conv}\big(MIX^{DN} \cap \{x \in \mathbb{R}^n : Bx \geq d\}\big) = \operatorname{conv}\big(MIX^{DN}\big) \cap \{x \in \mathbb{R}^n : Bx \geq d\}.$$

Proof. Since all variables appearing with nonzero coefficient in the inequalities of system $Bx \geq d$ are integer variables, Observation 2.15 (v) implies that an extended formulation of $\operatorname{conv}\big(MIX^{DN} \cap \{x \in \mathbb{R}^n : Bx \geq d\}\big)$ consists of constraints (2.62)–(2.68) together with the inequalities of the system $Bx \geq d$. It can be easily shown (e.g. by using Theorem 1.16) that the projection of such an extended formulation onto the space of the x-variables is given by the projection of (2.62)–(2.68) along with the inequalities of the system $Bx \geq d$. This proves the result, as the projection of (2.62)–(2.68) is $\operatorname{conv}\big(MIX^{DN}\big)$. □

A similar result for some specific mixed-integer sets was proven by Miller and Wolsey [45], Van Vyve [65] and Conforti, Di Summa and Wolsey [13].

Chapter 3

On the length of a complete list

As shown in Chapter 2, any mixed-integer set MIX^{2TU} admits an extended formulation of the type (2.36)–(2.42). We also observed that there is a *compact* extended formulation of this type if and only if MIX^{2TU} admits a complete list of fractional parts that is compact.

We show in Section 3.1 that there exist mixed-integer sets of the type MIX^{2TU} that do not admit a complete list of fractional parts that is compact. This implies that for such sets, no extended formulation of the form (2.36)–(2.42) is compact.

On the other hand, we give in Section 3.2 some sufficient conditions ensuring that a mixed-integer set MIX^{2TU} admits a complete list of polynomial length, thus proving that under these conditions the extended formulation of the type (2.36)–(2.42) is polynomial in the original description of the set. The list of fractional parts is explicitly given through a construction based on a graph associated with the set.

The results of this chapter are joint work with Michele Conforti, Friedrich Eisenbrand and Laurence A. Wolsey and are also summarized in [11].

3.1 A non-compact example

As remarked in Section 2.4, given an arbitrary mixed-integer set $MIX^{TU} := \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$ defined by a totally unimodular constraint matrix A, the list $\mathcal{F}$ consisting of all fractional parts $f\big(\sum_{j=1}^{m} \sigma_j b_j\big)$ for $\sigma_j \in \{0, \pm 1\}$ is complete for the set. Therefore this holds in particular for the sets MIX^{2TU}.

It is easy to choose the components of b so that the list $\mathcal{F}$ defined above contains an exponential number of elements. However, this does not prove that a set MIX^{2TU} associated with such a vector b does not admit a compact extended formulation of the form (2.36)–(2.42), as $\mathcal{F}$ may contain superfluous elements, i.e. fractional parts that do not appear the vertices of $\text{conv}\big(MIX^{2TU}\big)$.

We show here that in fact there are sets of the type MIX^{2TU} for which any complete list of fractional parts is exponentially long. This implies that our extended formulation cannot be compact for such sets.

The result that we prove is the following:

Theorem 3.1 *In the set of vertices of the polyhedron P defined by the inequalities*

$$s_i + r_j \geq \frac{3^{(j-1)n+i}}{3^{n^2+1}}, \quad 1 \leq i, j \leq n, \tag{3.1}$$
$$s_i \geq 0,\ r_j \geq 0, \qquad 1 \leq i, j \leq n, \tag{3.2}$$

the number of distinct fractional parts taken by variable s_n is exponential in n.

We remark the Theorem 3.1 implies the following fact:

Observation 3.2 *Since the constraint matrix of inequalities (3.1)–(3.2) is a totally unimodular matrix with at most two nonzero entries per row, there exists a mixed-integer set X of the type MIX^{2TU}, which is defined on continuous variables s_i, r_j, for $1 \leq i, j \leq n$ and integer variables z_h for $h \in I$, such that the polyhedron $\mathrm{conv}(M) \cap \{(s, r, z) : z_h = 0$ for $h \in I\}$ is a nonempty face of $\mathrm{conv}(X)$ described by inequalities (3.1)–(3.2). Therefore Theorem 3.1 shows that any extended formulation of $\mathrm{conv}(X)$ that* explicitly *takes into account a list of all possible fractional parts of the continuous variables will not be compact in the description of X.*

The remainder of this section is entirely devoted to proving Theorem 3.1.

Let b_{ij} be as in the theorem, i.e. $b_{ij} = 3^{(j-1)n+i}/3^{n^2+1}$ for $1 \leq i, j \leq n$. The following observation is immediate.

Observation 3.3 *$b_{ij} < b_{i'j'}$ if and only if $(j, i) \prec (j', i')$, where $\prec$ denotes the lexicographic order. Thus $b_{11} < b_{21} < \cdots < b_{n1} < b_{12} < \cdots < b_{nn}$.*

Lemma 3.4 *The two properties below hold:*

(i) *Let $\alpha \in \mathbb{Z}^q_+$ with $\alpha_t < \alpha_{t+1}$ for $1 \leq t \leq q-1$. Define*

$$\Phi(\alpha) := \sum_{t=1}^{q} (-1)^{q-t} 3^{\alpha_t}.$$

Then $\Phi(\alpha)$ satisfies the following inequalities:

$$\frac{1}{2} 3^{\alpha_q} < \Phi(\alpha) < \frac{3}{2} 3^{\alpha_q}.$$

(ii) *Suppose that α is as above and $\beta \in \mathbb{Z}^{q'}_+$ satisfies $\beta_t < \beta_{t+1}$ for $1 \leq t \leq q'-1$. Then $\Phi(\alpha) = \Phi(\beta)$ if and only if $\alpha = \beta$.*

Proof. First of all note that

$$\sum_{t=0}^{\alpha_q - 1} 3^t = \frac{3^{\alpha_q} - 1}{3 - 1} < \frac{1}{2} 3^{\alpha_q}.$$

This implies the following chains of inequalities, which prove (i):

$$\Phi(\alpha) \geq 3^{\alpha_q} - \sum_{t=0}^{\alpha_q - 1} 3^t > 3^{\alpha_q} - \frac{1}{2}3^{\alpha_q} = \frac{1}{2}3^{\alpha_q},$$
$$\Phi(\alpha) \leq 3^{\alpha_q} + \sum_{t=0}^{\alpha_q - 1} 3^t < 3^{\alpha_q} + \frac{1}{2}3^{\alpha_q} = \frac{3}{2}3^{\alpha_q}.$$

To prove (ii), suppose $\alpha \neq \beta$. Without loss of generality we assume $q \geq q'$. Assume first that $(\alpha_{q-q'+1}, \ldots, \alpha_q) = \beta$. Then $q > q'$ (otherwise $\alpha = \beta$) and, after defining $\bar{\alpha} := (\alpha_1, \ldots, \alpha_{q-q'})$, we have $\Phi(\alpha) - \Phi(\beta) = \Phi(\bar{\alpha}) > 0$ by (i). Now assume $(\alpha_{q-q'+1}, \ldots, \alpha_q) \neq \beta$. Define $h = \min\{\tau : \alpha_{q-\tau} \neq \beta_{q'-\tau}\}$ and suppose $\alpha_{q-h} > \beta_{q'-h}$ (the other case is similar). If we define the vectors $\bar{\alpha} := (\alpha_1, \ldots, \alpha_{q-h})$ and $\bar{\beta} := (\beta_1, \ldots, \beta_{q'-h})$, (i) gives

$$\Phi(\alpha) - \Phi(\beta) = \Phi(\bar{\alpha}) - \Phi(\bar{\beta}) > \frac{1}{2}3^{\alpha_{q-h}} - \frac{3}{2}3^{\beta_{q'-h}} \geq 0,$$

as $\alpha_{q-h} > \beta_{q'-h}$. This proves that $\Phi(\alpha) \neq \Phi(\beta)$ whenever $\alpha \neq \beta$. □

We now give a construction of an exponential family of vertices of P such that at each vertex variable s_n takes a distinct fractional part. Therefore this construction proves Theorem 3.1.

Let $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_{m-1})$ be two increasing sequences of indices in $\{1, \ldots, n\}$ with $i_1 = 1$ and $i_m = n$. For $1 \leq i, j \leq n$, define $p(i) := \max\{t : i_t \leq i\}$ and $q(j) := \max\{t : j_t \leq j\}$, with $q(j) = 0$ if $j < j_1$.

Consider the following system of equations:

$$s_{i_1} = 0, \tag{3.3}$$
$$s_{i_t} + r_{j_t} = b_{i_t j_t}, \qquad 1 \leq t \leq m-1, \tag{3.4}$$
$$s_{i_{t+1}} + r_{j_t} = b_{i_{t+1} j_t}, \qquad 1 \leq t \leq m-1, \tag{3.5}$$
$$s_{i_{q(j)+1}} + r_j = b_{i_{q(j)+1} j}, \qquad j \notin \{j_1, \ldots, j_{m-1}\}, \tag{3.6}$$
$$s_i + r_{j_{p(i)}} = b_{i j_{p(i)}}, \qquad i \notin \{i_1, \ldots, i_m\}. \tag{3.7}$$

The unique solution to this system is:

$$s_{i_1} = 0, \tag{3.8}$$
$$s_{i_t} = \sum_{\ell=1}^{t-1} b_{i_{\ell+1} j_\ell} - \sum_{\ell=1}^{t-1} b_{i_\ell j_\ell}, \quad 2 \leq t \leq m, \tag{3.9}$$
$$r_{j_t} = \sum_{\ell=1}^{t} b_{i_\ell j_\ell} - \sum_{\ell=1}^{t-1} b_{i_{\ell+1} j_\ell}, \quad 1 \leq t \leq m-1, \tag{3.10}$$
$$s_i = b_{i j_{p(i)}} - r_{j_{p(i)}}, \qquad i \notin \{i_1, \ldots, i_m\}, \tag{3.11}$$
$$r_j = b_{i_{q(j)+1} j} - s_{i_{q(j)+1}}, \qquad j \notin \{j_1, \ldots, j_{m-1}\}. \tag{3.12}$$

Lemma 3.5 *The vector defined by (3.8)–(3.12) is a vertex of P.*

Proof. We start by showing that the vector defined above is feasible in P. First, as each of the variables s_i, r_j takes a value of the form $\Phi(\alpha)/3^{n^2+1}$, by Lemma 3.4 (i) we have that $s_{i_t} > \frac{1}{2}b_{i_t j_{t-1}} > 0$ for $2 \leq t \leq m$, $r_{j_t} > \frac{1}{2}b_{i_t j_t} > 0$ for $1 \leq t \leq m-1$, $s_i > \frac{1}{2}b_{i j_{p(i)}} > 0$ for $i \notin \{i_1, \dots, i_m\}$, and $r_j > \frac{1}{2}b_{i_{q(j)+1} j} > 0$ for $j \notin \{j_1, \dots, j_{m-1}\}$. Therefore the nonnegativity constraints (3.2) are satisfied.

We now show that inequalities (3.1) are satisfied as well. Consider the i, j constraint with $j \notin \{j_1, \dots, j_{m-1}\}$. We distinguish some cases.

1. $p(i) \leq q(j)$. In this case

$$s_i + r_j \geq r_j > \frac{1}{2}b_{i_{q(j)+1} j} \geq \frac{1}{2}b_{i_{p(i)+1} j} \geq \frac{3}{2}b_{ij} > b_{ij}.$$

2. $p(i) > q(j)$ and $i \notin \{i_1, \dots, i_m\}$. Then

$$s_i + r_j \geq s_i > \frac{1}{2}b_{i j_{p(i)}} \geq \frac{1}{2}b_{i j_{q(j)+1}} \geq \frac{3^n}{2}b_{ij} > b_{ij}.$$

3. $p(i) = q(j) + 1$ and $i = i_t$ for some $1 \leq t \leq m$ (thus $p(i) = t = q(j) + 1$). In this case the i, j constraints is satisfied at equality by construction.

4. $p(i) > q(j) + 1$ and $i = i_t$ for some $1 \leq t \leq m$ (thus $p(i) = t > q(j) + 1$). Then

$$s_i + r_j \geq s_i > \frac{1}{2}b_{i j_{t-1}} \geq \frac{1}{2}b_{i j_{q(j)+1}} \geq \frac{3^n}{2}b_{ij} > b_{ij}.$$

The argument with $i \notin \{i_1, \dots, i_m\}$ is similar.

Finally suppose that $i = i_t$ and $j = j_u$ with $u \notin \{t-1, t\}$. If $u > t$ then $s_i + r_j \geq r_j > \frac{1}{2}b_{i_u j_u} \geq \frac{3}{2}b_{i_t j_u} > b_{ij}$. If $u < t-1$ then $s_i + r_j \geq s_i > \frac{1}{2}b_{i_t j_{t-1}} \geq \frac{3}{2}b_{i_t j_u} > b_{ij}$.

This shows that the vector defined by (3.8)–(3.12) is feasible. Since this vector is the unique solution to system (3.3)–(3.7), it defines a vertex of P. $\square$

Now let $a_{ij} = (j-1)n + i$, so that $b_{ij} = 3^{a_{ij}}/3^{n^2+1}$ and take

$$\alpha := (a_{i_1 j_1}, a_{i_2 j_1}, a_{i_2 j_2}, a_{i_3 j_2}, \dots, a_{i_m j_{m-1}}).$$

As $s_n = \Phi(\alpha)/3^{n^2+1}$, it follows from Lemma 3.4 (ii) that in any two vertices constructed as above by different sequences $(i_1, \dots, i_m)$, $(j_1, \dots, j_{m-1})$ and $(i'_1, \dots, i'_{m'})$, $(j'_1, \dots, j'_{m'-1})$, the values of s_n are distinct numbers in the interval $(0, 1)$. As the number of such sequences is exponential in n, this proves Theorem 3.1.

3.2 Sufficient conditions for the compactness of a complete list

The previous section shows that a formulation of the type (2.36)–(2.42) is not guaranteed to be compact in the original description of the set. We describe here some conditions that ensure the existence of a complete list which is compact for a mixed-integer set of the type

MIX^{2TU}, thus proving that the corresponding extended formulation (2.36)–(2.42) is compact under these assumptions.

Let X be a mixed-integer set of the type MIX^{2TU}. Since X is described by a linear system $Ax \geq b$ where A is a totally unimodular matrix with at most two nonzero entries per row, the constraints defining X are of the following type:

$$x_i + x_j \geq l_{ij}^{++}, \quad (i,j) \in N^{++}, \tag{3.13}$$
$$x_i - x_j \geq l_{ij}^{+-}, \quad (i,j) \in N^{+-}, \tag{3.14}$$
$$-x_i - x_j \geq l_{ij}^{--}, \quad (i,j) \in N^{--}, \tag{3.15}$$
$$x_i \geq l_i, \quad i \in N^l, \tag{3.16}$$
$$x_i \leq u_i, \quad i \in N^u, \tag{3.17}$$
$$x_i \text{ integer}, \quad i \in I, \tag{3.18}$$

where $N^{++}, N^{+-}, N^{--} \subseteq N \times N$ and $N^l, N^u, I \subseteq N$. The sets N^{++}, N^{+-}, N^{--} do not contain any pair of the type (i,i) for $i \in N$. Without loss of generality we assume that if $(i,j) \in N^{++}$ then $(j,i) \notin N^{++}$ and if $(i,j) \in N^{--}$ then $(j,i) \notin N^{--}$.

We construct a graph $\mathcal{G}_X = (V,E)$ associated with the mixed-integer set X. The node set of $\mathcal{G}_X$ is $V := L := N \setminus I$ and corresponds to the continuous variables of X. E contains an edge ij for each inequality of types (3.13)–(3.15) with $i,j \in L$ appearing in the linear system that defines X. The total unimodularity of A implies the following: for fixed i,j, if the system $Ax \geq b$ contains an inequality of type (3.14), then it does not contain any inequality of type (3.13) or (3.15). Therefore, for each pair of nodes $i,j \in V$, E contains at most two parallel edges connecting i and j.

We impose a *bi-orientation* ω on $\mathcal{G}_X$: with each edge $e \in E$ (corresponding to an inequality $a_i x_i + a_j x_j \geq l_{ij}$) and each endnode i of e, we associate the value

$$\omega(e,i) := \begin{cases} \textit{tail} & \text{if } a_i = 1, \\ \textit{head} & \text{if } a_i = -1. \end{cases}$$

Thus each edge of $\mathcal{G}_X$ might have one head and one tail (if corresponding to an inequality (3.14)), two tails (if corresponding to an inequality (3.13)) or two heads (if corresponding to an inequality (3.15)).

Given a path $P = (v_0, e_1, v_1, e_1, \ldots, v_t)$ in $\mathcal{G}_X$, where $v_0, \ldots, v_t \in V$ and $e_1, \ldots, e_t \in E$, we want to define the *ω-length* of P, denoted $l_\omega(P)$. To do this, we first define the *reverse* of an edge $e \in E$ as the edge obtained by turning each head of e into a tail and each tail into a head.

We construct a path $P' = (v_0, e'_1, v_1, e'_1, \ldots, v_t)$ from P by reversing some of its edges, so that v_0 is a tail of e_1, and every node v_j for $1 \leq j < t$ is a head of one edge of P' and a tail of the other. Note that given P, the path P' is uniquely determined.

Now we define $l_\omega(P) := \sum_{j=1}^{t} \sigma(P, e_j) l_{e_j}$, where for $e \in E$, l_e is the right-hand side of the inequality corresponding to edge e and

$$\sigma(P, e_j) := \begin{cases} -1 & \text{if } e_j \text{ has been reversed in } P', \\ +1 & \text{otherwise.} \end{cases}$$

We also define a list $\mathcal{L}$ as the set of values $f(l_\omega(P))$ for all paths P in $\mathcal{G}_X$.

Theorem 3.6 *Let X be a mixed-integer set of the type MIX^{2TU} and define the list $\mathcal{L}$ as above. Then X admits a complete list whose length is $\mathcal{O}(mh)$, where m is the number of inequalities in the description of X and $h := |\mathcal{L}|$.*

Proof. We assume that X is nonempty, otherwise the above statement is trivial. This proof is a refinement of that of Lemma 2.11. Let F be a minimal face of conv(X) and $\hat{x}$ be a point in $F \cap X$. We choose J and construct a nonsingular system of linear equations

$$A'x = b', \quad x_i = \hat{x}_i \text{ for } i \in I, \quad x_i = 0 \text{ for } i \in J \tag{3.19}$$

as described in the proof of Lemma 2.11. Recall that $J \cap I = \varnothing$.

Let $\bar{x}$ be the unique solution to system (3.19). Equations $x_i = \hat{x}_i$ for $i \in I$ can be used to eliminate variables x_i for $i \in I$ from system (3.19). After such elimination, system (3.19) has the following form:

$$x_i + x_j = l_{ij}^{++}, \quad (i,j) \in N_{\bar{x}}^{++}, \tag{3.20}$$

$$x_i - x_j = l_{ij}^{+-}, \quad (i,j) \in N_{\bar{x}}^{+-}, \tag{3.21}$$

$$-x_i - x_j = l_{ij}^{--}, \quad (i,j) \in N_{\bar{x}}^{--}, \tag{3.22}$$

$$x_i = d_i, \qquad i \in N_{\bar{x}}, \tag{3.23}$$

where $N_{\bar{x}}^{++} \subseteq N^{++}$, $N_{\bar{x}}^{+-} \subseteq N^{+-}$, $N_{\bar{x}}^{--} \subseteq N^{--}$ and the three sets $N_{\bar{x}}^{++}, N_{\bar{x}}^{+-}, N_{\bar{x}}^{--}$ only contain pairs of indices (i,j) with both $i, j \in L$. It is easily checked that $N_{\bar{x}} \subseteq L$. For each $i \in N_{\bar{x}}$, the value d_i satisfies one of the following conditions:

(a) either $d_i \in \{l_i, u_i\}$,

(b) or $d_i = 0$ and $i \in J$,

(c) or $f(d_i) \in \{f(l_{ij}^{++}), f(l_{ij}^{+-}), f(-l_{ij}^{--})\}$ for some $j \in I \cup J$.

Observe that if we construct the bi-oriented graph corresponding to the above system, we obtain a subgraph of the graph $\mathcal{G}_X$ associated with the original set X.

Recall that system (3.20)–(3.23) consists of $|L|$ linearly independent equations. It is well-known (and easy to see) that the edges of $\mathcal{G}_X$ corresponding to inequalities of type (3.20)–(3.22) define a forest $F_{\bar{x}}$ in $\mathcal{G}_X$. Let $C_{\bar{x}} = (V(C_{\bar{x}}), E(C_{\bar{x}}))$ be a connected component of such a forest. Since $|V(C_{\bar{x}})| = |E(C_{\bar{x}})| + 1$, $C_{\bar{x}}$ contains a unique node r whose value is determined by one of equations (3.23). Then (a)–(c) imply that the fractional part of $\bar{x}_r$ can only take $\mathcal{O}(m)$ possible values, where m is the number of inequalities in the description of X.

If v is a node of $C_{\bar{x}}$ distinct from r, then the value of $\bar{x}_v$ is determined by the value of $\bar{x}_r$ and the inequalities (3.20)–(3.22) corresponding to the edges in the path P_{vr} in $C_{\bar{x}}$ having v as first node and r as last node: if e is the edge in P_{vr} incident with r and P'_{vr} is constructed from P_{vr} as described above, we have

$$\bar{x}_v = \begin{cases} l_\omega(P_{vr}) + \bar{x}_r & \text{if } r \text{ is a head of } e, \\ l_\omega(P_{vr}) - \bar{x}_r & \text{otherwise.} \end{cases} \tag{3.24}$$

Since the list $\mathcal{L}$ has h elements, this shows that the fractional part of each variable x_v at a vertex can take at most $\mathcal{O}(mh)$ values. □

The following easy observation will be used in the next chapter.

Observation 3.7 *If* conv(X) *is a pointed polyhedron, the set J of the above proof is empty. In this case, given $i \in N_{\bar{x}}$, the value d_i satisfies one of the following conditions:*

(a) *either* $d_i \in \{l_i, u_i\}$,

(b) *or* $f(d_i) \in \left\{f(l_{ij}^{++}), f(l_{ij}^{+-}), f(-l_{ij}^{--})\right\}$ *for some* $j \in I$.

We now show how Theorem 3.6 can be applied in some special cases.

Corollary 3.8 *Assume that a mixed-integer set X of the type MIX^{2TU} (with rational right-hand side) satisfies at least one of the following conditions:*

(i) *The number of paths in $\mathcal{G}_X$ is bounded by a polynomial function of the size of the description of X;*

(ii) *The number of elements in the sets* $\left\{f(l_{ij}^{++}) : (i,j) \in N^{++}\right\}$, $\left\{f(l_{ij}^{+-}) : (i,j) \in N^{+-}\right\}$ *and* $\left\{f(l_{ij}^{--}) : (i,j) \in N^{--}\right\}$ *is bounded by a constant.*

(iii) *$\mathcal{G}_X$ is a bipartite graph with vertex classes U, V and the inequalities defining X which contain two continuous variables x_u, x_v ($u \in U, v \in V$) have the form $x_u + x_v \geq b_v - b_u$ for some fixed vector b with indices in $U \cup V$.*

Then X admits a complete list of fractional parts that is compact.

Proof. If condition (i) holds, the length of the list $\mathcal{L}$ is bounded by a polynomial function of the size of the description of X. Then Theorem 3.6 implies that there is a complete list for X which is compact.

Now suppose that condition (ii) holds and assume that $\{f_1, \ldots, f_t\}$ is the set of all elements of type $f(l_{ij}^{++})$, $f(l_{ij}^{+-})$ and $f(l_{ij}^{--})$. Each value $f(l_\omega(P_{vr}))$ can be expressed as

$$f(l_\omega(P_{vr})) = f\Big(\textstyle\sum_{\ell=1}^{t} \alpha_\ell f_\ell\Big), \tag{3.25}$$

where α_ℓ is an integer for $1 \leq \ell \leq t$. Since $\mathcal{G}_X$ has $|L|$ nodes, the maximum length of a path in $\mathcal{G}_X$ is $|L|-1$. This implies $|\alpha_\ell| \leq |L|-1$ for $1 \leq \ell \leq t$. Then the length of the list $\mathcal{L}$ is at most $(2|L|-1)^t$. Thus by Theorem 3.6 there is a complete list for X of size $\mathcal{O}\big(m|L|^t\big) = \mathcal{O}\big(mn^t\big)$, as t is a constant by assumption.

Finally assume that condition (iii) holds. In this case it is easy to verify that for $v \in U \cup V$,

$$l_\omega(P_{vr}) = b_r - b_v \tag{3.26}$$

and thus X admits a complete list which is compact. □

We remark that if the size of each connected components of $\mathcal{G}_X$ is bounded by a constant, then X satisfies condition (i) of the above corollary.

Finally it is interesting to note that if one of the conditions of Corollary 3.8 is satisfied, the knowledge of the structure of $\mathcal{G}_X$ allows one to explicitly compute a complete list of fractional parts which is compact (see Chapter 4 for some examples).

Chapter 4

Examples of formulations of dual network sets

In this chapter we show that several mixed-integer sets that have been studied in the literature can be *transformed* into sets of the type MIX^{2TU} and thus admit an extended formulation of the type introduced in Chapter 2. For many of these sets, one of the conditions of Corollary 3.8 is satisfied and thus a complete list of fractional parts which is compact can be explicitly given. Therefore the extended formulation is compact for such sets.

We will see that most of the mixed-integer sets considered in this chapter have application in real-word problems, such as production planning. Our results provide a unified framework for the extended formulations of these sets found in the last years.

Before presenting the examples, we need to explain precisely the meaning of the word *transformed* used above. This is done in Section 4.1.

The results of this chapter are joint work with Michele Conforti, Friedrich Eisenbrand and Laurence A. Wolsey and are also summarized in [11].

4.1 Mixed-integer linear mappings

Given a polyhedron $P \subseteq \mathbb{R}^n$ and an invertible linear transformation of the space $\mathbb{R}^n$, with associated matrix A (thus A is an $n \times n$ nonsingular matrix), it is well-known that the polyhedra P and $P' := \{Ax : x \in P\}$ are *equivalent*. This means that the polyhedral structure (faces, facets, vertices, etc.) of P and P' are identical under the change of coordinates $x \mapsto Ax$.

Now assume that we are interested in the convex hull of the integral points in P. In other words, we want to study the pure integer set defined by the inequalities that describe P (plus the integrality conditions). If we apply an arbitrary invertible linear transformation, we could loose information about the integral points of P: more specifically, there is no guarantee that $\mathrm{conv}(P' \cap \mathbb{Z}^n)$ is the transformed of $\mathrm{conv}(P \cap \mathbb{Z}^n)$. Thus studying the original pure integer set or the transformed set is not the same at all.

However, if the matrix A associated with the linear transformation is a *unimodular* matrix, i.e. A has integer entries and $\det(A) = \pm 1$, then the transformation is a bijection on $\mathbb{Z}^n$ (see e.g. [38]). Thus in this case there is a one-to-one correspondence between the integral points in P and those in P'.

When dealing with mixed-integer sets, say with continuous variables y_i and integer variables z_i, it is natural to wonder which invertible linear transformations preserve the integrality of the z-variables. The following result fully answers this question.

Theorem 4.1 *Consider the linear transformation defined by* $\begin{pmatrix} y' \\ z' \end{pmatrix} := A \begin{pmatrix} y \\ z \end{pmatrix}$, *where* $(y,z) \in \mathbb{R}^{m+n}$, $(y',z') \in \mathbb{R}^{m'+n'}$, $m+n = m'+n'$ *and* A *is an* $(m+n)\times(m'+n')$ *nonsingular matrix. The following are equivalent:*

(i) *For each* $(y,z) \in \mathbb{R}^{m+n}$, z *is integral if and only if* z' *is integral.*

(ii) $m = m'$, $n = n'$ *and* $A = \begin{bmatrix} A_1 & A_2 \\ \mathbb{O} & U \end{bmatrix}$, *where* A_1 *is an* $m \times m$ *nonsingular matrix,* A_2 *is an* $m \times n$ *matrix and* U *is an* $n \times n$ *unimodular matrix.*

Proof. We first prove that (i) implies (ii). Suppose $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_1 \in \mathbb{R}^{m'\times m}$, $A_2 \in \mathbb{R}^{m'\times n}$, $A_3 \in \mathbb{R}^{n'\times m}$ and $A_4 \in \mathbb{R}^{n'\times n}$. If $A_3 \neq \mathbb{O}$, one of the entries of A_3 is a nonzero number a. Without loss of generality we assume that this entry is in the first row and first column of A_3. Then the vector $A \begin{pmatrix} e_1/2a \\ \mathbf{0} \end{pmatrix}$, where e_1 denotes the m-vector with 1 in the first entry and 0 elsewhere, contains a component equal to 1/2 in the entry corresponding to z'_1, contradicting (i). Thus $A_3 = \mathbb{O}$.

If $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ is the inverse of A (where $B_1 \in \mathbb{R}^{m\times m'}$, $B_2 \in \mathbb{R}^{m\times n'}$, $B_3 \in \mathbb{R}^{n\times m'}$ and $B_4 \in \mathbb{R}^{n\times n'}$), a similar argument shows that $B_3 = \mathbb{O}$.

Thus we obtain $z' = A_4 z$ and $z = B_4 z'$ for all $z \in \mathbb{R}^n$. We now prove that this implies $n = n'$. Equation $z = B_4 A_4 z$ for all $z \in \mathbb{R}^n$ yields $B_4 A_4 = I_n$ (where I_n denotes the $n \times n$ identity matrix), thus $\mathrm{rk}(A_4) \geq n$. Since A_4 is $n' \times n$, this implies $n' \geq n$. Similarly, starting from $z' = A_4 B_4 z'$ for all z', one obtains $n \geq n'$. Thus $n = n'$ and consequently $m = m'$. (i) then implies that A_4 is unimodular.

To prove that (ii) implies (i), note that if (ii) holds then the transformation and its inverse are

$$\begin{cases} y' := A_1 y + A_2 z \\ z' := U y \end{cases} \quad \text{and} \quad \begin{cases} y := A_1^{-1}\left(y' - A_2 U^{-1} z'\right) \\ z := U^{-1} z' \end{cases}.$$

Since U is a unimodular matrix, these two transformations preserve the integrality of z and z'. □

We call a transformation of the type described in Theorem 4.1 a *mixed-integer linear mapping*. Theorem 4.1 shows that if the description of a mixed-integer set is given (as usual) as the set of mixed-integer points belonging to a polyhedron P, then, after applying a mixed-integer linear mapping, we can equivalently study the mixed-integer set defined by P' (the transformed of P).

Taking the above theorem for $n = 0$ or $m = 0$ shows that in the linear case (no integer variables) the mixed-integer linear mappings are precisely the invertible linear transformations, while in the pure integer case we find the unimodular transformations. Thus in the extreme cases Theorem 4.1 matches the known results.

Consider an arbitrary mixed-integer set X and let $\mathcal{F}$ be a complete list of fractional parts for X having compact size. In general, if we apply a linear mapping of the kind described in Theorem 4.1, the transformed mixed-integer set X' may not have a complete list which is compact. For instance, choose

$$X := \{x \in \mathbb{R}^n : 0 \le x_i \le 2^{-i} \text{ for } i \in N\}$$

(so here $I = \varnothing$; similar examples with $I \neq \varnothing$ can be easily derived from this instance). The list $\mathcal{F} := \{0;\ 2^{-i} : i \in N\}$ is complete for X and its size is linear in the size of the description of X. The mixed-integer linear mapping

$$x'_1 := x_2 + \cdots + x_n, \ \ x'_i := x_i \text{ for } i \in N \setminus \{1\}$$

transforms X into

$$X' := \left\{x' \in \mathbb{R}^n : 0 \le x'_1 - x'_2 - \cdots - x'_n \le 2^{-1},\ 0 \le x'_i \le 2^{-i} \text{ for } i \in N \setminus \{1\}\right\}.$$

Now, for each subset $S \subseteq N \setminus \{1\}$ the vector defined by

$$x'_i := \begin{cases} 2^{-i} & \text{if } i \in S, \\ 0 & \text{if } i \in (N \setminus \{1\}) \setminus S, \\ \sum_{j \in S} 2^{-j} & \text{if } i = 1 \end{cases}$$

is a vertex of X'. Since for each S the value of the sum $\sum_{j \in S} 2^{-j}$ is a different number in the interval $[0, 1)$, any complete list for X' contains a number of fractional parts which is exponential in the size of the description of X.

However, for the mixed-integer sets that we study below (except those considered in Sections 4.3 and 4.5), we will apply mixed-integer linear mappings which give rise to mixed-integer sets of the type MIX^{2TU} satisfying at least one of the conditions of Corollary 3.8. Thus in these cases the existence of a complete list which is compact is guaranteed. Furthermore, for these sets such a list is explicitly given.

4.2 The mixing set and its variants

Günlük and Pochet [31] introduced a mixed-integer set that is now referred to as *mixing set* (the authors do not give a name to such a set in [31]):

$$s + z_i \ge b_i, \quad 1 \le i \le n, \tag{4.1}$$

$$s \ge 0, \tag{4.2}$$

$$z_i \text{ integer}, \quad 1 \le i \le n, \tag{4.3}$$

where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$. The mixing set was introduced as an abstraction arising from some mixed-integer sets that have application in practical problems, such as production planning [55]. More specifically, the mixing set provides a relaxation for a number of lot-sizing problems (see e.g. [21, 45, 55, 63]).

Despite the simple structure of constraints (4.1)–(4.3), the convex hull of the mixing set is described by an exponential number of facet-defining inequalities. The name of the set originates from the fact that Günlük and Pochet [31] used this set to demonstrate the strength of a technique that they called *mixing procedure*: given a mixed-integer set, such a procedure consists in *mixing* the original inequalities that describe the set to obtain a new valid inequality. In fact the mixing procedure allowed the authors to compute the linear inequality description of the convex hull of the mixing set (4.1)–(4.3).

Several variants of the mixing set (4.1)–(4.3) have been introduced. Some of them are considered in this section, others are discussed in Chapter 7. As we explain below, all these variants are important in practical problems. For the sake of convenience, the variants of the mixing set studied in this section are treated starting with the most complicated one and ending with the mixing set itself.

4.2.1 The continuous mixing set with flows

The *continuous mixing set with flows*, studied in [12], is defined as follows:

$$s + r_i + y_i \geq b_i, \qquad 1 \leq i \leq n, \tag{4.4}$$

$$y_i \leq z_i, \qquad 1 \leq i \leq n, \tag{4.5}$$

$$s \geq 0,\ r_i \geq 0,\ y_i \geq 0, \quad 1 \leq i \leq n, \tag{4.6}$$

$$z_i \text{ integer}, \qquad 1 \leq i \leq n, \tag{4.7}$$

where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$.

Before proving that the continuous mixing set with flows can be transformed into a set of the type MIX^{2TU} that admits a complete list of fractional parts whose length is polynomial, we demonstrate the practical usefulness of this set by showing two links with lot-sizing.

The first link is to the single-item constant-capacity lot-sizing problem with backlogging over n periods, which can be formulated (including redundant equations) as:

$$\begin{aligned} s_{j-1} + \textstyle\sum_{l=j}^{i} x_l + r_i = \textstyle\sum_{l=j}^{i} d_l + s_i + r_{j-1}, &\quad 1 \leq j \leq i \leq n,\\ x_l \leq C w_l, &\quad 1 \leq l \leq n,\\ s_i \geq 0,\ r_i \geq 0,\ x_l \geq 0,\ w_l \in \{0,1\}, &\quad 1 \leq i \leq n,\ 0 \leq l \leq n. \end{aligned}$$

Here d_l is the demand in period l, s_l and r_l are the stock and backlog at the end of period l, w_l takes value 1 if there is a set-up in period l allowing production to take place, x_l is the production in period l and C is the capacity (i.e. the maximum production allowed). To see that this set has a relaxation as the intersection of n continuous mixing sets with flows, take $C = 1$ without loss of generality, fix j, set $s := s_{j-1}$, $y_i := \sum_{l=j}^{i} x_l$, $z_i := \sum_{l=j}^{i} w_l$ and

$b_i := \sum_{l=j}^{i} d_l$, giving a first relaxation:

$$s + r_i + y_i \geq b_i, \qquad j \leq i \leq n, \tag{4.8}$$

$$0 \leq y_l - y_{l-1} \leq z_l - z_{l-1} \leq 1, \quad j \leq l \leq n \tag{4.9}$$

$$s \geq 0,\ r_i \geq 0,\ y_l \geq 0, \qquad j \leq i \leq n,\ j-1 \leq l \leq n, \tag{4.10}$$

$$z_l \text{ integer}, \qquad j-1 \leq l \leq n. \tag{4.11}$$

Now summing (4.9) over $j \leq l \leq i$ (for each fixed $i = j, \dots, n$) and dropping the upper bound on z_i,[1] one obtains precisely a set of the type (4.4)–(4.7).

The continuous mixing set with flows (4.4)–(4.7) also provides an exact model for the two-stage stochastic lot-sizing problem with constant capacities and backlogging. The problem is as follows. At time 0 one must choose to produce a quantity s at a per unit cost of h. Then in period 1, there are n possible outcomes. For each $1 \leq i \leq n$, the probability of event i is ϕ_i, the demand is b_i and the unit production cost is p_i, with production in batches of size up to C; there are also a fixed cost of q_i per batch and a possible bound m_i on the number of batches. As an alternative to production there is a linear backlog (recovery) cost e_i. Finally the goal is to satisfy the demands in all possible outcomes and minimize the total expected cost. The resulting problem is

$$\min \quad hs + \textstyle\sum_{i=1}^{n} \phi_i (p_i y_i + q_i z_i + e_i r_i)$$

$$\text{subject to} \quad s + r_i + y_i \geq b_i, \qquad 1 \leq i \leq n, \tag{4.12}$$

$$y_i \leq C z_i,\ z_i \leq m_i, \qquad 1 \leq i \leq n, \tag{4.13}$$

$$s \geq 0,\ r_i \geq 0,\ y_i \geq 0, \qquad 1 \leq i \leq n, \tag{4.14}$$

$$z_i \text{ integer}, \qquad 1 \leq i \leq n. \tag{4.15}$$

When $m_i = 1$ for all $1 \leq i \leq n$, this is a standard lot-sizing problem, and in general, assuming $C = 1$ without loss of generality, (4.12)–(4.15) is the continuous mixing set with flows (4.4)–(4.7) plus inequalities $z_i \leq m_i$ for $1 \leq i \leq n$, which can be treated as shown by Proposition 2.16.

We now prove the existence of a compact extended formulation for the continuous mixing set with flows (4.4)–(4.7) (provided that b is a rational vector).

Note that the mixed-integer linear mapping

$$s' := s;\ \ \sigma_i := s + r_i,\ \ y'_i := y_i,\ \ z'_i := z_i \text{ for } 1 \leq i \leq n \tag{4.16}$$

transforms (4.4)–(4.7) into the following mixed-integer set:

$$\sigma_i + y'_i \geq b_i, \qquad 1 \leq i \leq n, \tag{4.17}$$

$$y'_i \leq z'_i, \qquad 1 \leq i \leq n, \tag{4.18}$$

$$s' \geq 0,\ \sigma_i - s' \geq 0,\ y'_i \geq 0, \quad 1 \leq i \leq n, \tag{4.19}$$

$$z'_i \text{ integer}, \qquad 1 \leq i \leq n. \tag{4.20}$$

[1] The only reason for dropping the upper bound on z_i is to obtain a set of the type (4.4)–(4.7). If the upper bound is kept, an extended formulation for the resulting set can be obtained immediately from that of the set (4.4)–(4.7) by applying Proposition 2.16.

Since the constraint matrix of inequalities (4.17)–(4.19) is a totally unimodular matrix with at most two nonzero entries per row, (4.17)–(4.20) is a mixed-integer set of the type MIX^{2TU}.

If we let X denote the mixed-integer set (4.17)–(4.20), then the graph $\mathcal{G}_X$ (as defined in Section 3.2) is a tree, with leaves corresponding to variables y'_i for $1 \le i \le n$. Therefore $\mathcal{G}_X$ satisfies condition (i) of Corollary 3.8 and X admits a complete list of compact size. Below we explicitly give such a list.

Lemma 4.2 *The list of fractional parts $\mathcal{F} := \{0;\ f(b_i) : 1 \le i \le n;\ f(b_i - b_j) : 1 \le i, j \le n\}$ is complete for the mixed-integer set* (4.17)–(4.20).

Proof. We use the notation of the proof of Theorem 3.6. Note that $\text{conv}(X)$ is a pointed polyhedron (as all variables are bounded from below), thus Observation 3.7 applies. Given a vertex $\bar{x} = (\bar{s}', \bar{\sigma}, \bar{y}', \bar{z}')$ of $\text{conv}(X)$ and a connected component $C_{\bar{x}}$ of $F_{\bar{x}}$, Observation 3.7 shows that node r corresponds to a variable that assumes an integer value. Then by equation (3.24) we only need to compute the values $f(l_\omega(P))$ for all paths P in $\mathcal{G}_X$. It is easy to check that the list $\mathcal{F} := \{0;\ f(b_i) : 1 \le i \le n;\ f(b_i - b_j) : 1 \le i, j \le n\}$ includes all these values. □

Therefore the result of Section 2.4 provides a compact extended formulation of the convex hull of the set (4.17)–(4.20). Applying the inverse of linear transformation (4.16) gives a compact extended formulation of the continuous mixing set with flows. Since $|\mathcal{F}| = \mathcal{O}(n^2)$, Corollary 2.12 shows that such an extended formulation uses $\mathcal{O}(n^3)$ variables and constraints.

The formulation can be made more compact if one uses the approach described in Section 2.5.1. Specifically, the following result holds:

Lemma 4.3 (i) *The list of fractional parts $\mathcal{F}_{s'} := \{0;\ f(b_j) : 1 \le j \le n\}$ is complete for the mixed-integer set* (4.17)–(4.20) *with respect to variable s'.*

(ii) *For each $1 \le i \le n$, the list of fractional parts $\mathcal{F}_{\sigma_i} := \{0;\ f(b_j) : 1 \le j \le n\}$ is complete for the mixed-integer set* (4.17)–(4.20) *with respect to variable σ_i.*

(iii) *For each $1 \le i \le n$, the list of fractional parts $\mathcal{F}_{y'_i} := \{0;\ f(b_i - b_j) : 1 \le j \le n\}$ is complete for the mixed-integer set* (4.17)–(4.20) *with respect to variable y'_i.*

Proof. The proof is just a refinement of the proof of Lemma 4.2: for instance, if v is the node in $\mathcal{G}_X$ corresponding to variable y'_i for some $1 \le i \le n$, the list $\mathcal{F}_{y'_i}$ given above contains all values of the type $f(l_\omega(P_{vr}))$, where r is a node in $\mathcal{G}_X$ and P is a path in $\mathcal{G}_X$ with r as last node. □

Since all the lists given in the above lemma contain $\mathcal{O}(n)$ elements, Corollary 2.14 implies the following result:

Proposition 4.4 *The continuous mixing set with flows* (4.4)–(4.7) *admits an extended formulation with $\mathcal{O}(n^2)$ variables and constraints.*

Conforti, Di Summa and Wolsey [12] gave two less compact extended formulations of (4.4)–(4.7): one, using an approach quite similar to that described here, involves $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^3)$ constraints; the other, which is based on the approach of Conforti and Wolsey [16] described in Section 1.5.4, uses $\mathcal{O}(n^3)$ variables and $\mathcal{O}(n^4)$ constraints. Such results are also presented in Section 8.3.

The linear inequality description of the convex hull of the continuous mixing set with flows in the original space is not known.

4.2.2 The mixing set with flows

The *mixing set with flows* is defined as follows:

$$s + y_i \geq b_i, \quad 1 \leq i \leq n, \tag{4.21}$$

$$y_i \leq z_i, \quad 1 \leq i \leq n, \tag{4.22}$$

$$s \geq 0,\ y_i \geq 0, \quad 1 \leq i \leq n, \tag{4.23}$$

$$z_i \text{ integer}, \quad 1 \leq i \leq n, \tag{4.24}$$

where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$.

This mixed-integer set is obtained from the continuous mixing set with flows (4.4)–(4.7) by setting $r_i = 0$ for all $1 \leq i \leq n$. We showed in Section 4.2.1 that the continuous mixing set with flows (4.4)–(4.7) provides relaxations for two kinds of lot-sizing problems with backlogging. Since in those formulations variables r_i represented the backlogged amount, it follows that the mixing set with flows (4.21)–(4.24) provides a relaxation for the single-item constant-capacity lot-sizing problems (without backlogging) and an exact formulation for the two-stage stochastic lot-sizing problem with constant capacities (see also [13]).

Since the convex hull of (4.21)–(4.24) is the face of the convex hull of (4.4)–(4.7) defined by the equations $r_i = 0$ for $1 \leq i \leq n$, an extended formulation for the mixing set with flows (4.21)–(4.24) is obtained by including equations $r_i = 0$ for $1 \leq i \leq n$ in any extended formulation of the continuous mixing set with flows (4.4)–(4.7). Then Proposition 4.4 implies the following result:

Proposition 4.5 *The mixing set with flows (4.21)–(4.24) admits an extended formulation with $\mathcal{O}(n^2)$ variables and constraints.*

In Section 5.3 we construct the extended formulation and then project it onto the original space, thus obtaining a linear inequality description of the convex hull of the set in its space of definition.

A different extended formulation, which also uses $\mathcal{O}(n^2)$ variables and constraints, was given by Conforti, Di Summa and Wolsey [13]. Furthermore they gave a linear inequality description of the convex hull of the set in its original (s, y, z)-space without using projections. Such results are also presented in Section 8.2.

In [13] a complete characterization of the extreme points and extreme rays of the convex hull of this set was also given. This was used to derive a simple algorithm for optimizing a rational linear function over the mixing set with flows (4.21)–(4.24) (with rational right-hand side).

4.2.3 The continuous mixing set

The *continuous mixing set*, introduced by Miller and Wolsey [45], is the mixed-integer set defined as follows:

$$s + r_i + z_j \geq b_i, \quad 1 \leq i \leq n, \tag{4.25}$$

$$s \geq 0,\ r_i \geq 0, \quad 1 \leq i \leq n, \tag{4.26}$$

$$z_i \text{ integer}, \quad 1 \leq i \leq n, \tag{4.27}$$

where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$.

It is immediate to see that system (4.25)–(4.27) is a relaxation of the feasible region of the continuous mixing set with flows (4.4)–(4.7). Therefore the continuous mixing set (4.25)–(4.27) itself is a relaxation of both the single-item constant-capacity lot-sizing problem with backlogging and the two-stage stochastic lot-sizing problem with constant capacities and backlogging. Other possible applications of the continuous mixing set (e.g. in chemistry) are described in [64].

Since the convex hull of (4.25)–(4.27) is the face of the convex hull of (4.4)–(4.7) defined by the equations $y_i = z_i$ for $1 \leq i \leq n$, an extended formulation for the continuous mixing set (4.25)–(4.27) is obtained by including equations $y_i = z_i$ for $1 \leq i \leq n$ in any extended formulation of the continuous mixing set with flows (4.4)–(4.7). Then Proposition 4.4 implies the following result:

Proposition 4.6 *The continuous mixing set (4.21)–(4.24) admits an extended formulation with $\mathcal{O}(n^2)$ variables and constraints.*

Miller and Wolsey [45] gave a different compact extended formulation which also uses $\mathcal{O}(n^2)$ variables and constraints, and so did Van Vyve [65]. The formulation by Van Vyve also works when an additional system of the type $Bz \geq d$, where B is a dual network matrix and d is an integral vector, is included in the original description of the set. In a different paper, Van Vyve [64] gave a more compact extended formulation involving only $\mathcal{O}(n)$ variables and $\mathcal{O}(n^2)$ constraints. He also gave a linear inequality description of the convex hull of the set in its original space, as well as an $\mathcal{O}(n^3)$ algorithm for the separation problem in the original space.

4.2.4 The mixing set

Recall that the *mixing set* is defined by constraints (4.1)–(4.3). Clearly this set is a relaxation of each of the sets considered in Sections 4.2.1–4.2.3, thus it provides relaxations for lot-sizing problems. In fact the mixing set appears as a substructure in many production planning problems [21, 45, 55, 63].

Since the convex hull of the mixing set (4.1)–(4.3) is the face of the convex hull of (4.25)–(4.27) defined by the equations $r_i = 0$ for $1 \leq i \leq n$, an extended formulation for the mixing set is obtained by including equations $r_i = 0$ for $1 \leq i \leq n$ in any extended formulation of the continuous mixing set (4.25)–(4.27). This observation, together with Proposition 4.6, shows that the mixing set (4.1)–(4.3) admits an extended formulation with $\mathcal{O}(n^2)$ variables and constraints. However, a better result can be achieved, as the mixing set admits a shorter complete list of fractional parts.

Lemma 4.7 *The list of fractional parts* $\mathcal{F}_s := \{0;\ f(b_i) : 1 \le i \le n\}$ *is complete for the mixing set* (4.1)–(4.3) *with respect to variable* s.

Proof. Let $(\bar{s}, \bar{z})$ be a vertex of the convex hull of (4.1)–(4.3). Since $\bar{z}$ is an integral vector, if $f(\bar{s})$ were not in the list $\mathcal{F}$ defined above then both points $(\bar{s} \pm \varepsilon, \bar{z})$ would satisfy (4.1)–(4.3) for some $\varepsilon \neq 0$. However, this contradicts the assumption that $(\bar{s}, \bar{z})$ is a vertex. □

Note that the mixing set (4.1)–(4.3) is a set of the type MIX^{2TU} and the above list contains $\mathcal{O}(n)$ elements. If one uses the approach described in Section 2.5 to deal with integer variables, the following result is easily obtained:

Proposition 4.8 *The mixing set* (4.1)–(4.3) *admits an extended formulation with* $\mathcal{O}(n)$ *variables and constraints.*

Such a formulation, which is essentially the same as that proposed by Miller and Wolsey in [45], is given in Section 5.2 in a more general context. Miller and Wolsey [45] also proved that if one intersects the mixing set with a system of inequalities $Bz \ge d$, where B is a dual network matrix and d is an integral vector, an extended formulation of the resulting set is obtained by just including the system $Bz \ge d$ in the extended formulation of the mixing set. Note that this result also follows from our study (see Proposition 2.16).

The convex hull of the mixing set in its original space, which was first described by Günlük and Pochet [31], is obtained in Section 5.2 as a consequence of the characterization of the convex hull of any set of the type MIX^{2TU} having a single continuous variable. Such a characterization is found by projecting the extended formulation onto the original space of variables.

4.3 The intersection set

The following mixed-integer set was studied in [12] under the name of *intersection set*:

$$s_i + r_j + z_j \ge b_{ij}, \qquad 1 \le i, j \le n, \tag{4.28}$$

$$s_i \ge 0,\ r_i \ge 0,\ z_i \ge 0, \quad 1 \le i \le n, \tag{4.29}$$

$$z_i \text{ integer}, \qquad 1 \le i \le n, \tag{4.30}$$

where $b_{ij} \in \mathbb{R}$ for $1 \le i, j \le n$. Note that this set is the intersection of n continuous mixing sets (4.25)–(4.27), each having its own s_i variable but all sharing the same (r, z) variables. Conforti, Di Summa and Wolsey [12] analyzed this set as an instrument to study the continuous mixing set with flows defined in Section 4.2.1.

By applying the mixed-integer linear mapping

$$s'_i := s_i,\ \ \rho_i := r_i + z_i,\ \ z'_i := z_i \text{ for } 1 \le i \le n,$$

the set (4.28)–(4.30) is transformed into the following mixed-integer set:

$$s'_i + \rho_j \geq b_{ij}, \qquad 1 \leq i, j \leq n, \tag{4.31}$$

$$s_i \geq 0,\ \rho_i - z'_i \geq 0,\ z'_i \geq 0, \quad 1 \leq i \leq n, \tag{4.32}$$

$$z'_i \text{ integer}, \qquad 1 \leq i \leq n. \tag{4.33}$$

The above mixed-integer set is of the type MIX^{2TU}. If we denote it by X, the graph $\mathcal{G}_X$ (as defined in Section 3.2) is the complete bipartite graph with n nodes in each class of the bipartition, where all edges have two tails.

Lemma 4.9 *Given two sequences of indices $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ in $\{1, \ldots, n\}$, where each sequence consists of pairwise distinct elements, define*

$$\varphi(i_1, \ldots, i_m; j_1, \ldots, j_{m-1}) := \sum_{t=1}^{m-1} (b_{i_t j_t} - b_{i_t j_{t+1}}),$$

$$\psi(i_1, \ldots, i_m; j_1, \ldots, j_m) := \sum_{t=1}^{m-1} (b_{i_t j_t} - b_{i_t j_{t+1}}) + b_{i_m j_m}.$$

Then the list of fractional parts $\mathcal{F}$ consisting of all values of the types

$$f\big(\varphi(i_1, \ldots, i_m; j_1, \ldots, j_{m-1})\big), \quad f\big(\psi(i_1, \ldots, i_m; j_1, \ldots, j_m)\big)$$

is complete for the mixed-integer set (4.31)–(4.33).

Proof. We use again the notation of the proof of Theorem 2.10. Given a vertex $\bar{x} = (\bar{s}', \bar{\rho}, \bar{z}')$ of conv(X) and a connected component $C_{\bar{x}}$ of $F_{\bar{x}}$, node r corresponds to a variable that assumes an integer value (this follows from Observation 3.7, as conv(X) is a pointed polyhedron). Then by equation (3.24) we only need to compute the values $f(l_\omega(P))$ for all paths P in $\mathcal{G}_X$. It is easy to check that the list $\mathcal{F}$ defined above includes all these values. □

The number of distinct fractional parts contained in the list $\mathcal{F}$ given above depends on the values b_{ij} for $1 \leq i, j \leq n$. Note that the face of conv(X) defined by equations $z'_j = 0$ for $1 \leq j \leq n$ is a polyhedron of the same form as (3.1)–(3.2). This, together with Observation 3.2, shows that there exists a choice of the values b_{ij} for $1 \leq i, j \leq n$ such that any complete list for the set (4.31)–(4.33) contains an exponential number of distinct fractional parts.

4.3.1 The difference set

Conforti, Di Summa and Wolsey [12] payed particular attention to instances of the intersection set (4.28)–(4.30) with $b_{ij} = d_i - d_j$ for some fixed vector $d \in \mathbb{R}^n$. The motivation for the study of this type of set, called *difference set* in [12], relied again on the fact that the difference set can be useful in the study of the continuous mixing set with flows (see also Section 8.3)

It is readily checked that if $b_{ij} = d_i - d_j$ for some fixed vector $d \in \mathbb{R}^n$, then the corresponding transformed set X defined by (4.31)–(4.33) satisfies condition (iii) of Corollary 2.12, thus in this case the existence of a complete list of polynomial length is guaranteed.

Lemma 4.10 *If $b_{ij} = d_i - d_j$ for some fixed vector $d \in \mathbb{R}^n$, the list of fractional parts $\mathcal{F} := \{0;\ f(d_i) : 1 \le i \le n;\ f(d_i - d_j) : 1 \le i, j \le n\}$ is complete for (4.31)–(4.33).*

Proof. Directly from equations (3.24) and (3.26). □

Since the above list contains $\mathcal{O}(n^2)$ elements, Corollary 2.12 implies that the difference set admits an extended formulation with $\mathcal{O}(n^3)$ variables and $\mathcal{O}(n^4)$ constraints. However, a better result can be obtained if one uses the approach described in Section 2.5. Specifically, observe first that the following improvement of Lemma 4.10 holds:

Lemma 4.11 *If $b_{ij} = d_i - d_j$ for some fixed vector $d \in \mathbb{R}^n$, then for each index $1 \le i \le n$ the list of fractional parts $\mathcal{F}_i := \{0;\ f(d_j) : 1 \le j \le n;\ f(d_i - d_j) : 1 \le j \le n\}$ is complete for (4.31)–(4.33) with respect to each of variables s'_i, ρ_i.*

Proof. Again directly from equations (3.24) and (3.26). □

The following results is then implied:

Proposition 4.12 *The difference set admits an extended formulation that uses $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^3)$ constraints.*

Proof. From Lemma 4.11 and Corollary 2.14. □

Two kinds of extended formulations of the difference set were given in [12] (and also here in Section 8.3): one is essentially the same as that described here, while the other, which is based on the technique by Conforti and Wolsey [16] described in Section 1.5.4, involves $\mathcal{O}(n^4)$ variables and constraints.

4.4 Lot-sizing

Van Vyve [65] studied a mixed-integer set of the following form:

$$s_i + r_j + C \sum_{t=i}^{j} z_t \ge d_j - d_i, \quad 1 \le i < j \le n, \tag{4.34}$$

$$s_i \ge 0,\ r_i \ge 0,\ 0 \le z_i \le m_i, \quad 1 \le i \le n, \tag{4.35}$$

$$z_i \text{ integer}, \quad 1 \le i \le n. \tag{4.36}$$

He showed that optimizing a linear function over the above set is equivalent to solving a certain lot-sizing problem, provided that the costs satisfy the Wagner-Whitin conditions recalled in Section 1.5.5. In such lot-sizing problem the capacity is a constant C and backlogging is allowed. There is also a bound m_j on the number of batches that can be produced in period j. The value d_j is the cumulative demand up to period j. Van Vyve [65] provided an extended formulation for the convex hull of (4.34)–(4.36) which uses $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^3)$ constraints.

Assuming $C = 1$ without loss of generality, the mixed-integer linear mapping

$$w_i := \sum_{t=1}^{i} z_t, \;\; \sigma_i := s_i - w_{i-1}, \;\; \rho_i := r_i + w_i \text{ for } 1 \le i \le n, \tag{4.37}$$

where $w_0 := 0$, maps (4.34)–(4.36) into the following mixed-integer set:

$$\sigma_i + \rho_j \ge d_j - d_i, \qquad 1 \le i < j \le n, \tag{4.38}$$
$$\sigma_i + w_{i-1} \ge 0, \qquad 1 \le i \le n, \tag{4.39}$$
$$\rho_i - w_i \ge 0, \; 0 \le w_i - w_{i-1} \le m_i, \qquad 1 \le i \le n, \tag{4.40}$$
$$w_i \text{ integer}, \qquad 1 \le i \le n. \tag{4.41}$$

The above is a set of the type MIX^{2TU}. If we denote it by X, the graph $\mathcal{G}_X$, as defined in Section 3.2, is a bipartite graph where all edges have two tails.

Lemma 4.13 *The list $\mathcal{F} := \{0; f(d_i) : 1 \le i \le n;\ f(d_i - d_j) : 1 \le i, j \le n\}$ is complete for (4.38)–(4.41).*

Proof. After noting that Observation 3.7 can be applied (as $\mathrm{conv}(X)$ is a pointed polyhedron) and condition (iii) of Corollary 2.12 holds, the result follows directly from equations (3.24) and (3.26). □

The above lemma, together with the result of Section 2.4, yields a compact extended formulation of (4.34)–(4.36) with $\mathcal{O}(n^3)$ constraints and $\mathcal{O}(n^4)$ variables. However, a property similar to Lemma 4.11 holds and thus the result can be improved:

Proposition 4.14 *The set (4.34)–(4.36) admits an extended formulation that uses $\mathcal{O}(n^2)$ constraints and $\mathcal{O}(n^3)$ variables.*

Proof. Just observe that for each index $1 \le i \le n$, the list of fractional part $\mathcal{F}_i := \{0;\ f(d_j) : 1 \le j \le n;\ f(d_i - d_j) : 1 \le j \le n\}$ is complete for X with respect to each of variables σ_i, ρ_i (this follows from equations (3.24) and (3.26)). The result now follows from Corollary 2.14. □

Such an extended formulation is essentially the same as that given by Van Vyve [65].

4.5 Bipartite cover inequalities

Given a bipartite graph $\mathcal{G} = (V_1, V_2; E)$, let (I, L) be a partition of $V_1 \cup V_2$ with $I \neq \varnothing$. We consider here the following mixed-integer set:

$$x_i + x_j \ge b_{ij}, \quad ij \in E, \tag{4.42}$$
$$x_i \ge 0, \quad i \in V_1 \cup V_2, \tag{4.43}$$
$$x_i \text{ integer}, \quad i \in I, \tag{4.44}$$

where $b_{ij} \in \mathbb{R}$ for $ij \in E$.

The above is obviously a set of the type MIX^{2TU}. The example of Section 3.1 shows that this set does not admit in general a complete list of polynomial length. However, such a list exists in the following two special cases.

4.5.1 The intersection of mixing sets

The first case is the set (4.42)–(4.44) with $I = V_1$ and $L = V_2$ (i.e. the integer variables correspond to the nodes of one side of the bipartition of $\mathcal{G}$). Note that in this case the set (4.42)–(4.44) is the intersection of $|V_2|$ mixing sets (see Section 4.2), each one having its own continuous variable but all sharing the same integer variables. (Here we also require nonnegativity of the integer variables.)

This set was studied by Miller and Wolsey in [45], where a compact extended formulation was given. Their result can be easily reobtained by using our approach, as we now show.

Lemma 4.15 *If $I = V_1$ and $L = V_2$, then for each $j \in V_2$ the list of fractional parts $\mathcal{F}_j := \{0;\ f(b_{ij}) : i \in V_1\}$ is complete for the set (4.42)–(4.44) with respect to variable x_j.*

Proof. Let X denote the mixed-integer set defined by conditions (4.42)–(4.44). The graph $\mathcal{G}_X$ (see Section 3.2) has no edges. Since $\text{conv}(X)$ is a pointed polyhedron, it follows immediately by Observation 3.7 that the list given above is complete for the set with respect to variable x_j. □

Proposition 4.16 *If $I = V_1$ and $L = V_2$, the set (4.42)–(4.44) admits an extended formulation with $\mathcal{O}(|V_1||V_2|)$ variables and constraints.*

Proof. Just count the variables and the constraints of the extended formulation (2.62)–(2.68) corresponding to the set (4.42)–(4.44) and the list given above (the general bound provided by Corollary (2.14) is weaker than $\mathcal{O}(|V_1||V_2|)$). □

Miller and Wolsey [45] gave a formulation of this set in its original space of variables. They showed that such a formulation is obtained by just intersecting the linear inequality descriptions of the $|V_2|$ mixing sets that form (4.42)–(4.44).

4.5.2 Constant number of fractional parts

The second case we consider is the set (4.42)–(4.44) with the additional condition that for some integer constant K, the value Kb_{ij} is an integer for all $ij \in E$; in other words, $f(b_{ij}) \in \{0, 1/K, \ldots, 1 - 1/K\}$ for all $ij \in E$. Note that this set satisfies condition (ii) of Corollary 3.8.

As stated in Corollary 3.8, every set of the type MIX^{2TU} such that the number of distinct fractional parts taken by the right-hand sides is bounded by a constant admits a compact extended formulation. Thus one might wonder why we pay particular attention to the sets (4.42)–(4.44) with the above property. The reason for this is the fact that the special case $K = 2$ was studied recently by Conforti, Gerards and Zambelli [15]. They first gave a compact extended formulation (of the same type as that described in Chapter 2) and then computed the linear inequality description of the set in the original space by projecting the extended formulation.

Chapter 5

Projections onto the original space of variables

Recall that given an extended formulation of a mixed-integer set, Theorem 1.17 can be used in principle to compute the projection of the extended formulation onto the original space of variables, thus obtaining a linear inequality description of the convex hull of the set in its space of definition.

In this chapter we consider the problem of carrying out explicitly the projection of an extended formulation of a mixed-integer set with dual network constraint matrix. Since computing the projection onto the x-space of a general polyhedron of the type (2.36)–(2.42) or (2.62)–(2.68) seems to be an extremely hard task, we only consider a few special cases for which we can explicitly find an inequality description in the original space.

Except for equations (2.62), which define the original variables, the constraint matrix of a formulation of the type (2.62)–(2.68) is a dual network matrix. Thus, when using Theorem 1.17 to compute the projection, one essentially has to solve a family of circulation problems on a network depending on continuous parameters. In fact, some flow techniques are used in this chapter to compute the projections. This is discussed in Section 5.1.

In Section 5.2 we consider a general mixed-integer set of the type MIX^{DN} (or MIX^{2TU}) with a single continuous variable. We construct an extended formulation of the form (2.62)–(2.68) for such a set and then project it onto the original space of variables. This will provide a linear inequality description of the set in its space of definition. The "opposite" case, i.e. a single integer variable, is treated in Chapter 6.

In Section 5.3 we reconsider the mixing set with flows (see Section 4.2.2), which is of the type MIX^{2TU} and therefore admits an extended formulation (2.62)–(2.68). We explicitly give such a formulation and then project it onto the original space. As we will see, while the projection is computed quite easily for the family of sets considered in Section 5.2, much more effort is required for the mixing set with flows studied in Section 5.3.

A further example of explicit computation of the projection of an extended formulation which is essentially of the type (2.36)–(2.42) was carried out recently by Conforti, Gerards and Zambelli in [15], where the set described in Section 4.5.2 with $K = 2$ was studied.

5.1 Circulation problems

A linear system of the type (2.36)–(2.42) or (2.62)–(2.68) has the following form:

$$Ix = B\mu, \tag{5.1}$$

$$M\mu \geq d, \tag{5.2}$$

where I is the identity matrix of suitable dimension, M is a dual network matrix and μ is the vector of all additional variables. By Theorem 1.17, the projection of the above polyhedron onto the space of the x-variables is described by inequalities $wx - ud \geq 0$ for all row vectors (w, u) that are extreme rays[1] of the following cone:

$$-wB + uM = 0, \tag{5.3}$$

$$w \text{ free}, \; u \geq \mathbf{0}. \tag{5.4}$$

Since M is a dual network matrix, for each fixed vector w the above conditions define the feasible region of a *circulation* (or *b-flow*) *problem.*[2] Therefore computing the projection of an extended formulation of a dual network set amounts to solving a family of circulation problems parameterized on w. The basic results about problems of this type that are used in the remainder of the chapter are now recalled.

Let $\mathcal{N} = (V, A)$ be a network with vertex set V and arc set A. For $v \in V$, we denote by $\delta^+(v)$ (resp. $\delta^-(v)$) the set of arcs entering (resp. leaving) node v.

Suppose we are assigned real numbers b_v for $v \in V$. We denote by $\mathcal{N}(b)$ the network $\mathcal{N}$ with the corresponding *circulation requirements* b_v assigned to its nodes. A (feasible) *circulation* in $\mathcal{N}(b)$ is a vector x with indices in A that satisfies the following constraints:

$$\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = b_v, \quad v \in V, \tag{5.5}$$

$$x_a \geq 0, \qquad a \in A. \tag{5.6}$$

Equations (5.5) require that at each node $v \in V$ the balance between entering and exiting flow is exactly the circulation requirement b_v. As inequalities (5.6) suggest, we allow any amount of flow on the arcs, provided that such a flow goes in the "correct direction". In a more general version of the circulation problem, a lower and an upper bound are assigned to the flow on each arc. However, for our purpose, we always take 0 as lower bound and $+\infty$ as upper bound.

Remark that the constraint matrix of the system of equations (5.5) has exactly one $+1$ and one -1 per column, while the constraint matrix of (5.3) (i.e. the transpose of M) may

[1] Since w is unbounded in (5.3)–(5.4), it is not obvious that such a cone does have extreme rays (i.e. is pointed). Note however that the structure of system (2.62)–(2.68) shows that each column of B has at most one nonzero entry and each row of B has at least one nonzero entry. This observation can be used to show that (5.3)–(5.4) is a pointed cone.

[2] Though many authors call circulation problems only the b-flow problems where $b = \mathbf{0}$, we give here the same meaning to the two terms.

also contain columns with only one nonzero entry. This aspect is discussed at the end of the section.

Summing all equations (5.5) gives

$$0 = \sum_{v \in V} b_v. \tag{5.7}$$

Therefore this is a necessary condition for the existence of a feasible circulation in $\mathcal{N}(b)$.

Conditions (5.5)–(5.6) define a polyhedron. In the next sections we will be interested in finding the extreme points and extreme rays of such a polyhedron. The following well-known characterization will be useful:

Theorem 5.1 *The following hold:*

(i) *the extreme points of* (5.5)–(5.6) *correspond to the acyclic circulations in* $\mathcal{N}(b)$*;*

(ii) *the extreme rays of* (5.5)–(5.6) *are the characteristic vectors of directed cycles in* $\mathcal{N}$*.*

In the above theorem "acyclic" means "not containing any undirected cycle". We will also need the following result:

Theorem 5.2 *Let* $\bar{x}$ *be a feasible circulation in* $\mathcal{N}(b)$*. Let* F *be a forest contained in the support of* $\bar{x}$ *and let* $\Delta \in \mathbb{R}^V$ *be a vector satisfying the following two conditions:*

(i) *the support of* Δ *is contained in the node set of* F*;*

(ii) *for each connected component* $C = (V(C), A(C))$ *of* F*,* $\sum_{v \in V(C)} \Delta_v = 0$*.*

If $\varepsilon > 0$ *is small enough, then there exists a unique circulation* $\tilde{x}$ *in* $\mathcal{N}(b + \varepsilon\Delta)$ *such that* $\bar{x}$ *and* $\tilde{x}$ *coincide on all arcs not belonging to* F*.*

Proof. Note that it is sufficient to prove that the statement holds when F is connected (i.e. it is a tree). For fixed $\varepsilon > 0$, consider the following linear system:

$$\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = \varepsilon\Delta_v, \quad v \in V(F), \tag{5.8}$$

$$x_e = 0, \qquad e \notin A(F). \tag{5.9}$$

Define $m := |V(F)|$. Since F is a tree, it is well-known that the constraint matrix of equations (5.8), restricted to variables x_e for $e \in A(F)$, is an $m \times (m-1)$ matrix with rank $m-1$. Summing up all equations (5.8) and using (5.9) gives equation $0 = \sum_{v \in V(F)} \Delta_v$. Since this condition is satisfied by assumption, one of equations (5.8) is redundant. After removing this redundant equation, (5.8)–(5.9) becomes a nonsingular system. Let $\xi(\varepsilon)$ be its unique solution and define $x(\varepsilon) := \bar{x} + \xi(\varepsilon)$. Note that $x(\varepsilon)$ satisfies equations (5.5) for all $v \in V$ and $x_e(\varepsilon) = \bar{x}_e$ for all $e \notin A(F)$. Since $x(\varepsilon)$ is a continuous function of ε and $x(0) = \bar{x}$, then for $\varepsilon > 0$ sufficiently small $x(\varepsilon)$ also satisfies conditions (5.6). □

Since some rows of M may have exactly one nonzero entry, we need to consider a more general version of a network, where some arcs may have only one endpoint in the network. If H (resp. T) denotes the set of arcs having only their head (resp. tail) in the network, summing all equations (5.5) now gives (after changing all the signs)

$$\sum_{a\in T} x_a - \sum_{a\in H} x_a = -\sum_{v\in V} b_v. \tag{5.10}$$

Such an equation can be viewed as a constraint of type (5.5) corresponding to a *dummy node* $d \notin V$, with associated balance $b_d := -\sum_{v\in V} b_v$. Such a dummy node d is the head of all arcs in T and tail of all arcs in H. Thus adding this node yields a network containing both endpoints of each of its arcs. Furthermore, equation (5.7) is now satisfied and therefore Theorem 5.1 can be applied to this new network.

Remark that the insertion of the dummy node does not change the feasible region (5.5)–(5.6), as equation (5.10) is implicit in that system.

5.2 Dual network sets with a single continuous variable

We study here mixed-integer sets with dual network constraint matrix and a single continuous variable. For such sets, we explicitly give an extended formulation of the type presented in Chapter 2 and then project it onto the original space of variables. This will give us a linear inequality description of the set in its space of definition. The results of this section are joint work with Michele Conforti and Laurence A. Wolsey.

We first explain why the projection can be carried out easily when there is a single continuous variable. As remarked in Section 5.1, an extended formulation of a dual network set has the form (5.1)–(5.2) and computing the projection amounts to detecting the extreme rays of the cone defined by (5.3)–(5.4). As observed in Section 2.5.2, it is not necessary to introduce any additional variables to model the integer variables of the set. It follows that when there is a single continuous variable in the original dual network set, system (5.1) actually consists of a single equation, thus the vector w in (5.3)–(5.4) has only one component. Then, given an extreme ray $(\bar{w}, \bar{u})$ of (5.3)–(5.4), one can assume (after normalization) that $\bar{w} \in \{0, \pm 1\}$. Since once the value of $\bar{w}$ is fixed we obtain a circulation problem on a network, we only have to study three different circulation problems. It will be then sufficient to apply Theorem 5.1 in the three cases.

Every mixed-integer set with dual network constraint matrix and a single continuous variable can be written as follows:

$$s - z_i \geq l_i, \quad i \in I^l, \tag{5.11}$$

$$s - z_i \leq u_i, \quad i \in I^u, \tag{5.12}$$

$$l_0 \leq s \leq u_0, \tag{5.13}$$

$$z_i \text{ integer}, \quad 1 \leq i \leq n, \tag{5.14}$$

$$Bz \geq d, \tag{5.15}$$

where $I^l, I^u \subseteq \{1, \dots, n\}$ and B is a dual network matrix. Note that to treat the most general case, each of the two inequalities in (5.13) may be omitted.

It is convenient to introduce a dummy variable z_0 whose value is always zero. This allows us to write the above constraints in a more homogeneous form:

$$s - z_i \geq l_i, \quad i \in J^l, \tag{5.16}$$

$$s - z_i \leq u_i, \quad i \in J^u, \tag{5.17}$$

$$z_i \text{ integer}, \quad 0 \leq i \leq n, \tag{5.18}$$

$$Bz \geq d, \tag{5.19}$$

$$z_0 = 0, \tag{5.20}$$

where

$$J^l := \begin{cases} I^l \cup \{0\} & \text{if inequality } s \geq l_0 \text{ appears in (5.13)}, \\ I^l & \text{otherwise} \end{cases}$$

and

$$J^u := \begin{cases} I^u \cup \{0\} & \text{if inequality } s \leq u_0 \text{ appears in (5.13)}, \\ I^u & \text{otherwise.} \end{cases}$$

Since $z_1, \dots, z_n$ are integer variables, we can assume without loss of generality that all components of d are integer (otherwise round them up). By Proposition 2.16, we only need to compute the convex hull of the set (5.16)–(5.18): inequalities (5.19)–(5.20) will be then added to the formulation of that convex hull.

5.2.1 The extended formulation

Let $f_1, \dots, f_k$ be the k distinct elements in $\{f(l_i) : i \in J^l\} \cup \{f(u_i) : i \in J^u\} \cup \{0\}$, with $f_1 > \cdots > f_k = 0$, and define $f_0 := 1$ and $f_{k+1} := 0$. For each index $i \in J^l$, we denote by $p(i)$ the unique index in $\{1, \dots, k\}$ such that $f_{p(i)} = f(l_i)$. Similarly, for each index $i \in J^u$, we denote by $q(i)$ the unique index in $\{1, \dots, k\}$ such that $f_{q(i)} = f(u_i)$.

Lemma 5.3 *The list of fractional parts $\mathcal{F}_s := \{f_1, \dots, f_k\}$ is complete for the set (5.16)–(5.18) with respect to variable s.*

Proof. Let $(\bar{s}, \bar{z})$ be a vertex of the convex hull of (5.16)–(5.18). Since $\bar{z}$ is an integral vector, if $f(\bar{s})$ were not in the list $\mathcal{F}$ defined above then both points $(\bar{s} \pm \varepsilon, \bar{z})$ would satisfy (5.16)–(5.18) for some $\varepsilon \neq 0$. However, this contradicts the assumption that $(\bar{s}, \bar{z})$ is a vertex. □

Note that unless $f(l_i) = 0$ or $f(u_i) = 0$ for some index i, it is not necessary to include the value 0 in $\mathcal{F}_s$. However, in the following we find useful to have $f_k = 0$.

By Theorem 2.13 and using Observation 2.15 to model inequalities (5.16)–(5.17), an extended formulation of the set (5.16)–(5.18) is given by the following linear system:

$$s = \sum_{\ell=0}^{k} (f_\ell - f_{\ell+1})\mu_\ell, \tag{5.21}$$

$$\mu_k - \mu_0 = 1, \tag{5.22}$$

$$\mu_\ell - \mu_{\ell-1} \geq 0, \qquad 1 \leq \ell \leq k, \tag{5.23}$$

$$\mu_{p(i)} - z_i \geq \lfloor l_i \rfloor + 1, \quad i \in J^l, \tag{5.24}$$

$$\mu_{q(i)-1} - z_i \leq \lfloor u_i \rfloor, \qquad i \in J^u. \tag{5.25}$$

Instead of immediately projecting the above system, it is useful to write it in a slightly different form. To do this, we first need to introduce some new notation.

Given a real number α, $f'(\alpha)$ will denote the fractional part of α, except that $f'(\alpha) = 1$ if α is an integer. That is,

$$f'(\alpha) := \begin{cases} f(\alpha) = \alpha - \lfloor \alpha \rfloor & \text{if } \alpha \notin \mathbb{Z}, \\ 1 & \text{if } \alpha \in \mathbb{Z}. \end{cases} \tag{5.26}$$

Also, for each index $i \in J^l$, we denote by $p'(i)$ the unique index in $\{0, \ldots, k-1\}$ such that $f_{p'(i)} = f'(l_i)$. Note that

$$p'(i) = \begin{cases} p(i) & \text{if } l_i \notin \mathbb{Z}, \\ 0 & \text{if } l_i \in \mathbb{Z}. \end{cases}$$

In other words $p'(i) = p(i)$ if $0 \leq p(i) \leq k-1$, while $p'(i) = 0$ if $p(i) = k$. We also set $p'(n+1) := k$.

Using equation (5.22), one can readily verify that for all indices $i \in J^l$, inequality $\mu_{p(i)} - z_i \geq \lfloor l_i \rfloor + 1$ is equivalent to inequality $\mu_{p'(i)} - z_i \geq \lceil l_i \rceil$. System (5.21)–(5.25) can then be rewritten as follows:

$$s = \sum_{\ell=0}^{k} (f_\ell - f_{\ell+1})\mu_\ell, \tag{5.27}$$

$$\mu_k - \mu_0 = 1, \tag{5.28}$$

$$\mu_\ell - \mu_{\ell-1} \geq 0, \qquad 1 \leq \ell \leq k, \tag{5.29}$$

$$\mu_{p'(i)} - z_i \geq \lceil l_i \rceil, \qquad i \in J^l, \tag{5.30}$$

$$\mu_{q(i)-1} - z_i \leq \lfloor u_i \rfloor, \quad i \in J^u. \tag{5.31}$$

Equation (5.28) can be used to eliminate variable μ_k from the above system. Note that the coefficient of μ_k in equation (5.27) is equal to zero, as $f_k = f_{k+1} = 0$. Furthermore, none of inequalities (5.30)–(5.31) contains variable μ_k in its support, as $p'(i) < k$ for $i \in J^l$ and $q(i) \leq k$ for $i \in J^u$. System (5.27)–(5.31) is then equivalent to the following (we assign dual variables to the constraints as indicated on the left):

$$w: \quad s = \sum_{\ell=0}^{k-1} (f_\ell - f_{\ell+1})\mu_\ell, \tag{5.32}$$

$$u_\ell: \quad \mu_\ell - \mu_{\ell-1} \geq 0, \qquad 1 \leq \ell \leq k-1, \tag{5.33}$$

$$u_0: \quad \mu_0 - \mu_{k-1} \geq -1, \tag{5.34}$$

$$v_i^l: \quad \mu_{p'(i)} - z_i \geq \lceil l_i \rceil, \quad i \in J^l, \tag{5.35}$$

$$v_i^u: \quad \mu_{q(i)-1} - z_i \leq \lfloor u_i \rfloor, \quad i \in J^u. \tag{5.36}$$

Note that except for the first equation, the constraint matrix of the above system is still a dual network matrix.

5.2.2 The projection

By Theorem 1.17, a linear inequality description of the convex hull of (5.16)–(5.18) in its original space is given by inequalities

$$\bar{w}s - \sum_{i \in J^l} \bar{v}_i^l (z_i + \lceil l_i \rceil) + \sum_{i \in J^u} \bar{v}_i^u (z_i + \lfloor u_i \rfloor) + \bar{u}_0 \geq 0 \tag{5.37}$$

for all vectors $(\bar{w}, \bar{u}, \bar{v}^l, \bar{v}^u)$ that are extreme rays of the following polyhedral cone (beside each constraint, the corresponding primal variable is indicated):

$$\mu_\ell: \quad u_\ell - u_{\ell+1} + \sum_{i \in J^l : p'(i) = \ell} v_i^l + \sum_{i \in J^u : q(i) = \ell+1} v_i^u = (f_\ell - f_{\ell+1})w, \quad 0 \leq \ell \leq k-2, \tag{5.38}$$

$$\mu_{k-1}: \quad u_{k-1} - u_0 + \sum_{i \in J^l : p'(i) = k-1} v_i^l + \sum_{i \in J^u : q(i) = k} v_i^u = (f_{k-1} - f_k)w, \tag{5.39}$$

$$w \text{ free}, \ u \geq \mathbf{0}, \ v^l \geq \mathbf{0}, \ v^u \geq \mathbf{0}. \tag{5.40}$$

In the following we study the extreme rays of the polyhedral cone defined by inequalities (5.38)–(5.40).

Recall that the constraint matrix of inequalities (5.33)–(5.36) is a dual network matrix. This implies that for each fixed $w \in \mathbb{R}$, system (5.38)–(5.40) defines the feasible region of a circulation problem on a network $\mathcal{N}$. The value of w determines the requirement of the nodes of the network, but the structure of the network (nodes and arcs) is independent of w. This structure is now described.

For each $0 \leq \ell \leq k-1$, the corresponding equation (5.38) or (5.39), which is associated with the primal variable μ_ℓ, corresponds to a node of $\mathcal{N}$ which we also call μ_ℓ. The arcs of $\mathcal{N}$ inherit the name of the corresponding variables of system (5.38)–(5.40). The structure of network $\mathcal{N}$ is depicted in Figure 5.1, where $w > 0$ is assumed. Note that a dummy node d has been added to the network as described in Section 5.1: node d is the tail of arcs v_i^l for $i \in J^l$ and the head of arcs v_i^u for $i \in J^u$. For each $i \in J^l$, the head of arc v_i^l is node $\mu_{p'(i)}$. For each $i \in J^u$, the tail of arc v_i^u is node $\mu_{q(i)-1}$. We also remark that the thick arrows in the figure do not represent arcs of the network, but circulation requirements.

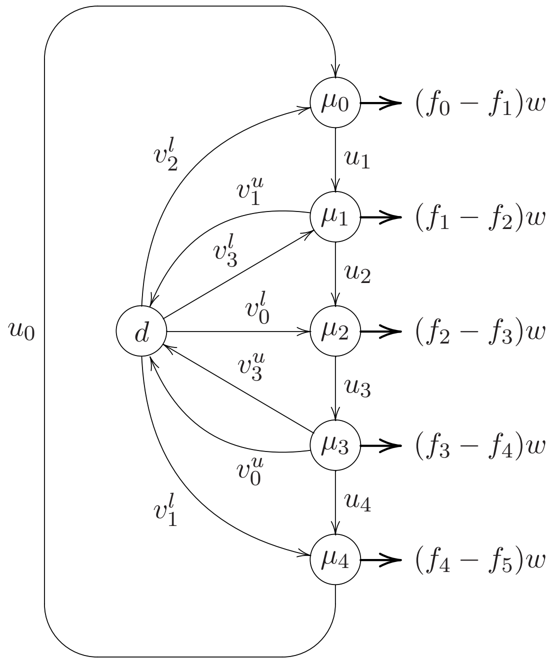


Figure 5.1: The network corresponding to a possible instance of problem (5.38)–(5.40). Here $n = 3$ and $k = 4$. Also $J^l = \{0, 1, 2, 3\}$, $J^u = \{0, 1, 3\}$, $p'(0) = 2$, $p'(1) = 4$, $p'(2) = 0$, $p'(3) = 1$, $q(0) = 4$, $q(1) = 2$ and $q(3) = 4$.

The case $\bar{w} = 0$

Let $(\bar{w}, \bar{u}, \bar{v})$ be an extreme ray of cone (5.38)–(5.40) with $\bar{w} = 0$. Then $(\bar{u}, \bar{v})$ is an extreme ray of the polyhedral cone obtained by setting $w = 0$ in (5.38)–(5.40). Theorem 5.1 shows that $(\bar{u}, \bar{v})$ defines a directed cycle in network $\mathcal{N}$. In the following we use $(\bar{u}, \bar{v})$ to denote both the vector and the corresponding cycle.

The structure of $\mathcal{N}$ immediately shows that every directed cycle in $\mathcal{N}$ consists of an arc v_i^l for some $i \in J^l$, a (possibly zero-length) path formed by arcs of type u_ℓ, and an arc v_j^u for some $j \in J^u$. More specifically, if $f'(l_i) > f(u_j)$ then arc u_0 is not contained in the support of the cycle and the corresponding inequality (5.37) is $z_j - z_i \geq \lceil l_i \rceil - \lfloor u_j \rfloor$. If $f'(l_i) \leq f(u_j)$ then arc u_0 is part of the cycle and the corresponding inequality (5.37) is $z_j - z_i \geq \lceil l_i \rceil - \lfloor u_j \rfloor - 1$. It is easy to check that in both cases the inequality is

$$z_j - z_i \geq \lceil l_i - u_j \rceil. \tag{5.41}$$

The case $\bar{w} > 0$

Let $(\bar{w}, \bar{u}, \bar{v})$ be an extreme ray of cone (5.38)–(5.40) with $\bar{w} > 0$. Without loss of generality we can assume $\bar{w} = 1$. In this case $(\bar{u}, \bar{v})$ is an extreme point of the polyhedron obtained by setting $w = 1$ in (5.38)–(5.40). By Theorem 5.1, this implies that $(\bar{u}, \bar{v})$ defines an acyclic circulation in the corresponding network $\mathcal{N}$.

Note that $\bar{v}_i^u = 0$ for all $i \in J^u$, as otherwise the circulation $(\bar{u}, \bar{v})$ would necessarily contain a cycle (of the type described in the analysis of the case $\bar{w} = 0$) and $(\bar{u}, \bar{v})$ would not be an extreme point.

We clearly have $\bar{v}_i^l > 0$ for at least an index $i \in J^l$, as otherwise the circulation requirements would not be satisfied. Let $i_1, \dots, i_r$ be the indices in J^l such that $\bar{v}_{i_t}^l > 0$ for $1 \leq t \leq r$. Note that there do not exist two distinct indices t, t', with $1 \leq t, t' \leq r$, such that $p'(i_t) = p'(i_{t'})$, as otherwise the arcs $v_{i_t}^l, v_{i_{t'}}^l$ would form a cycle contained in the support of circulation $(\bar{u}, \bar{v})$. So we can assume without loss of generality that $p'(i_1) < \cdots < p'(i_r)$ (in other words, $f'(b_{i_1}) > \cdots > f'(b_{i_q})$). We also define $i_{r+1} := n + 1$ (thus $p'(i_{r+1}) = k$).

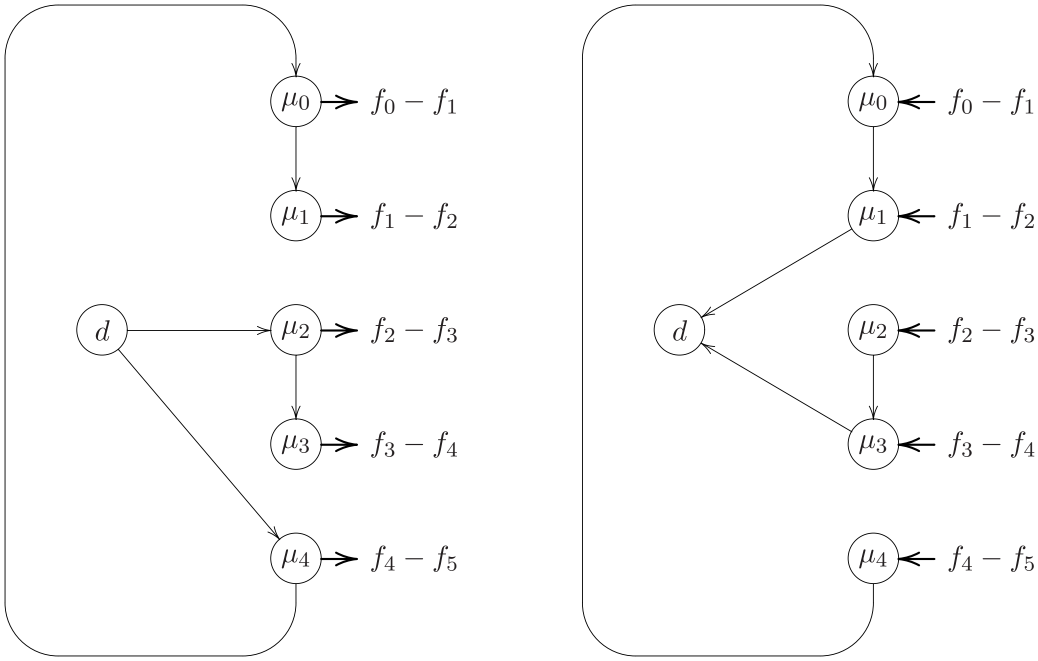
The structure of the network easily implies that the nonzero entries of $\bar{v}^l$ are (see the example in Figure 5.2 (a))

$$\bar{v}_{i_t}^l = \sum_{\ell = p'(i_t)}^{p'(i_{t+1})-1} (f_\ell - f_{\ell+1}) = f_{p'(i_t)} - f_{p'(i_{t+1})} \quad \text{for } 1 \leq t \leq r-1,$$

$$\bar{v}_{i_r}^l = \sum_{\ell = p'(i_r)}^{k-1} (f_\ell - f_{\ell+1}) + \sum_{\ell=0}^{p'(i_1)-1} (f_\ell - f_{\ell+1}) = f_{p'(i_r)} + \left(1 - f_{p'(i_1)}\right),$$

while $\bar{u}_0 = 1 - f_{p'(i_1)}$. The corresponding inequality (5.37) is then

$$s - \sum_{t=1}^{r} \left(f_{p'(i_t)} - f_{p'(i_{t+1})}\right) \left(z_{i_t} + \lceil l_{i_t} \rceil\right) - \left(1 - f_{p'(i_1)}\right) \left(z_{i_r} + \lceil l_{i_r} \rceil - 1\right) \geq 0,$$



(a) Case $\bar{w} = 1$.

(b) Case $\bar{w} = -1$.

Figure 5.2: Acyclic circulations in network $\mathcal{N}$.

which can be equivalently be written as

$$s - \sum_{t=1}^{r} \left(f'(l_{i_t}) - f'(l_{i_{t+1}})\right)\left(z_{i_t} + \lceil l_{i_t} \rceil\right) - \left(1 - f'(l_{i_1})\right)\left(z_{i_r} + \lceil l_{i_r} \rceil - 1\right) \geq 0, \tag{5.42}$$

where $f'(l_{i_{r+1}}) := 0$.

The case $\bar{w} < 0$

Let $(\bar{w}, \bar{u}, \bar{v})$ be an extreme ray of cone (5.38)–(5.40) with $\bar{w} < 0$. Without loss of generality we can assume $\bar{w} = -1$. In this case $(\bar{u}, \bar{v})$ is an extreme point of the polyhedron obtained by setting $w = -1$ in (5.38)–(5.40). By Theorem 5.1, this implies that $(\bar{u}, \bar{v})$ defines an acyclic circulation in the corresponding network $\mathcal{N}$. Such a network has the same structure as that depicted in Figure 5.1, except that the thick arrows should be reversed (i.e. there are supplies instead of demand on the nodes).

Similarly to the case $\bar{w} > 0$, one proves that $\bar{v}_i^u = 0$ for all $i \in J^l$ and $\bar{v}_i^u > 0$ for at least an index $i \in J^u$. Let $i_1, \ldots, i_r$ be the indices in J^u such that $\bar{v}_{i_t}^u > 0$ for $1 \leq t \leq r$. Note that there do not exist two distinct indices t, t', with $1 \leq t, t' \leq r$, such that $q(i_t) = q(i_{t'})$, as otherwise the arcs $v_{i_t}^u, v_{i_{t'}}^u$ would form a cycle contained in the support of circulation $(\bar{u}, \bar{v})$. So we can assume without loss of generality $q(i_1) > \cdots > q(i_r)$ (in other words, $f(b_{i_1}) < \cdots < f(b_{i_r})$).

The structure of the network easily implies that the nonzero entries of $\bar{v}^u$ are (see the example in Figure 5.2 (b))

$$\bar{v}^u_{i_t} = \sum_{\ell=q(i_{t+1})}^{q(i_t)-1} (f_\ell - f_{\ell+1}) = f_{q(i_{t+1})} - f_{q(i_t)} \quad \text{for } 1 \le t \le r-1,$$
$$\bar{v}^u_{i_r} = \sum_{\ell=0}^{q(i_r)-1} (f_\ell - f_{\ell+1}) + \sum_{\ell=q(i_1)}^{k-1} (f_\ell - f_{\ell+1}) = \big(1 - f_{q(i_r)}\big) + f_{q(i_1)},$$

while $\bar{u}_0 = f_{q(i_1)}$. The corresponding inequality (5.37) is then

$$s + \sum_{t=1}^{r} \big(f_{q(i_{t+1})} - f_{q(i_t)}\big)\,(z_{i_t} + \lfloor u_{i_t} \rfloor) + f_{q(i_1)}\,(z_{i_r} + \lfloor u_{i_r} \rfloor + 1) \ge 0,$$

where $q(i_{r+1}) := 0$. The above inequality can be equivalently written as

$$s + \sum_{t=1}^{r} \big(f(u_{i_{t+1}}) - f(u_{i_t})\big)\,(z_{i_t} + \lfloor u_{i_t} \rfloor) + f(u_{i_1})\,(z_{i_r} + \lfloor u_{i_r} \rfloor + 1) \ge 0, \tag{5.43}$$

where $f(u_{i_{r+1}}) := 1$.

We have proven the following result:

Theorem 5.4 *The convex hull of* (5.11)–(5.15)*, a general mixed-integer set with dual network constraint matrix and a single continuous variable, is given by the following linear inequalities (where each occurrence of* z_0 *should be replaced by* 0*):*

- (5.41) *for all* $i \in J^l$ *and* $j \in J^u$*;*
- (5.42) *for all sequences of indices* $i_1, \ldots, i_r$ *in* J^l *such that* $f'(b_{i_1}) > \cdots > f'(b_{i_r})$*;*
- (5.43) *for all sequences of indices* $i_1, \ldots, i_r$ *in* J^u *such that* $f(b_{i_1}) < \cdots < f(b_{i_r})$*;*
- *the inequalities of the system* $Bz \ge d$*.*

5.2.3 The mixing set

We recall the definition of the mixing set given in Section 4.2:

$$s + z_i \ge b_i, \quad 1 \le i \le n, \tag{5.44}$$
$$s \ge 0, \tag{5.45}$$
$$z_i \text{ integer}, \quad 1 \le i \le n, \tag{5.46}$$

where $b_i \in \mathbb{R}$ for $1 \le i \le n$. The importance of this set in the context of lot-sizing was discussed in Section 4.2. The convex hull of the above set was given by Günlük and Pochet [31]. Here we obtain the convex hull as an application of Theorem 5.4.

Proposition 5.5 *The convex hull of the mixing set (5.44)–(5.46) is described by $s \geq 0$ along with the linear inequalities*

$$s + \sum_{t=1}^{r} \left(f'(b_{i_t}) - f'(b_{i_{t+1}})\right)\left(z_{i_t} - \lceil b_{i_t} \rceil\right) \geq 0, \tag{5.47}$$

$$s + \sum_{t=1}^{r} \left(f'(b_{i_t}) - f'(b_{i_{t+1}})\right)\left(z_{i_t} - \lceil b_{i_t} \rceil\right) + \left(1 - f'(b_{i_1})\right)\left(z_{i_r} - \lceil b_{i_r} \rceil + 1\right) \geq 0 \tag{5.48}$$

for all sequences of indices $i_1, \ldots, i_r$ in $\{1, \ldots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_q})$, where $f'(b_{i_{r+1}}) := 0$.

Proof. The set (5.44)–(5.46) can be transformed into a mixed-integer set with dual network constraint matrix by applying the following mixed-integer linear mapping:

$$s' := s, \ \ z'_i := -z_i \ \text{ for } 1 \leq i \leq n. \tag{5.49}$$

The transformed set is

$$s' - z'_i \geq b_i, \quad 1 \leq i \leq n, \tag{5.50}$$
$$s' \geq 0, \tag{5.51}$$
$$z'_i \text{ integer}, \quad 1 \leq i \leq n. \tag{5.52}$$

The set (5.50)–(5.52) is of the type (5.11)–(5.15), with $J^l = \{0, \ldots, n\}$ and $J^u = \varnothing$. By Theorem 5.4, a linear inequality description of the convex hull of this set is given by inequalities (5.42) for all sequences of indices $i_1, \ldots, i_r$ in $\{0, \ldots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_r})$ (where each occurrence of z_0 should be replaced by 0).

Assume first that the sequence $i_1, \ldots, i_r$ does not contain index 0. After applying the inverse of (5.49), the corresponding inequality (5.42) is precisely inequality (5.48).

Now assume that the sequence $i_1, \ldots, i_r$ contains index 0. Since the lower bound l_0 is 0 for the mixing set, $f'(0) = 1$ and thus $i_1 = 0$. If $r = 1$ then the corresponding inequality (5.42) is $s' - z_0 \geq 0$, i.e. $s \geq 0$. If $r > 1$ then after applying the inverse of (5.49) and setting $z_0 = 0$, inequality (5.42) becomes

$$s + \sum_{t=2}^{r} \left(f'(b_{i_t}) - f'(b_{i_{t+1}})\right)\left(z_{i_t} - \lceil b_{i_t} \rceil\right) \geq 0.$$

Renumbering the indices gives inequality (5.47). □

Inequalities (5.47)–(5.48) are called *mixing inequalities*, as they can be obtained from the original inequalities (5.44) through a *mixing* procedure (see [31]). An $\mathcal{O}(n \log n)$ separation algorithm for the mixing inequalities is known [53].

When $r = 1$, the mixing inequality (5.47) is the simple MIR-inequality by Nemhauser and Wolsey [49] (see also Theorem 1.11), while the mixing inequality (5.48) coincides with the original inequality $s + z_{i_1} \geq b_{i_1}$.

Miller and Wolsey [45] showed that if a system $Bz \geq d$, where B is a dual network matrix and d is an integral vector, is added to constraints (5.44)–(5.46), a linear inequality description of the resulting set in its original space is obtained by just including the system $Bz \geq d$ in the description of the mixing set given by the above proposition. This result is also implied by Theorem 5.4 or Proposition 2.16 (in fact the proof of Proposition 2.16 uses the same technique as that adopted by Miller and Wolsey).

5.3 The mixing set with flows

We recall the definition of the mixing set with flows given in Section 4.2.2:

$$s + y_i \geq b_i, \quad 1 \leq i \leq n, \tag{5.53}$$
$$y_i \leq z_i, \quad 1 \leq i \leq n, \tag{5.54}$$
$$s \geq 0,\ y_i \geq 0, \quad 1 \leq i \leq n, \tag{5.55}$$
$$z_i \text{ integer}, \quad 1 \leq i \leq n, \tag{5.56}$$

where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$. Since all variables are nonnegative (as $z_i \geq y_i \geq 0$ for $1 \leq i \leq n$), we can assume without loss of generality that $b_i \geq 0$ for $1 \leq i \leq n$. We discussed in Section 4.2.2 the relevance of this set in the context of lot-sizing.

As shown in Section 4.2.2, this set admits an extended formulation with $\mathcal{O}(n^2)$ variables and constraints (see Proposition 4.5). In this section, after transforming the above set into a mixed-integer set with dual network constraint matrix, we explicitly give the extended formulation and then project it onto the original space of variables.

The computation of this projection will be more difficult and technical than that carried out in Section 5.2.

5.3.1 The extended formulation

To transform (5.53)–(5.56) into a dual network set, we apply the following mixed-integer linear mapping:

$$y'_0 := s; \ \ y'_i := -y_i, \ \ z'_i := -z_i \ \text{ for } 1 \leq i \leq n.$$

The transformed set is then

$$y'_0 - y'_i \geq b_i, \quad 1 \leq i \leq n, \tag{5.57}$$
$$y'_i - z'_i \geq 0, \quad 1 \leq i \leq n, \tag{5.58}$$
$$y'_0 \geq 0,\ y'_i \leq 0, \quad 1 \leq i \leq n, \tag{5.59}$$
$$z'_i \text{ integer}, \quad 1 \leq i \leq n. \tag{5.60}$$

Let $f^0_1, \dots, f^0_k$ be the k distinct elements in $\{0, f(b_1), \dots, f(b_n)\}$, with $f^0_1 > \cdots > f^0_k = 0$. For each index $1 \leq i \leq n$, let $f^i_1, \dots, f^i_k$ be the k elements in $\{f\left(f^0_1 - b_i\right), \dots, f\left(f^0_k - b_i\right)\}$, with $f^i_1 > \cdots > f^i_k$. (Note that $f^i_1, \dots, f^i_k$ are pairwise distinct because so are $f^0_1, \dots, f^0_k$.) We set $f^i_0 := 1$ and $f^i_{k+1} := 0$ for $0 \leq i \leq n$.

Lemma 5.6 *For each index $0 \le i \le n$, the list of fractional parts $\mathcal{F}_i := \{f_1^i, \dots, f_k^i\}$ is complete for (5.57)–(5.60) with respect to variable y_i'.*

Proof. We use the notation of the proof of Theorem 3.6. If we let X denote the mixed-integer set (5.57)–(5.60), the graph $\mathcal{G}_X$ is a star with center node corresponding to variable y_0'. For $1 \le i \le n$, there is an arc leaving the center node and entering the node corresponding to variable y_i'. Given a vertex $\bar{x} = (\bar{y}', \bar{z}')$ of $\text{conv}(X)$ and a connected component $C_{\bar{x}}$ of $F_{\bar{x}}$, node r correspond to a variable that takes an integer value (this follows from Observation 3.7). The result is then a consequence of equation (3.24). □

Similarly to Section 5.2, for each index $1 \le i \le n$ we define $p(i)$ to be the unique index in $\{1, \dots, k\}$ such that $f^0_{p(i)} = f(b_i)$. One can check that for each index $1 \le i \le n$,

$$f_\ell^i = \begin{cases} f^0_{p(i)+\ell} - f^0_{p(i)} + 1 & \text{if } 0 \le \ell \le k - p(i), \\ f^0_{p(i)+\ell-k} - f^0_{p(i)} & \text{if } k - p(i) + 1 \le \ell \le k. \end{cases} \tag{5.61}$$

By Theorem 2.13 and using Observation 2.15 to model inequalities (5.58), an extended formulation for (5.57)–(5.60) is given by the following linear system:

$$\begin{aligned}
y_i' &= \sum_{\ell=0}^{k} (f_\ell^i - f_{\ell+1}^i)\mu_\ell^i, && 0 \le i \le n, && (5.62)\\
\mu_k^i - \mu_0^i &= 1, && 0 \le i \le n, && (5.63)\\
\mu_\ell^i - \mu_{\ell-1}^i &\ge 0, && 0 \le i \le n,\ 1 \le \ell \le k, && (5.64)\\
\mu^0_{p(i)+\ell} - \mu_\ell^i &\ge \lfloor b_i \rfloor + 1, && 1 \le i \le n,\ 1 \le \ell \le k - p(i), && (5.65)\\
\mu^0_{p(i)+\ell-k} - \mu_\ell^i &\ge \lfloor b_i \rfloor, && 1 \le i \le n,\ k - p(i) + 1 \le \ell \le k, && (5.66)\\
\mu_k^i - z_i' &\ge 1, && 1 \le i \le n, && (5.67)\\
\mu_k^0 &\ge 1, && && (5.68)\\
\mu_{k-1}^i &\le 0, && 1 \le i \le n. && (5.69)
\end{aligned}$$

Before computing the projection onto the original space of variables, we write the above system in a more convenient form.

Similarly to Section 5.2, for each index $1 \le i \le n$ we denote by $p'(i)$ the unique index in $\{0, \dots, k-1\}$ such that $f^0_{p'(i)} = f'(b_i)$, where notation f' is defined in (5.26). We set $p'(0) := 0$ and $p'(n+1) := k$.

Using equations (5.63), one can check that inequalities (5.65)–(5.66) are equivalent to the inequalities

$$\begin{aligned}
\mu^0_{p(i)+\ell} - \mu_\ell^i &\ge \lfloor b_i \rfloor + 1, && 1 \le i \le n,\ 0 \le \ell \le k - p(i) - 1,\\
\mu^0_{p(i)+\ell-k} - \mu_\ell^i &\ge \lfloor b_i \rfloor, && 1 \le i \le n,\ k - p(i) \le \ell \le k - 1.
\end{aligned}$$

It is not difficult to see that the above inequalities are in turn equivalent to the following (the case $b_i \notin \mathbb{Z}$ is trivial as $p(i) = p'(i)$, the case $b_i \in \mathbb{Z}$ is less trivial but easy —just recall that $p(i) = k$ and $p'(i) = 0$):

$$\begin{aligned} \mu^0_{p'(i)+\ell} - \mu^i_\ell &\geq \lceil b_i \rceil, && 1 \leq i \leq n,\ 0 \leq \ell \leq k - p'(i) - 1, \\ \mu^0_{p'(i)+\ell-k} - \mu^i_\ell &\geq \lceil b_i \rceil - 1, && 1 \leq i \leq n,\ k - p'(i) \leq \ell \leq k - 1. \end{aligned}$$

If for an integer α we write $[\alpha]$ to denote the remainder of the division of α by k, system (5.62)–(5.69) is then equivalent to the following:

$$y'_i = \sum_{\ell=0}^{k} (f^i_\ell - f^i_{\ell+1})\mu^i_\ell, \quad 0 \leq i \leq n, \tag{5.70}$$

$$\mu^i_k - \mu^i_0 = 1, \quad 0 \leq i \leq n, \tag{5.71}$$

$$\mu^i_\ell - \mu^i_{\ell-1} \geq 0, \quad 0 \leq i \leq n,\ 1 \leq \ell \leq k, \tag{5.72}$$

$$\mu^0_{[p'(i)+\ell]} - \mu^i_\ell \geq \lceil b_i \rceil, \quad 1 \leq i \leq n,\ 0 \leq \ell \leq k - p'(i) - 1, \tag{5.73}$$

$$\mu^0_{[p'(i)+\ell]} - \mu^i_\ell \geq \lceil b_i \rceil - 1, \quad 1 \leq i \leq n,\ k - p'(i) \leq \ell \leq k - 1, \tag{5.74}$$

$$\mu^i_k - z'_i \geq 1, \quad 1 \leq i \leq n, \tag{5.75}$$

$$\mu^0_k \geq 1, \tag{5.76}$$

$$\mu^i_{k-1} \leq 0, \quad 1 \leq i \leq n. \tag{5.77}$$

Equations (5.71) can be used to eliminate variables μ^i_k for $0 \leq i \leq n$. Note that for $0 \leq i \leq n$, the coefficient of μ^i_k in equation (5.70) is equal to zero, as $f^i_k = f^i_{k+1} = 0$. Furthermore, none of inequalities (5.73)–(5.74) contains variable μ^i_k in its support.

System (5.70)–(5.77) is then equivalent to the following one (we assign dual variables to the constraints as indicated on the left):

$$\begin{aligned} w^i: &\quad y'_i = \sum_{\ell=0}^{k-1} (f^i_\ell - f^i_{\ell+1})\mu^i_\ell, && 0 \leq i \leq n, \\ u^i_\ell: &\quad \mu^i_\ell - \mu^i_{\ell-1} \geq 0, && 0 \leq i \leq n,\ 1 \leq \ell \leq k-1, \\ u^i_0: &\quad \mu^i_0 - \mu^i_{k-1} \geq -1, && 0 \leq i \leq n, \\ \xi^i_\ell: &\quad \mu^0_{[p'(i)+\ell]} - \mu^i_\ell \geq \lceil b_i \rceil, && 1 \leq i \leq n,\ 0 \leq \ell \leq k - p'(i) - 1, \\ \xi^i_\ell: &\quad \mu^0_{[p'(i)+\ell]} - \mu^i_\ell \geq \lceil b_i \rceil - 1, && 1 \leq i \leq n,\ k - p'(i) \leq \ell \leq k - 1, \\ v^i: &\quad \mu^i_0 - z'_i \geq 0, && 1 \leq i \leq n, \\ v^0: &\quad \mu^0_0 \geq 0, && \\ \vartheta^i: &\quad -\mu^i_{k-1} \geq 0, && 1 \leq i \leq n. \end{aligned}$$

Note that except for the equations on the first line, the constraint matrix of the above system is still a dual network matrix.

5.3.2 The projection

By Theorem 1.17, a linear inequality description of the convex hull of (5.57)–(5.60) in its original space is given by inequalities

$$\sum_{i=0}^{n} \bar{w}^i y_i' - \sum_{i=1}^{n} \bar{v}^i z_i' \geq -\sum_{i=0}^{n} \bar{u}_0^i + \sum_{i=1}^{n} \left(\sum_{\ell=0}^{k-p'(i)-1} \bar{\xi}_\ell^i \lceil b_i \rceil + \sum_{\ell=k-p'(i)}^{k-1} \bar{\xi}_\ell^i (\lceil b_i \rceil - 1) \right) \tag{5.78}$$

for all vectors $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ that are rays of the following cone (beside each constraint the corresponding primal variable is indicated):

$$\begin{aligned}
\mu_0^0: &\quad u_0^0 - u_1^0 + \textstyle\sum_{i=1}^n \xi^i_{[-p'(i)]} + v^0 = \left(f_0^0 - f_1^0\right) w^0, && && (5.79)\\
\mu_\ell^0: &\quad u_\ell^0 - u_{\ell+1}^0 + \textstyle\sum_{i=1}^n \xi^i_{[\ell-p'(i)]} = \left(f_\ell^0 - f_{\ell+1}^0\right) w^0, && 1 \leq \ell \leq k-2, && (5.80)\\
\mu_{k-1}^0: &\quad u_{k-1}^0 - u_0^0 + \textstyle\sum_{i=1}^n \xi^i_{k-1-p'(i)} = \left(f_{k-1}^0 - f_k^0\right) w^0, && && (5.81)\\
\mu_0^i: &\quad u_0^i - u_1^i - \xi_0^i + v^i = \left(f_0^i - f_1^i\right) w^i, && 1 \leq i \leq n, && (5.82)\\
\mu_\ell^i: &\quad u_\ell^i - u_{\ell+1}^i - \xi_\ell^i = \left(f_\ell^i - f_{\ell+1}^i\right) w^i, && 1 \leq i \leq n,\ 1 \leq \ell \leq k-2, && (5.83)\\
\mu_{k-1}^i: &\quad u_{k-1}^i - u_0^i - \xi_{k-1}^i - \vartheta^i = \left(f_{k-1}^i - f_k^i\right) w^i, && 1 \leq i \leq n, && (5.84)\\
&\quad w^i \text{ free; } u_0^i, \dots, u_{k-1}^i \geq 0,\ v^i \geq 0, && 0 \leq i \leq n, && (5.85)\\
&\quad \xi_0^i, \dots, \xi_{k-1}^i \geq 0,\ \vartheta^i \geq 0, && 1 \leq i \leq n. && (5.86)
\end{aligned}$$

In the original variables, inequality (5.78) reads

$$\bar{w}^0 s - \sum_{i=1}^{n} \bar{w}^i y_i + \sum_{i=1}^{n} \bar{v}^i z_i \geq -\sum_{i=0}^{n} \bar{u}_0^i + \sum_{i=1}^{n} \left(\sum_{\ell=0}^{k-p'(i)-1} \bar{\xi}_\ell^i \lceil b_i \rceil + \sum_{\ell=k-p'(i)}^{k-1} \bar{\xi}_\ell^i (\lceil b_i \rceil - 1) \right),$$

or equivalently

$$\bar{w}^0 s - \sum_{i=1}^{n} \bar{w}^i y_i + \sum_{i=1}^{n} \bar{v}^i z_i \geq -\sum_{i=0}^{n} \bar{u}_0^i + \sum_{i=1}^{n} \left(\sum_{\ell=0}^{k-1} \bar{\xi}_\ell^i \lceil b_i \rceil - \sum_{\ell=k-p'(i)}^{k-1} \bar{\xi}_\ell^i \right). \tag{5.87}$$

Let $\mathcal{C}$ denote the polyhedral cone defined by inequalities (5.79)–(5.86). In the following we study the rays of $\mathcal{C}$ generating inequalities (5.87) that are non-redundant in the description of the convex hull of (5.53)–(5.56). This will reveal simpler than characterizing the extreme rays of $\mathcal{C}$ (as we did in Section 5.2), and will also allow us to ignore a large number of redundant inequalities (5.87) arising from the extreme rays of $\mathcal{C}$.

Note that summing up all equations (5.79)–(5.81) gives

$$-w^0 + v^0 + \sum_{i=1}^{n} \sum_{\ell=0}^{k-1} \xi_\ell^i = 0, \tag{5.88}$$

which implies $w^0 \geq 0$, as all other variables appearing in the above equation are nonnegative.

Let $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ be a ray of cone $\mathcal{C}$. If $\bar{w}^i = 0$ for all $0 \leq i \leq n$, the above equation shows that $\bar{\xi}_\ell^i = 0$ for all $1 \leq i \leq n$ and $0 \leq \ell \leq k-1$, and the corresponding inequality (5.87) is

$\sum_{i=1}^{n} \bar{v}^i z_i \geq -\sum_{i=0}^{n} \bar{u}_0^i$. Among the inequalities of this form, the only non-redundant ones are $z_i \geq 0$ for $1 \leq i \leq n$, which are clearly superfluous in the description of the convex hull of (5.53)–(5.56). Thus from now on we assume that $\bar{w} \neq \mathbf{0}$ (and $\bar{w}^0 \geq 0$).

The network

For each fixed vector $\bar{w} \neq \mathbf{0}$, let $\mathcal{C}(\bar{w})$ be the polyhedron obtained from $\mathcal{C}$ by setting $w^i = \bar{w}^i$ for $0 \leq i \leq n$. That is,

$$\mathcal{C}(\bar{w}) := \{(u, \xi, v, \vartheta) : (\bar{w}, u, \xi, v, \vartheta) \in \mathcal{C}\}.$$

Note that $\mathcal{C}(\bar{w})$ is the feasible region of a circulation problem on a network $\mathcal{N}$ which is independent of $\bar{w}$. Similarly to Section 5.2, we use the primal variables μ_ℓ^i to denote the nodes of $\mathcal{N}$ and the dual variables $u_\ell^i, \xi_\ell^i, v^i, \vartheta^i$ to denote the arcs. The structure of $\mathcal{N}$ is now described.

For $0 \leq i \leq n$, let S^i be the subnetwork of $\mathcal{N}$ induced by nodes $\mu_0^i, \ldots, \mu_{k-1}^i$ (arcs having a node μ_ℓ^i and the dummy node d as endpoints belong to S^i). We call $S^0, \ldots, S^n$ the *sectors* of $\mathcal{N}$. Note that every arc whose endnodes lie on two distinct sectors of $\mathcal{N}$ has its head in S^0 and its tail in S^i for some $1 \leq i \leq n$.

Figure 5.3 represents the structure of a sector S^i for some $1 \leq i \leq n$ and sector S^0, as well as the connections between S^i and S^0. Note that the nodes of each sector are aligned on a vertical line. The k positions on such a line are called *levels*: the highest position correspond to level 0, the lowest one to level $k-1$. For each $0 \leq i \leq n$ and $0 \leq \ell \leq k-1$, node μ_ℓ^i is located at level $[p'(i) + \ell]$. There are at least two good reasons for such a choice.

The first good reason for locating node μ_ℓ^i at level $[p'(i) + \ell]$ is that this simplifies the representation of the network, as all arcs ξ_ℓ^i are horizontal.

To illustrate the second reason, let $\mathcal{N}(\bar{w})$ denote network $\mathcal{N}$ with the circulation requirements corresponding to $\bar{w}$. It is readily checked that for $0 \leq i \leq n$, the total requirement of all nodes in sector S^i in $\mathcal{N}(\bar{w})$ is $\bar{w}^i$. For $0 \leq i \leq n$ and $0 \leq \ell \leq k-1$, the requirement of node μ_ℓ^i in $\mathcal{N}(\bar{w})$ is $\left(f_\ell^i - f_{\ell+1}^i\right) \bar{w}^i$. Using equation (5.61) and recalling that $[p(i)] = [p'(i)]$ for $0 \leq i \leq n$, one can check that

$$\left(f_\ell^i - f_{\ell+1}^i\right) \bar{w}^i = \left(f^0_{[p(i)+\ell]} - f^0_{[p(i)+\ell]+1}\right) \bar{w}^i = \left(f^0_{[p'(i)+\ell]} - f^0_{[p'(i)+\ell]+1}\right) \bar{w}^i \tag{5.89}$$

for all indices $0 \leq i \leq n$ and $0 \leq \ell \leq k-1$. Since node μ_ℓ^i is located at level $[p'(i) + \ell]$, this shows that nodes of distinct sectors located at the same level are associated with the same fraction of the total requirement of their sectors.

It is clear that a vector $\left(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ belongs to $\mathcal{C}$ if and only if $\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ corresponds to a feasible circulation in $\mathcal{N}(\bar{w})$. Similarly to Section 5.2, we use $\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ to denote both the vector and the corresponding circulation.

We say that a cycle in $\mathcal{N}$, possibly containing the dummy node d, is a *heavy cycle* if the corresponding inequality (5.87) is anything but $0 \geq 0$.

The following observations will be used several times in the remainder of the section:

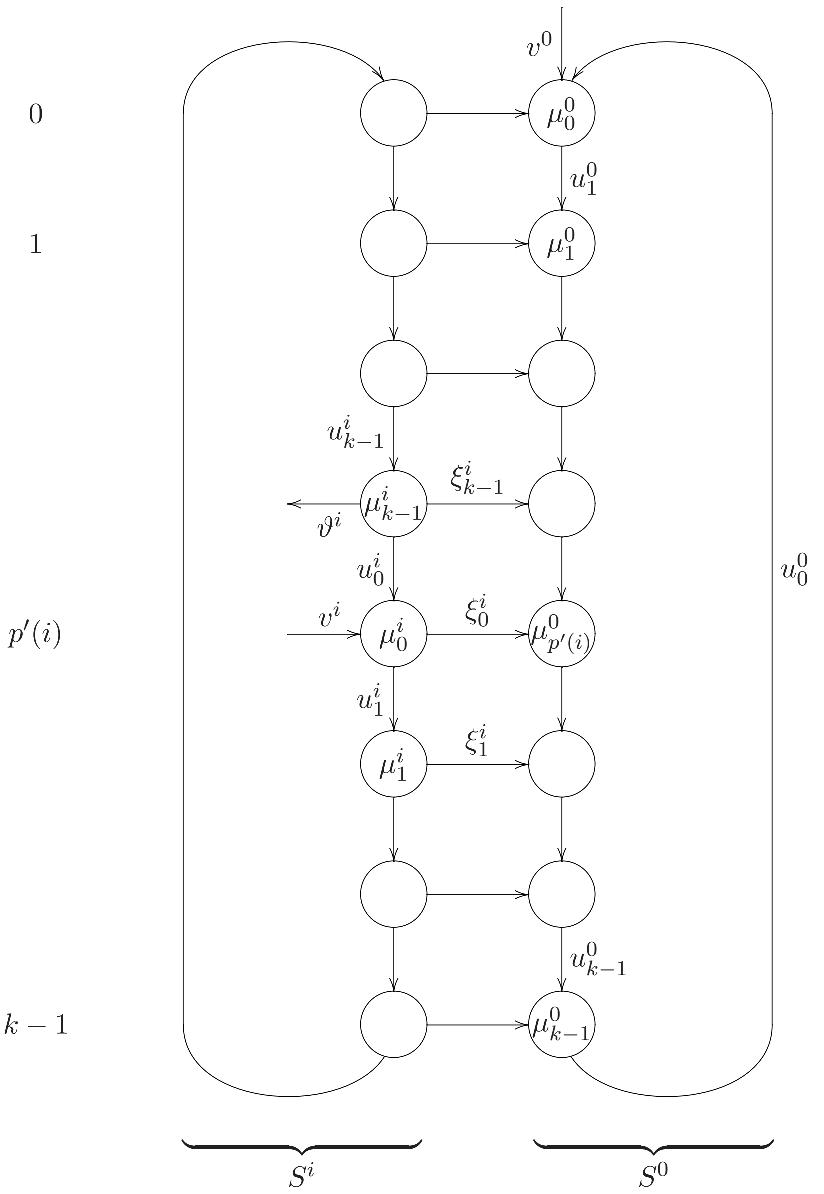


Figure 5.3: The structure of a sector S^i for $1 \leq i \leq n$ and sector S^0, and the connections between them. Levels are indicated on the left. Circulation requirements are not represented.

Lemma 5.7 *For $\bar{w} \neq \mathbf{0}$, let $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ be a vector in $\mathcal{C}$ generating an inequality (5.87) that is non-redundant in the linear inequality description of the convex hull of (5.53)–(5.56). The following hold:*

(i) *The support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ contains no heavy cycle in $\mathcal{N}$.*

(ii) *Fix $1 \leq i \leq n$ and assume that the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ contains a forest F in $\mathcal{N}$ which spans all nodes in sectors S^0 and S^i. Also suppose that each connected component of F is*

  (a) *either a tree containing the dummy node d,*

  (b) *or a single arc ξ^i_ℓ for some $0 \leq \ell \leq k-1$.*

  Then $\bar{w}^0 = -\bar{w}^i$ and $\bar{w}^j = 0$ for all $j \notin \{0, i\}$.

(iii) *Fix $1 \leq i \leq n$ and assume that the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ contains a tree T in $\mathcal{N}$ which spans all nodes in sector S^i as well as the dummy node d. Then the inequality (5.87) corresponding to vector $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ is implied by inequalities $y_i \geq 0$, $y_i \leq z_i$ and $z_j \geq 0$ for $1 \leq j \leq n$.*

Proof. (i) Assume that $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ contains a heavy cycle C in its support. For $\varepsilon > 0$ small enough, let $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ be the feasible circulation in $\mathcal{N}(\bar{w})$ obtained from $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ by increasing by ε the variables corresponding to the arcs of C. Similarly, let $(\check{u}, \check{\xi}, \check{v}, \check{\vartheta})$ be the feasible circulation in $\mathcal{N}(\bar{w})$ obtained from $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ by decreasing by ε the variables corresponding to the arcs of C. Clearly $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}) = \frac{1}{2}(\bar{w}, \hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta}) + \frac{1}{2}(\bar{w}, \check{u}, \check{\xi}, \check{v}, \check{\vartheta})$. Since the inequality (5.87) corresponding to $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ is non-redundant in the description of the convex hull of (5.53)–(5.56), it follows that such inequality is identical (up to multiplication by a positive number) to those corresponding to $(\bar{w}, \hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ and $(\bar{w}, \check{u}, \check{\xi}, \check{v}, \check{\vartheta})$. However this contradicts the fact that C is a heavy cycle in $\mathcal{N}(\bar{w})$.

(ii) Assume that there is a forest F as above (note that F has at most one connected component of type (a)). Let $\varepsilon > 0$ be a sufficiently small number. Define

$$\hat{w}^0 := \bar{w}^0 + \varepsilon, \;\; \hat{w}^i := \bar{w}^i - \varepsilon, \;\; \hat{w}^j := \bar{w}^j \text{ for } j \notin \{0, i\}.$$

It can be checked that in each connected component of F the total requirement of the nodes is unchanged. Then by Theorem 5.2 there exists a unique circulation $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ in $\mathcal{N}(\hat{w})$ that coincides with $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ on all arcs not belonging to F. Similarly, if one defines

$$\check{w}^0 := \bar{w}^0 - \varepsilon, \;\; \check{w}^i := \bar{w}^i + \varepsilon, \;\; \check{w}^j := \bar{w}^j \text{ for } j \notin \{0, i\},$$

there exists a unique circulation $(\check{u}, \check{\xi}, \check{v}, \check{\vartheta})$ in $\mathcal{N}(\check{w})$ that coincides with $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ on all arcs not belonging to F.

It is easy to see that $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}) = \frac{1}{2}(\hat{w}, \hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta}) + \frac{1}{2}(\check{w}, \check{u}, \check{\xi}, \check{v}, \check{\vartheta})$. As in (i), this implies that these three vectors generate the same inequality (5.87) (up to multiplication by a positive number). The coefficients of variables s and y_i in the inequality (5.87) corresponding to vector $(\hat{w}, \hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ are $\bar{w}^0 + \varepsilon$ and $-\bar{w}^i - \varepsilon$ respectively. On the other hand, if $\alpha > 0$ is the real number such that $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}) = \alpha(\hat{w}, \hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$, then such coefficients are also equal to $\alpha\bar{w}^0$ and $-\alpha\bar{w}^i$ respectively. However, this is possible only if $\bar{w}^0 = -\bar{w}^i$.

Similarly, for $j \notin \{0, i\}$ the coefficient of variable y_j in the inequality (5.87) corresponding to vector $(\hat{w}, \hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ is $-\bar{w}^j$ on the one hand and $-\alpha\bar{w}^j$ on the other hand. This is possible only if $\bar{w}^j = 0$.

The proof of (iii) begins as that of (ii), except that now one has to define

$$\hat{w}^i := \bar{w}^i - \varepsilon, \;\; \hat{w}^j := \bar{w}^j \text{ for } j \neq i, \;\; \check{w}^i := \bar{w}^i + \varepsilon, \;\; \check{w}^j := \bar{w}^j \text{ for } j \neq i.$$

As in (ii) one defines circulations $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ in $\mathcal{N}(\hat{w})$ and $(\check{u}, \check{\xi}, \check{v}, \check{\vartheta})$ in $\mathcal{N}(\check{w})$ that coincide with $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ on all arcs not belonging to T. The same argument as that used above shows that $\bar{w}^j = 0$ for all $j \neq i$. Note in particular that condition $\bar{w}^0 = 0$ and equation (5.88) imply that $\bar{\xi}^j_\ell = 0$ for $1 \le j \le n$ and $0 \le \ell \le k-1$. Then the inequality (5.87) corresponding to vector $(\bar{w}, \bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ is

$$-\bar{w}^i y_i + \sum_{j=1}^{n} \bar{v}^j z_j \ge -\sum_{j=0}^{n} \bar{u}^j_0. \tag{5.90}$$

If $\bar{w}^i \le 0$ then the above inequality is implied by inequalities $y_i \ge 0$ and $z_j \ge 0$ for $1 \le j \le n$. So we assume $\bar{w}^i > 0$, say $\bar{w}^i = 1$ without loss of generality. In this case summing up equations (5.82)–(5.84) and using $\bar{\xi}^i_\ell = 0$ for $0 \le \ell \le k-1$ shows that $\bar{v}^i \ge 1$. Then inequality (5.90) is implied by inequalities $y_i \le z_i$ and $z_j \ge 0$ for $1 \le j \le n$. □

Assume that there is an index $1 \le i \le n$ such that $\bar{w}^i > 0$, say $\bar{w}^i = 1$ without loss of generality. Since v^i is the only arc entering sector S^i, then all arcs $v^i, u^i_1, \dots, u^i_{k-1}$ belong to the support of circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$. By Lemma 5.7 (iii) we can then ignore this case.

Therefore from now on we assume that $\bar{w}^i \le 0$ for all $1 \le i \le n$ (and recall that we have already shown that $\bar{w}^0 \ge 0$). Note that $\bar{w}^0$ is the total demand of the nodes in sector S^0, and for $1 \le i \le n$, $-\bar{w}^i$ is the total supply of the nodes in sector S^i.

Standard circulations

For a circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$ and an index $1 \le j \le k-1$, we define $\beta_j\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ as the total balance of flow of the set of nodes $\{\mu^0_\ell : j \le \ell \le k-1\}$, where the flow carried by arcs u^0_0 and u^0_j is ignored. After recalling that for $1 \le i \le n$ and $0 \le \ell \le k-1$ the arc leaving sector S^i end entering node μ^0_ℓ is arc $\xi^i_{[\ell - p'(i)]}$, we can write

$$\beta_j\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right) := \sum_{\ell=j}^{k-1} \left(\sum_{i=1}^{n} \bar{\xi}^i_{[\ell - p'(i)]} - (f_\ell - f_{\ell+1})\bar{w}^0 \right). \tag{5.91}$$

Lemma 5.8 *Any circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$ generating an inequality (5.87) that is non-redundant in the linear inequality description of the convex hull of (5.53)–(5.56) satisfies*

$$\bar{u}^0_0 = \max\left\{0, \max_{1 \le j \le k-1} \beta_j\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)\right\}. \tag{5.92}$$

Proof. Since $\beta_j\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ is the total balance of flow of the set of nodes $\left\{\mu^0_\ell : j \le \ell \le k-1\right\}$, where the flow carried by arcs u^0_0 and u^0_j is ignored, then clearly $\bar{u}^0_0 = \bar{u}^0_j + \beta_j\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ for all $1 \le j \le k-1$. Since $\bar{u}^0_j \ge 0$ for all $1 \le j \le k-1$, we see that $\bar{u}^0_0 \ge \max_{1\le j\le k-1} \beta_j\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$. Assume that $\bar{u}^0_0 > \max_{1\le j\le k-1} \beta_j\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$. Then clearly $\bar{u}^0_j > 0$ for all $1 \le j \le k-1$. Then, if also $\bar{u}^0_0 > 0$, the support of $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ contains the heavy cycle $u^0_1,\dots,u^0_{k-1},u^0_0$. The conclusion now follows from Lemma 5.7 (i). □

We say that two circulations in $\mathcal{N}(\bar{w})$ are *equivalent* if they give rise to the same inequality (5.87). Similarly we say that circulation $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ *dominates* circulation $\left(\hat{u},\hat{\xi},\hat{v},\hat{\vartheta}\right)$ if the inequality (5.87) corresponding to $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ dominates that corresponding to $\left(\hat{u},\hat{\xi},\hat{v},\hat{\vartheta}\right)$.

Lemma 5.9 *Any circulation in $\mathcal{N}(\bar{w})$ is equivalent to a circulation $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ in $\mathcal{N}(\bar{w})$ satisfying the following conditions for all $1 \le i \le n$:*

(i) *If $\bar{u}^i_\ell > 0$ for some $0 \le \ell \le k-1$, then $\bar{\xi}^i_\ell = 0$.*

(ii) *If $\bar{u}^i_\ell > 0$ for some $0 \le \ell \le k-1$, then $\bar{u}^i_l > 0$ for all $\ell \le l \le k-1$.*

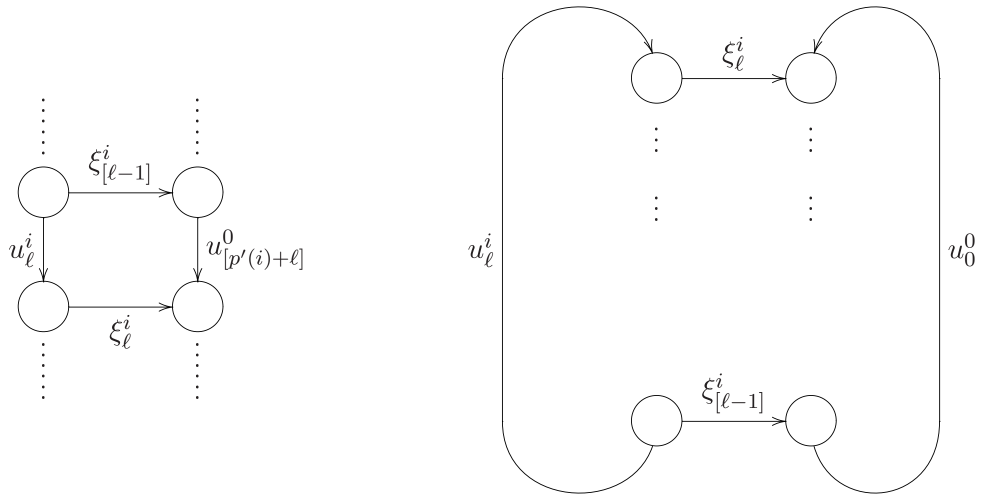
(iii) *If $\bar{\xi}^i_\ell = 0$ for some $0 \le \ell \le k-1$, then $\bar{\xi}^i_l = 0$ for all $\ell \le l \le k-1$.*

Proof. Let $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ be a circulation in $\mathcal{N}(\bar{w})$ that violates condition (i) and let ℓ be the maximum index in $\{0,\dots,k-1\}$ such that $\bar{u}^i_\ell > 0$ and $\bar{\xi}^i_\ell > 0$. Define $\rho := \min\{\bar{u}^i_\ell, \bar{\xi}^i_\ell\}$. Note that arcs $u^i_\ell, \xi^i_\ell, \xi^i_{[\ell-1]}, u^0_{[p'(i)+\ell]}$ are as in Figure 5.4 (a) or (b), depending on the value of ℓ. If we decrease the flow on arcs u^i_ℓ, ξ^i_ℓ by a quantity equal to ρ and increase the flow on arcs $\xi^i_{[\ell-1]}, u^0_{[p'(i)+\ell]}$ by the same amount, the resulting feasible circulation gives rise to the same inequality (5.87) as before. Furthermore, at least one of the arcs u^i_ℓ, ξ^i_ℓ now carries a flow of value 0. By iterating this procedure, we eventually find an equivalent circulation satisfying condition (i).

Now assume that condition (ii) is violated. Then there exists an index $0 \le \ell \le k-2$ such that $\bar{u}^i_\ell > 0$ and $\bar{u}^i_{\ell+1} = 0$. Note that arc ξ^i_ℓ necessarily carries a positive flow, that is, condition (i) is not satisfied. Thus any circulation satisfying condition (i) also satisfies condition (ii).

Finally we show that (i) implies (iii). Assume that condition (iii) is violated. Then there exists an index $0 \le \ell \le k-2$ such that $\bar{\xi}^i_\ell = 0$ and $\bar{\xi}^i_{\ell+1} > 0$. By (i), $\bar{u}^i_{\ell+1} = 0$, thus equation (5.83) for the indices $i, \ell+1$ (or equation (5.84) if $\ell = k-2$) implies $\bar{w}^i < 0$. Now equation (5.83) for the indices i, ℓ gives $\bar{\xi}^i_\ell > 0$, which contradicts our assumption. □

Let $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$ be a circulation in $\mathcal{N}(\bar{w})$ satisfying conditions (i)–(iii) of Lemma 5.7 and assume $\bar{u}^i_1 > 0$ for some index $1 \le i \le n$. Condition (ii) then implies that all arcs $u^i_1,\dots,u^i_{k-1}$ belong to the support of $\left(\bar{u},\bar{\xi},\bar{v},\bar{\vartheta}\right)$. Furthermore, by condition (i), $\bar{\xi}^i_\ell = 0$ for $1 \le \ell \le k-1$, thus necessarily $\bar{\vartheta}^i > 0$ (as all nodes in sector S^i have a nonnegative supply). Then arcs $u^i_1,\dots,u^i_{k-1},\vartheta^i$ form a tree in $\mathcal{N}$ satisfying the conditions of Lemma 5.7 (iii). Then this case can be ignored and we can assume $\bar{u}^i_1 = 0$ for $1 \le i \le n$, which also implies $\bar{u}^i_0 = 0$ for $1 \le i \le n$ (again by condition (ii) of Lemma 5.7).



(a) Case $\ell \neq [-p'(i)]$. (b) Case $\ell = [-p'(i)]$.

Figure 5.4: Illustration of the proof of Lemma 5.9, depending on the value of ℓ.

We say that a circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$ is a *standard* circulation if the following conditions hold for $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$:

- equation (5.92);
- conditions (i)–(iii) of Lemma 5.9;
- $\bar{u}^i_0 = 0$ for $1 \leq i \leq n$.

Figure 5.5 demonstrates the above definition.

The above discussion shows that every circulation that generates an inequality (5.87) which is non-redundant in the description of the convex hull of (5.53)–(5.56) is equivalent to a standard circulation. Thus from now on we only study the standard circulations in $\mathcal{N}(\bar{w})$.

It is easily checked that any circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$ satisfies conditions

$$\bar{v}^i - \bar{w}^i - \bar{\vartheta}^i \geq 0 \text{ for } 1 \leq i \leq n, \qquad \sum_{i=1}^{n} (\bar{v}^i - \bar{w}^i - \bar{\vartheta}^i) - \bar{w}^0 \leq 0 \tag{5.93}$$

(this can be deduced directly from conditions (5.79)–(5.86) or from the structure of the network.) Furthermore, given values of $(\bar{v}, \bar{\vartheta})$ satisfying the above inequalities, it is always possible to complete $(\bar{v}, \bar{\vartheta})$ to a feasible circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$.

Let $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ be a standard circulation in $\mathcal{N}(\bar{w})$. We claim that the knowledge of $\bar{v}^1, \dots, \bar{v}^n$ and $\bar{\vartheta}^1, \dots, \bar{\vartheta}^n$ is sufficient to completely determine $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$. To see this, observe the following: the values $\bar{\xi}^i_\ell, \bar{u}^i_\ell$ for $1 \leq i \leq n$ and $0 \leq \ell \leq k-1$ are determined by conditions (i)–(iii) of Lemma 5.9 together with conditions $\bar{u}^i_0 = 0$ for $1 \leq i \leq n$; the value of $\bar{u}^0_0$ is given by equation (5.92); the value of $\bar{v}^0$ can be obtained from equation (5.79).

This means that a standard circulation in $\mathcal{N}(\bar{w})$ is completely determined by nonnegative values of $\bar{v}^1, \dots, \bar{v}^n$ and $\bar{\vartheta}^1, \dots, \bar{\vartheta}^n$ satisfying conditions (5.93). Then when considering a

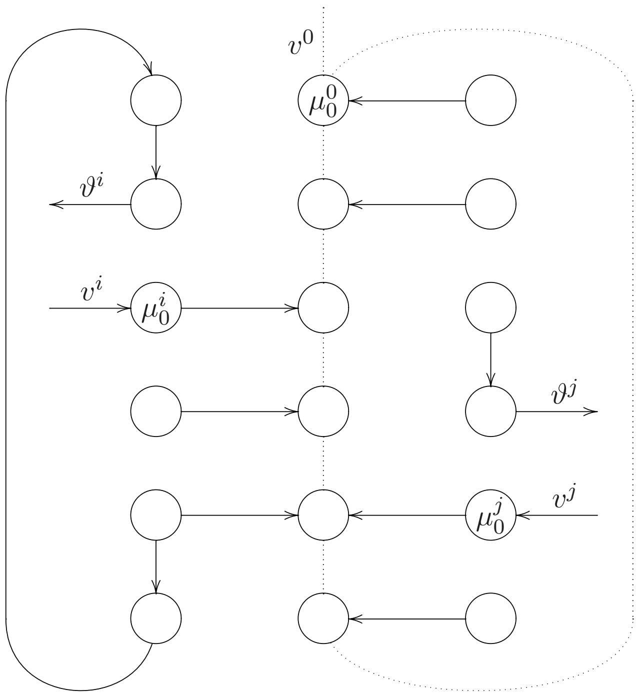


Figure 5.5: A standard circulation. Only nodes and arcs of sector S^0 (in the middle) and two other sectors S^i (on the left) and S^j (on the right) are depicted. Dotted lines indicate possible arcs. Circulation requirements are not represented (recall that nodes in S^i and S^j have a nonnegative supply, nodes in S^0 have a nonnegative demand).

standard circulation $\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$, we will use the short notation $\beta_j\left(\bar{v}, \bar{\vartheta}\right)$ instead of $\beta_j\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$, as this is not ambiguous. Similarly, in any further definition of notation relative to standard circulations we will only write the dependence on v, ϑ.

Define $J(\bar{w})$ as the set of indices in $\{1, \ldots, n\}$ such that $\bar{w}^i < 0$.

Let $\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ be a standard circulation in $\mathcal{N}(\bar{w})$. For $i \in J(\bar{w})$, we define

$$\lambda_i\left(\bar{v}, \bar{\vartheta}\right) := \max\left\{\ell : \bar{\xi}^i_{[\ell-p'(i)]} > 0\right\}. \tag{5.94}$$

To explain the above definition in words, recall that arc $\xi^i_{[\ell-p'(i)]}$ is located at level ℓ. Then $\lambda_i\left(\bar{v}, \bar{\vartheta}\right)$ is the maximum (i.e. the lowest) level of an arc that connects S^i and S^0 and carries a positive amount of flow.

Note that the above maximum is well defined, because for a standard circulation and an index $i \in J(\bar{w})$ one has $\bar{\xi}^i_0 = v^i - (f^i_0 - f^i_1)\bar{w}^i > 0$, as $\bar{w}^i < 0$.

It is also convenient to use notation

$$r_i\left(\bar{v}, \bar{\vartheta}\right) := \left[\lambda_i\left(\bar{v}, \bar{\vartheta}\right) - p'(i)\right],$$

so that

$$r_i\left(\bar{v}, \bar{\vartheta}\right) = \max\left\{\ell : \bar{\xi}^i_\ell > 0\right\}.$$

Our analysis has now to be divided into two cases: we first assume $J(\bar{w}) \neq \varnothing$ and then $J(\bar{w}) = \varnothing$.

The case $J(\bar{w}) \neq \varnothing$

To study the case $J(\bar{w}) \neq \varnothing$, another definition is needed. Given a standard circulation $\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ in $\mathcal{N}(\bar{w})$ and two indices $i, j \in J(\bar{w})$, we write $S^i \succ S^j$ *with respect to circulation* $\left(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\right)$ (or just $S^i \succ S^j$ if there is no ambiguity) to indicate that the following condition is satisfied:

$$\text{For each index } \ell \neq p'(j), \text{ if } \bar{\xi}^j_{[\ell-p'(j)]} > 0 \text{ then } \bar{\xi}^i_{[\ell-p'(i)]} > 0.$$

In order to make the above condition less odd, we remark that for each $0 \leq \ell \leq k-1$, arcs $\xi^i_{[\ell-p'(i)]}$ and $\xi^j_{[\ell-p'(j)]}$ are located at the same level. An example is depicted in Figure 5.6.

By using the fact that the circulation is standard, one can see that if $\bar{\vartheta}^i = 0$ then $\bar{\xi}^i_\ell > 0$ for $0 \leq \ell \leq k-1$ and thus $S^i \succ S^j$. Also, if $\vartheta^j = v^j - \bar{w}^j$ then $\bar{\xi}^j_\ell = 0$ for $0 \leq \ell \leq k-1$ and thus $S^i \succ S^j$.

Define h as an index in $J(\bar{w})$ such that $b_h = \max_{j \in J(\bar{w})} b_j$. The following result is crucial. Unfortunately, its proof is rather long, tedious and technical and is for patient readers only.

Lemma 5.10 *Any standard circulation in $\mathcal{N}(\bar{w})$ is dominated by a standard circulation in $\mathcal{N}(\bar{w})$ satisfying $S^h \succ S^j$ for all $j \in J(\bar{w})$.*

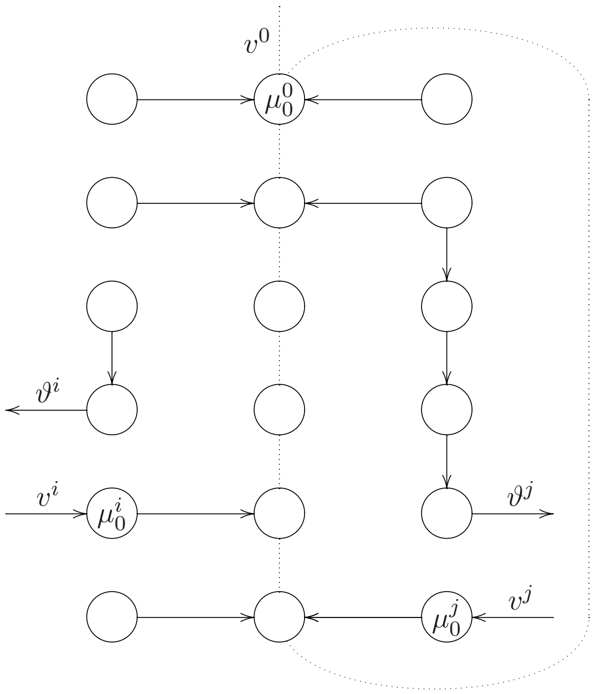


Figure 5.6: A standard circulation satisfying the condition $S^i \succ S^j$. Only nodes and arcs of sector S^0 (in the middle) and two other sectors S^i (on the left) and S^j (on the right) are depicted. Dotted lines indicate possible arcs. Circulation requirements are not represented.

Proof. Let $m\left(\bar v, \bar\vartheta\right)$ be the number of indices $j \in J(\bar w)$ such that $S^h \not\succ S^j$. We show that if $m\left(\bar v, \bar\vartheta\right) > 0$, it is possible to construct a circulation $\left(\hat u, \hat\xi, \hat v, \hat\vartheta\right)$ dominating $\left(\bar u, \bar\xi, \bar v, \bar\vartheta\right)$ such that $m(\hat v, \hat\vartheta) < m\left(\bar v, \bar\vartheta\right)$. Thus, by repeating this construction, one eventually finds a circulation dominating $\left(\bar u, \bar\xi, \bar v, \bar\vartheta\right)$ such that $S^h \succ S^j$ for all $j \in J(\bar w)$.

Pick any $j \in J(\bar w)$ such that $S^h \not\succ S^j$ and define $\rho_{\max} := \min\left\{\bar\vartheta^h, \bar v^j - \bar w^j - \bar\vartheta^j\right\}$. For every value $0 \le \rho \le \rho_{\max}$ we define a standard circulation $(u(\rho), \xi(\rho), v(\rho), \vartheta(\rho))$ in $\mathcal{N}(\bar w)$ by setting

$$\vartheta^h(\rho) := \bar\vartheta^h - \rho,\ \vartheta^j(\rho) := \bar\vartheta^j + \rho,\ \vartheta^i(\rho) := \bar\vartheta^i \text{ for } i \notin \{h, j\},\ v^i(\rho) := \bar v^i \text{ for } 1 \le i \le n. \quad (5.95)$$

Condition $0 \le \rho \le \rho_{\max}$ ensures that inequalities (5.93) are satisfied by the above values and thus the standard circulation $(u(\rho), \xi(\rho), v(\rho), \vartheta(\rho))$ is well defined.

In order to give the reader a better understanding of this proof, we find useful to point out how the standard circulation $(u(\rho), \xi(\rho), v(\rho), \vartheta(\rho))$ depends on ρ. Note that as ρ increases, the subset of arcs ξ_ℓ^h (for $0 \le \ell \le k-1$) that belongs to the support of the circulation either enlarges or does not change at all. In other words, $r_h(v(\rho), \vartheta(\rho))$ is a non-decreasing function of ρ. Symmetrically, $r_j(v(\rho), \vartheta(\rho))$ is a non-increasing function of ρ.

Conditions (5.95) easily imply that

$$\sum_{\ell=0}^{k-1}\left(\xi_\ell^h(\rho) - \bar\xi_\ell^h\right) = \rho = -\sum_{\ell=0}^{k-1}\left(\xi_\ell^j(\rho) - \bar\xi_\ell^j\right). \quad (5.96)$$

In words, the flow on arcs $\xi_0^h, \ldots, \xi_{k-1}^h$ (resp. $\xi_0^j, \ldots, \xi_{k-1}^j$) has been increased by ρ (resp. $-\rho$).

For $1 \le i \le n$, define $\alpha_i(\rho)$ as the total variation of flow on the arcs ξ_ℓ^i that are located at a level that is at least $p'(i)$. That is,

$$\alpha_i(\rho) := \sum_{\ell=p'(i)}^{k-1} \left(\xi^i_{[\ell-p'(i)]}(\rho) - \bar{\xi}^i_{[\ell-p'(i)]}\right) = \sum_{\ell=0}^{k-1-p'(i)} \left(\xi^i_\ell(\rho) - \bar{\xi}^i_\ell\right), \tag{5.97}$$

Clearly $0 \le \alpha_h(\rho) \le \rho$, $-\rho \le \alpha_j(\rho) \le 0$ and $\alpha_i(\rho) = 0$ for $i \notin \{h, j\}$.

Using the fact that $(u(\rho), \xi(\rho), v(\rho), \vartheta(\rho))$ is a standard circulation for all $0 \le \rho \le \rho_{\max}$, one can verify the following:

(i) There exists a value $0 \le \rho_h \le \rho_{\max}$ such that

$$\alpha_h(\rho) = \begin{cases} \rho & \text{if } 0 \le \rho \le \rho_h, \\ \rho_h & \text{if } \rho_h \le \rho \le \rho_{\max}. \end{cases} \tag{5.98}$$

Furthermore $\rho \le \rho_h$ if and only if $\lambda_h(v(\rho), \vartheta(\rho)) \ge p'(h)$.

(ii) There exists a value $0 \le \rho_j \le \rho_{\max}$ such that

$$\alpha_j(\rho) = \begin{cases} 0 & \text{if } 0 \le \rho \le \rho_j, \\ \rho_j - \rho & \text{if } \rho_j \le \rho \le \rho_{\max}. \end{cases} \tag{5.99}$$

Furthermore $\rho \ge \rho_j$ if and only if $\lambda_j(v(\rho), \vartheta(\rho)) \ge p'(j)$.

Recall that $\rho_{\max} = \min\left\{\bar{\vartheta}^h, \bar{v}^j - \bar{w}^j - \bar{\vartheta}^j\right\}$. Note that if $\rho_{\max} = \bar{\vartheta}^h$ then $\vartheta^h(\rho_{\max}) = 0$, and if $\rho_{\max} = \bar{v}^j - \bar{w}^j - \bar{\vartheta}^j$ then $\vartheta^j(\rho_{\max}) = \bar{v}^j - \bar{w}^j$. As observed before this lemma, in both cases this implies $S^h \succ S^j$ with respect to circulation $(u(\rho_{\max}), \xi(\rho_{\max}), v(\rho_{\max}), \vartheta(\rho_{\max}))$. We can then safely define a number $\hat{\rho}$ such that:

(a) $0 \le \hat{\rho} \le \rho_{\max}$;

(b) $S^h \succ S^j$ with respect to circulation $(u(\hat{\rho}), \xi(\hat{\rho}), v(\hat{\rho}), \vartheta(\hat{\rho}))$;

(c) under the above conditions, the number

$$\left|\left\{\ell : \xi^h_\ell(\hat{\rho}) > 0\right\}\right| + \left|\left\{\ell : \xi^j_\ell(\hat{\rho}) = 0\right\}\right| \tag{5.100}$$

is minimum.[3]

We now set $\left(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta}\right) := (u(\hat{\rho}), \xi(\hat{\rho}), v(\hat{\rho}), \vartheta(\hat{\rho}))$ and $\alpha_i(\hat{\rho}) := \hat{\alpha}_i$ for $1 \le i \le n$. We also shorten the notation by defining $\hat{\lambda}_i := \lambda_h\left(\hat{v}, \hat{\vartheta}\right)$ for $1 \le i \le n$.

The following observation will be useful: the definition of $\hat{\rho}$ implies that

$$\hat{\lambda}_j \ge \hat{\lambda}_h, \tag{5.101}$$

[3]Since both terms in (5.100) are nondecreasing functions of ρ, one might think that condition (c) could be replaced with the easier request that $\hat{\rho}$ is minimum. However this would produce some technical complications, as the existence of such a minimum is not guaranteed.

as otherwise condition (c) above would be violated (just decrease $\hat{\rho}$ by a suitable value).

In the following we show that $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ gives rise to an inequality (5.87) that dominates that corresponding to $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$. First of all, note that the left-hand side of inequality (5.87) is the same in both cases, thus we only need to show that the right-hand side corresponding to $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ is greater than or equal to that corresponding to $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$.

Let Δ be the difference between the right-hand side of inequality (5.87) corresponding to $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ and that corresponding to $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$. As observed above, we have to prove that $\Delta \geq 0$. If one writes down patiently the expression for Δ given by (5.87), recalling that $\bar{u}_0^i = 0$ for $1 \leq i \leq n$ (as the circulation is standard) and $\hat{\alpha}_i = 0$ for $i \notin \{h, j\}$, and then uses (5.96) for $\rho = \hat{\rho}$ together with the second expression for $\hat{\alpha}_h$ and $\hat{\alpha}_j$ in (5.97), one finds

$$\Delta = -\hat{u}_0^0 + \bar{u}_0^0 + \hat{\rho}\lceil b_h \rceil + \hat{\alpha}_h - \hat{\rho}\lceil b_j \rceil + \hat{\alpha}_j. \tag{5.102}$$

We now distinguish two cases.

Case 1: $p'(h) \leq p'(j)$ (in other words, $f'(b_h) \geq f'(b_j)$).
Assume that $\hat{\alpha}_h < \hat{\rho}$ and $\hat{\alpha}_j < 0$. Then (5.98) and (5.99) show that $\hat{\rho} > \max\{\rho_h, \rho_j\}$. One can verify that by setting $\tilde{\rho} := \max\{\rho_h, \rho_j\}$, conditions (a)–(b) above are satisfied and the corresponding number (5.100) is smaller than that corresponding to $\hat{\rho}$. This means that $\hat{\rho}$ does not satisfies condition (c), a contradiction. Therefore $\hat{\alpha}_j = 0$ whenever $\hat{\alpha}_h < \hat{\rho}$, which also implies that $\hat{\alpha}_h + \hat{\alpha}_j \geq 0$ (as $0 \leq \hat{\alpha}_h \leq \hat{\rho}$ and $-\rho \leq \hat{\alpha}_j \leq 0$).

If $\hat{u}_0^0 = 0$ then equation (5.102) shows that $\Delta \geq 0$, as $\bar{u}_0^0 \geq 0$, $b_h \geq b_j$ and $\hat{\alpha}_h + \hat{\alpha}_j \geq 0$. So we now assume $\hat{u}_0^0 > 0$. By equation (5.92), there is an index $1 \leq l \leq k-1$ such that $\hat{u}_0^0 = \beta_l(\hat{v}, \hat{\vartheta})$. Again by (5.92), $\bar{u}_0^0 \geq \beta_l(\bar{v}, \bar{\vartheta})$. Equation (5.102) then gives

$$\Delta \geq -\beta_l(\hat{v}, \hat{\vartheta}) + \beta_l(\bar{v}, \bar{\vartheta}) + \hat{\rho}(\lceil b_h \rceil - \lceil b_j \rceil) + \hat{\alpha}_h + \hat{\alpha}_j.$$

Since $b_h \geq b_j$ and $\hat{\rho} \geq 0$, the above inequality implies

$$\Delta \geq -\beta_l(\hat{v}, \hat{\vartheta}) + \beta_l(\bar{v}, \bar{\vartheta}) + \hat{\alpha}_h + \hat{\alpha}_j. \tag{5.103}$$

Using (5.91), one finds

$$-\beta_l(\hat{v}, \hat{\vartheta}) + \beta_l(\bar{v}, \bar{\vartheta}) = -\sum_{\ell=l}^{k-1}\left(\hat{\xi}^h_{[\ell-p'(h)]} - \bar{\xi}^h_{[\ell-p'(h)]}\right) - \sum_{\ell=l}^{k-1}\left(\hat{\xi}^j_{[\ell-p'(j)]} - \bar{\xi}^j_{[\ell-p'(j)]}\right). \tag{5.104}$$

By (5.96) for $\rho = \hat{\rho}$ and the fact that $\hat{\xi}^h_\ell \geq \bar{\xi}^h_\ell$ for all ℓ, the value of the first summation in the above equation does not exceed $\hat{\rho}$. Similarly, since $\hat{\xi}^j_\ell \leq \bar{\xi}^j_\ell$ for all ℓ, the value of the second summation is at most 0.

We consider two possibilities:

1.1 Assume first that $\hat{\alpha}_h = \hat{\rho}$, or in other words $\hat{\rho} \leq \rho_h$, or in other words $\hat{\lambda}_h \geq p'(h)$. Note that $\hat{\xi}^h_{[\ell-p'(h)]} = \bar{\xi}^h_{[\ell-p'(h)]}$ for all $\ell > \hat{\lambda}_h$, and $\hat{\xi}^j_\ell \leq \bar{\xi}^j_\ell$ for all ℓ. By equation (5.104), this implies that if $l > \hat{\lambda}_h$ then $-\beta_l(\hat{v}, \hat{\vartheta}) + \beta_l(\bar{v}, \bar{\vartheta}) \geq 0$. Together with (5.103) and inequality $\hat{\alpha}_h + \hat{\alpha}_j \geq 0$ proven above, this shows that $\Delta \geq 0$ if $l > \hat{\lambda}_h$.

So we assume $l \le \hat\lambda_h$. Since $\hat\lambda_j \ge \hat\lambda_h$ by (5.101), we have $l \le \hat\lambda_j$. Since $\hat\xi^j_\ell \le \bar\xi^j_\ell$ for all ℓ and $\hat\xi^j_{[\ell-p'(j)]} = \bar\xi^j_{[\ell-p'(j)]}$ for $p'(j) \le \ell \le \hat\lambda_j$, the second summation in (5.104) is (we also use the first expression for $\hat\alpha_j$ in (5.97))

$$\sum_{\ell=l}^{k-1}\left(\hat\xi^j_{[\ell-p'(j)]} - \bar\xi^j_{[\ell-p'(j)]}\right) \le \sum_{\ell=\hat\lambda_j}^{k-1}\left(\hat\xi^j_{[\ell-p'(j)]} - \bar\xi^j_{[\ell-p'(j)]}\right) = \hat\alpha_j. \tag{5.105}$$

Since the value of the first summation in (5.104) is at most $\hat\rho = \hat\alpha_h$, we then have $-\beta_l(\hat v,\hat\vartheta)+\beta_l(\bar v,\bar\vartheta) \ge -\hat\alpha_h - \hat\alpha_j$, which together with (5.103) shows that $\Delta \ge 0$.

1.2 Now assume $\hat\alpha_h < \hat\rho$, or in other words $\hat\lambda_h < p'(h)$. As remarked above, $\hat\alpha_j = 0$ in this case. Also note that $\hat\xi^h_{[\ell-p'(h)]} = \bar\xi^h_{[\ell-p'(h)]}$ for $\hat\lambda_h < \ell < p'(h)$. Then if $l > \hat\lambda_h$ then the value of the first summation in (5.104) is at most $\hat\alpha_h$. Since the value of the second summation in (5.104) is at 0, we then have $-\beta_l(\hat v,\hat\vartheta) + \beta_l(\bar v,\bar\vartheta) \ge -\hat\alpha_h = -\hat\alpha_h - \hat\alpha_j$. This, together with (5.103), shows that $\Delta \ge 0$.

So we assume $l \le \hat\lambda_h$. Since $\hat\lambda_j \ge \hat\lambda_h$ by (5.101), we have $l \le \hat\lambda_j$. Note that $\hat\xi^j_{[\ell-p'(j)]} = \bar\xi^j_{[\ell-p'(j)]}$ for $0 \le \ell < \hat\lambda_j$, thus the value of the second summation in (5.104) is exactly $-\hat\rho$. Since the value of the first summation in (5.104) is at most $\hat\rho$, we then have $-\beta_l(\hat v,\hat\vartheta) + \beta_l(\bar v,\bar\vartheta) \ge -\hat\rho + \hat\rho = 0$, thus $\Delta \ge 0$ by (5.103), as $\hat\alpha_h + \hat\alpha_j \ge 0$.

CASE 2: $p'(h) > p'(j)$ (in other words, $f'(b_h) < f'(b_j)$).
Note that since $b_h \ge b_j$ and $f'(b_h) < f'(b_j)$, then $\lceil b_h\rceil \ge \lceil b_j\rceil + 1$. If $\hat u^0_0 = 0$ then equation (5.102) shows that $\Delta \ge 0$, as $\bar u^0_0 \ge 0$, $\lceil b_h\rceil \ge \lceil b_j\rceil + 1$ and $\hat\alpha_h + \hat\alpha_j \ge -\hat\rho$. So we now assume $\hat u^0_0 > 0$. By equation (5.92), there is an index $1 \le l \le k-1$ such that $\hat u^0_0 = \beta_l(\hat v,\hat\vartheta)$. Again by (5.92), $\bar u^0_0 \ge \beta_l(\bar v,\bar\vartheta)$. Equation (5.102) then gives

$$\Delta \ge -\beta_l(\hat v,\hat\vartheta) + \beta_l(\bar v,\bar\vartheta) + \hat\rho(\lceil b_h\rceil - \lceil b_j\rceil) + \hat\alpha_h + \hat\alpha_j.$$

Since $b_h \ge b_j$ and $\hat\rho \ge 0$, the above inequality implies

$$\Delta \ge -\beta_l(\hat v,\hat\vartheta) + \beta_l(\bar v,\bar\vartheta) + \hat\rho + \hat\alpha_h + \hat\alpha_j. \tag{5.106}$$

Note that equation (5.104) still holds.

We consider two possibilities.

2.1 Assume first that $\hat\alpha_h = \hat\rho$. Since $-\beta_l(\hat v,\hat\vartheta)+\beta_l(\bar v,\bar\vartheta) \ge -\hat\rho$ by (5.104) and since $\hat\alpha_h+\hat\alpha_j = \hat\rho + \hat\alpha_j \ge 0$, we obtain $\Delta \ge 0$.

2.2 Now suppose that $\hat\alpha_h < \hat\rho$. As in Case 1.2, if $l > \hat\lambda_h$ then the value of the first summation in (5.104) is at most $\hat\alpha_h$. This implies that $\Delta \ge 0$, as $-\beta_l(\hat v,\hat\vartheta) + \beta_l(\bar v,\bar\vartheta) \ge -\hat\alpha_h$ and $\hat\rho + \hat\alpha_j \ge 0$.

So we assume $l \le \hat\lambda_h$, which together with $\hat\lambda_j \ge \hat\lambda_h$ implies $l \le \hat\lambda_j$. As in the second part of Case 1.1, the value of the second summation in (5.104) is at most $\hat\alpha_j$. Then $\Delta \ge 0$, as $-\beta_l(\hat v,\hat\vartheta) + \beta_l(\bar v,\bar\vartheta) \ge -\rho - \hat\alpha_j$ and $\hat\alpha_h \ge 0$.

We have proven that in all cases the inequality (5.87) corresponding to $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ dominates that corresponding to $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$. To conclude we have to show that $m(\hat{v}, \hat{\vartheta}) < m(\bar{v}, \bar{\vartheta})$. This follows from the following two observations: first, for any i, if $S^h \succ S^i$ with respect to $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ then $S^h \succ S^i$ with respect to $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$ as well; second, $S^h \not\succ S^j$ with respect to $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ but $S^h \succ S^j$ with respect to $(\hat{u}, \hat{\xi}, \hat{v}, \hat{\vartheta})$. □

From now on we only consider standard circulations in $\mathcal{N}(\bar{w})$ satisfying $S^h \succ S^j$ for all $j \in J(\bar{w})$.

For the next lemmas it is useful to introduce some simple notation. Given two indices $0 \le \ell, \ell' \le k-1$, we define $\langle \ell, \ell' \rangle$ as the set of indices ranging from ℓ to ℓ' in "circular" fashion. That is,

$$\langle \ell, \ell' \rangle := \begin{cases} \{\ell, \dots, \ell'\} & \text{if } 0 \le \ell \le \ell' \le k-1, \\ \{\ell, \dots, k-1\} \cup \{0, \dots, \ell'\} & \text{if } 0 \le \ell' < \ell \le k-1, \\ \varnothing & \text{if } \ell = k \text{ or } \ell' = k. \end{cases}$$

(The third case in the above definition is given for technical reasons.)

Given indices $0 \le i \le n$ and $0 \le \ell, \ell' \le k-1$, let $P^i(\ell, \ell')$ be the set of arcs in the unique directed path in S^i connecting nodes μ^i_ℓ and $\mu^i_{\ell'}$. That is,

$$P^i(\ell, \ell') := \begin{cases} \{u^i_l : l \in \langle \ell+1, \ell' \rangle\} & \text{if } \ell \neq \ell', \\ \varnothing & \text{if } \ell = \ell'. \end{cases}$$

In the following we will be considering a fixed standard circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$ such that $S^h \succ S^j$ for all $j \in J(\bar{w})$. Thus we can safely drop the dependence on $(\bar{v}, \bar{\vartheta})$ in notation $\lambda_i(\bar{v}, \bar{\vartheta})$ and just write λ_i. Similarly we write r_i for $r_i(\bar{v}, \bar{\vartheta})$.

Lemma 5.11 *If a standard circulation in $\mathcal{N}(\bar{w})$ satisfies $S^h \succ S^j$ for all $j \in J(\bar{w})$, then $\bar{w}^0 = -\bar{w}^h$ and $\bar{w}^j = 0$ for all $j \notin \{0, h\}$.*

Proof. It is sufficient to show that the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ contains a forest F as in Lemma 5.7 (ii), with $i := h$. The construction of F is divided into several steps, which are illustrated in Figure 5.7. Note that the picture represents only the forest F: other nodes and arcs have not been drawn.

STEP 1. Since $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ is a standard circulation, its support contains arcs ξ^h_ℓ for $0 \le \ell \le r_h$, which we include in F (solid arcs in Figure 5.7). Such arcs span nodes μ^h_ℓ for $0 \le \ell \le r_h$ and μ^0_ℓ for $\ell \in \langle p'(h), \lambda_h \rangle$. Thus, if $\lambda_h = p'(h) - 1$ (or in other words, $r_h = k-1$), the construction of F is complete. We then assume $\lambda_h \neq p'(h) - 1$ and go to the next step.

STEP 2. The support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ also contains arcs u^i_ℓ for $\ell \in \langle r_h + 2, k-1 \rangle$ and arc ϑ^h, which we add to F (dashed arcs in Figure 5.7). Now all nodes in S^h are spanned by F. It remains to cover nodes μ^0_ℓ for $\ell \in \big\langle [\lambda_h + 1], [p'(h) - 1] \big\rangle$. If $\bar{u}^0_{[\lambda_h+1]} > 0$ we go to Step 3, otherwise we skip to Step 4.

STEP 3. (To be executed if and only if $\bar{u}^0_{[\lambda_h+1]} > 0$.) Note that in this case we can assume without loss of generality that $\bar{u}^h_{[r_h+1]} > 0$: if not, we can decrease by a small $\varepsilon > 0$ the flow

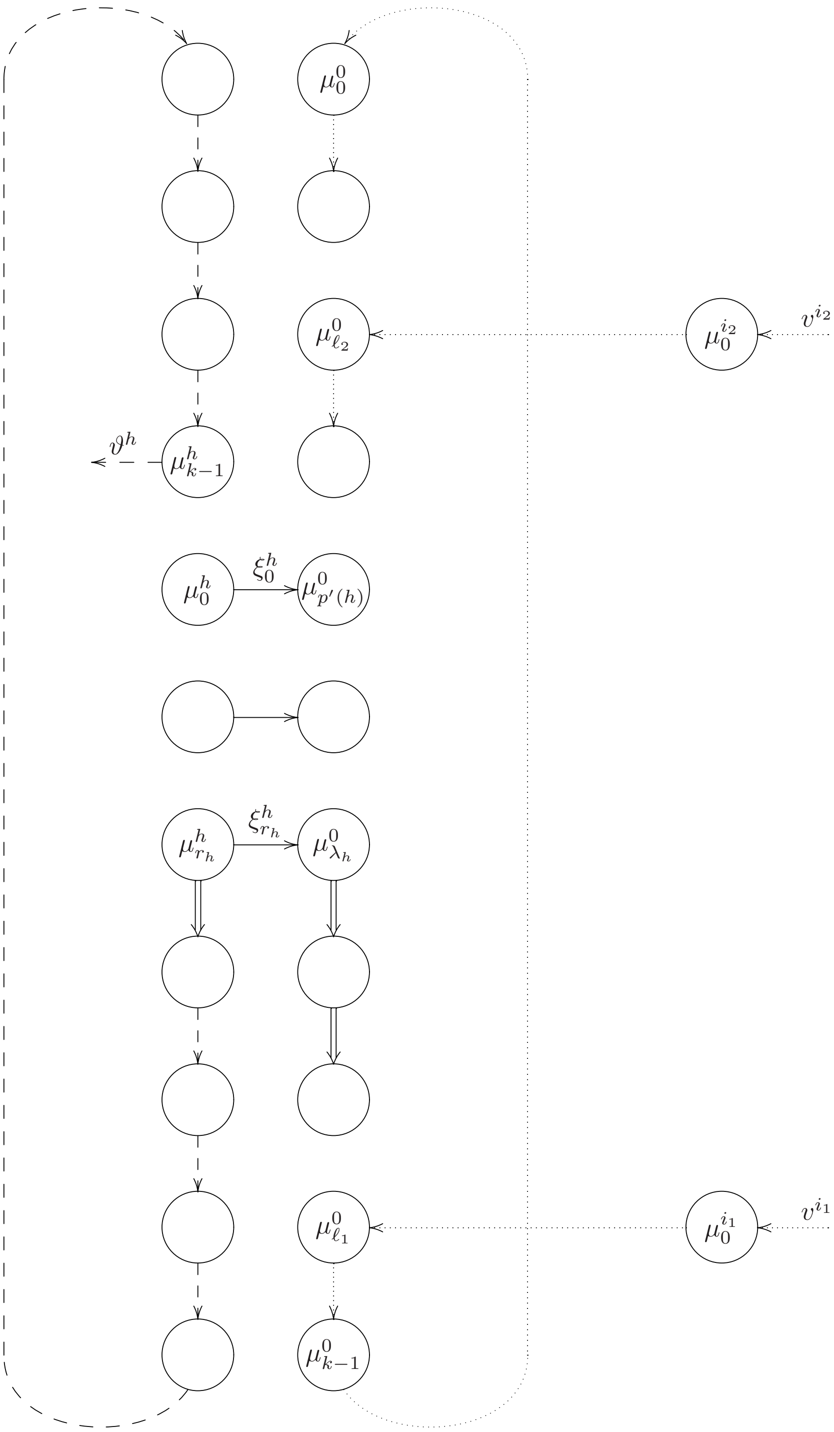


Figure 5.7: Illustration of the steps of the proof of Lemma 5.11. Solid arcs correspond to Step 1, dashed arcs to Step 2, double arcs to Step 3 and dotted arcs to Step 4.

on arcs $\xi^h_{r_h}, u^0_{[\lambda_h+1]}$ and increase by the same amount the flow on arcs $u^h_{[r_h+1]}, \xi^h_{[r_h+1]}$, thus obtaining an equivalent circulation.[4] Choose the index $\ell_1 \in \langle[\lambda_h+1], p'(h)\rangle$ such that the path $P^0(\lambda_h, [\ell_1 - 1])$ is contained in the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ and has maximum length. We add the arcs of such path to F (double arcs in Figure 5.7). If $\ell_1 = p'(h)$, the construction of F is complete. Otherwise we go to the next step.

Step 4. If $\bar{u}^0_{[\lambda_h+1]} > 0$, ℓ_1 has already been defined in the previous step. If $\bar{u}^0_{[\lambda_h+1]} = 0$, set $\ell_1 := [\lambda_h + 1]$. In both cases, it remains to cover nodes μ^0_ℓ for $\ell \in \langle \ell_1, [p'(h) - 1]\rangle$. Since $S^h \succ S^j$ for all $j \in J(\bar{w})$ and since $\bar{\xi}^h_{\ell_1} = 0$, node $\mu^0_{\ell_1}$ receive a positive amount of flow from

(a) either arc v^0 (clearly this is possible only if $\ell_1 = 0$),

(b) or an arc $\xi^{i_1}_0$ such that $p'(i_1) = \ell_1$.

In the former case we add arc v^0 to F, in the latter case we add $\xi^{i_1}_0$. Note that if (b) holds then arc v^{i_1} carries a positive flow as well, and we also add v^{i_1} to F. Now let ℓ_2 be the index in $\langle \ell_1, p'(h)\rangle$ such that the path $P^0(\ell_1, [\ell_2 - 1])$ is contained in the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ and has maximum length. We add the arcs of such path to F. If $\ell_2 = p'(h)$, the construction of F is complete. Otherwise we repeat this step with ℓ_2 in place of ℓ_1, and so forth. (The arcs added in this step are the dotted arcs in Figure 5.7.)

At the end of the above process, a forest F as in Lemma 5.7 (ii) is detected in the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$. $\square$

Note that for $i \notin \{0, h\}$, condition $\bar{w}^i = 0$ and the fact that the $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ is a standard circulation imply that $\bar{\xi}^i_\ell = 0$ for all $1 \le \ell \le k-1$, $\bar{u}^i_\ell = 0$ for all $0 \le \ell \le k-1$ and $\bar{\vartheta}^i = 0$. Therefore the network can now be simplified by removing all such arcs: the resulting reduced network consists of the following arcs:

- the arcs of sectors S^h and S^0;
- the arcs connecting sectors S^h and S^0, i.e. arcs $\xi^h_0, \ldots, \xi^h_{k-1}$;
- the arcs v^i, ξ^i_0 for all $1 \le i \le n$.

Also, using (5.89) and the fact that $\bar{w}^0 = -\bar{w}^h$, one sees that for each index $0 \le \ell \le k$ the demand of node μ^0_ℓ is exactly equal to the supply of the node of sector S^i placed at level ℓ, i.e. node $\mu^h_{[\ell - p'(h)]}$. The structure of a possible reduced network is depicted in Figure 5.8.

Therefore we can restrict to the reduced network our search for circulations generating non-redundant inequalities. Before showing explicitly such circulations we make a few final observations.

Lemma 5.12 *Assume $\bar{w}^0 = 1$, $\bar{w}^h = -1$ and $\bar{w}^i = 0$ for $i \notin \{0, h\}$. Every standard circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ in $\mathcal{N}(\bar{w})$ generating an inequality (5.87) that is non-redundant in the description of the convex hull of (5.53)–(5.56) satisfies the following conditions:*

[4] This circulation is non-standard, but the remainder of the proof still works.

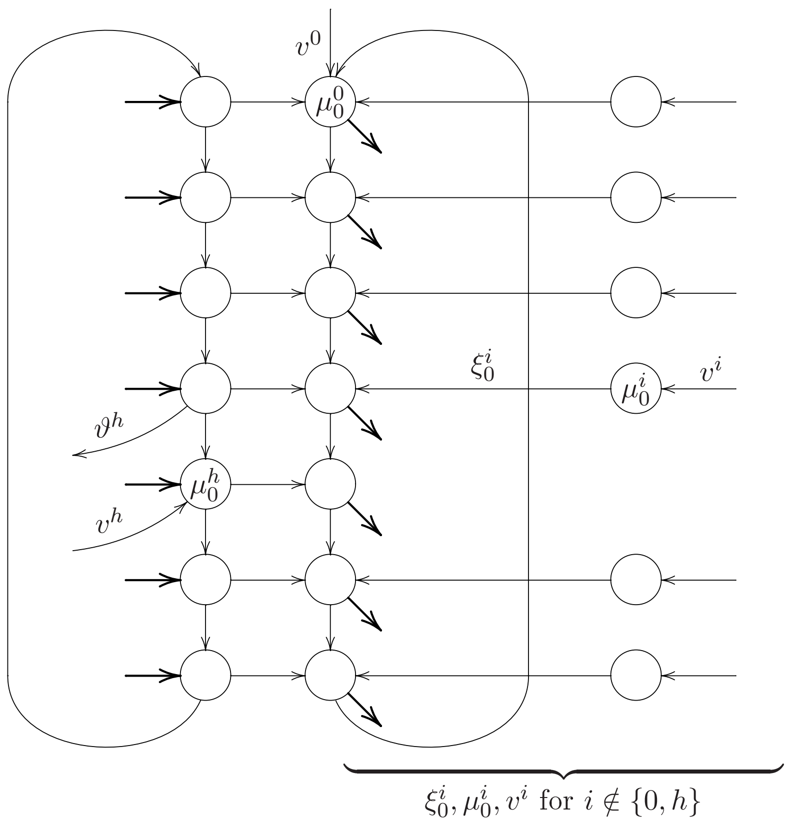


Figure 5.8: The reduced network when $\bar{w}^0 = 1$, $\bar{w}^h = -1$ and $\bar{w}^i = 0$ for $i \notin \{0, h\}$. Thick arrows represent circulation requirements. The supply of each node in S^h is equal to the demand of the node of S^0 located at the same level. Note that the nodes μ_0^i aligned on the same vertical line on the right actually belong to distinct sectors S^i for $i \in \{0, h\}$. This picture represents the special case in which the values $p'(i)$ for $1 \leq i \leq n$ are all distinct.

(i) $\bar{\xi}_0^h = \bar{v}^h + f_0^h - f_1^h$ *and* $\bar{\xi}_\ell^h = f_\ell^h - f_{\ell+1}^h$ *for* $1 \le \ell \le r_h$*;*

(ii) $\bar{u}_{[r_h+1]}^h = 0$*;*

(iii) $\bar{v}^i = \bar{\xi}_0^i$ *for all* $i \notin \{0, h\}$*;*

(iv) $\bar{v}^i = 0$ *for all indices* i *such that* $p'(i) \in \langle p'(h), \lambda_h \rangle$*;*

(v) $\bar{u}_\ell^0 = 0$ *for all* $\ell \in \big\langle p'(h), [\lambda_h + 1] \big\rangle$.

Proof. We prove the above statements in the case $r_h > 0$. If $r_h = 0$ the idea is the same but some notation used below is meaningless.

Since $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$ is a standard circulation, $\bar{u}_\ell^h = 0$ for $0 \le \ell \le r_h$. This immediately implies that $\bar{\xi}_0^h = \bar{v}^h + f_0^h - f_1^h$ and $\bar{\xi}_\ell^h = f_\ell^h - f_{\ell+1}^h$ for $1 \le \ell \le r_h - 1$, which partly proves (i).

Since $\bar{u}_{r_h}^h = 0$ and the supply of node $\mu_{r_h}^h$ is $f_{r_h}^h - f_{r_h+1}^h$, we have $\bar{u}_{[r_h+1]}^h + \bar{\xi}_{r_h}^h = f_{r_h}^h - f_{r_h+1}^h$. Also note that for $1 \le \ell \le r_h - 1$ the flow carried by arc ξ_ℓ^h is equal to the demand of its head-node, and the flow carried by arc ξ_0^i is at least as large as the demand of its head-node. In other words, nodes μ_ℓ^0 for $p'(h) \le \ell \le \lambda_h - 1$ are saturated by these arcs. Furthermore recall that arcs ξ_ℓ^h for $\ell > r_h$ do not carry any amount of flow (by definition of r_h). If we assume $\bar{\xi}_{r_h}^h < f_{r_h}^h - f_{r_h+1}^h$ (i.e. arc $\xi_{r_h}^h$ does not saturate node $\mu_{\lambda_h}^0$), all these considerations can be used to show the existence of a path Q contained in the support of circulation $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$ that connects node $\mu_{\lambda_h}^0$ with the dummy node d without using arc ϑ^h. In this case the arcs

$$\vartheta^h,\ P^h\big([r_h + 1], k - 1\big),\ \xi_{r_h}^h,\ Q$$

form a cycle contained in the support of $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$. It is easy to see that such a cycle is heavy, contradicting Lemma 5.11 (i). This completes the proof of (i) and also shows (ii), as $\bar{u}_{[r_h+1]}^h + \bar{\xi}_{r_h}^h = f_{r_h}^h - f_{r_h+1}^h$.

To see that (iii) holds, assume $\bar{v}^i > \bar{\xi}_0^i$ for some $i \notin \{0, h\}$. Since $\bar{w}^i = 0$ and the circulation is standard, then necessarily the support of $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$ contains the path $P^i(0, k-1)$. The arcs of this path, together with arcs v^i and ϑ^i, form a heavy cycle, contradicting Lemma 5.11 (i).

If $r_h = k - 1$ then (iv)-(v) can be checked easily, so we now assume $0 < r_h < k - 1$.

To prove (iv), let i be an index such that $\bar{v}^i > 0$ and $p'(i) \in \langle p'(h), \lambda_h \rangle$. Note that if $i \neq 0$ then we also have $\bar{\xi}_0^i > 0$, as $\bar{u}_0^i = 0$ in a standard circulation. Since for $0 \le \ell \le r_h$ the flow carried by arc ξ_ℓ^h is at least as large as the demand of its head-node (and thus nodes μ_ℓ^0 for $p'(h) \le \ell \le \lambda_h$ are saturated by these arcs), we see that all arcs in $P^0\big(p'(i), [\lambda_h + 1]\big)$ belong to the support of $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$. We can then decrease by a small $\varepsilon > 0$ the flow on arcs $\xi_{[p'(i)-p'(h)]}^h, P^0\big(p'(i), [\lambda_h + 1]\big)$ and increase by the same amount the flow on $P^h\big([p'(i) - p'(h)], [r_h + 1]\big), \xi_{[r_h+1]}^h$, thus obtaining an equivalent circulation. However, this new circulation contains in its support all arcs

$$v^i,\ \xi_0^i,\ \xi_{[p'(i)-p'(h)]}^h,\ P^h([p'(i) - p'(h)], k - 1),\ \vartheta^h,$$

where arc ξ_0^i must be removed from the above sequence if $i = 0$. This set of arcs forms (or contains, if $i = h$) a heavy cycle, contradicting Lemma 5.11 (i).

To prove (v), let $\ell \in \langle p'(h), [\lambda_h+1]\rangle$ be such that $\bar{u}^0_\ell > 0$. Similarly to the proof of part (iv), one shows that all arcs in $P^0\big(p'(h), [\lambda_h+1]\big)$ belong to the support of $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$. We can then decrease by a small $\varepsilon > 0$ the flow on arcs $\xi^h_0, P^0\big(p'(h), [\lambda_h+1]\big)$ and increase by the same amount the flow on $P^h\big(0, [r_h+1]\big), \xi^h_{[r_h+1]}$, thus obtaining an equivalent circulation. However, this new circulation contains in its support all arcs $u^h_0, \dots, u^h_{k-1}$, which form a tree (actually a path) as in Lemma 5.7 (iii). Thus $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$ is equivalent to a circulation generating a redundant inequality, that is, $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$ itself generates a redundant inequality. □

Since part (iv) of the above lemma applies to index $i = h$, the statement in (i) can be written this way: $\bar{\xi}^h_\ell = f^h_\ell - f^h_{\ell+1}$ for $0 \le \ell \le r_h$. In other words, for each $0 \le \ell \le r_h$ the supply of node μ^h_ℓ is entirely carried to node $\mu^0_{[p'(h)+\ell]}$ by arc ξ^h_ℓ, and this amount of flow satisfies precisely the demand of node $\mu^0_{[p'(h)+\ell]}$.

Using the fact that $v^i = \bar{\xi}^i_0$ and $\bar{\xi}^i_1 = \cdots = \bar{\xi}^i_{k-1} = 0$ for $i \notin \{0, h\}$, inequality (5.87) can now be rewritten as follows:

$$s + y_h + \sum_{i=1}^{n} \bar{v}^i(z_i - \lceil b_i \rceil) + \bar{u}^0_0 \ge \sum_{\ell=0}^{k-1} \bar{\xi}^h_\ell \lceil b_h \rceil - \sum_{\ell=k-p'(h)}^{k-1} \bar{\xi}^h_\ell. \tag{5.107}$$

By the above considerations, the right-hand side of inequality (5.107) is

$$\sum_{\ell=0}^{k-1} \bar{\xi}^h_\ell \lceil b_h \rceil - \sum_{\ell=k-p'(h)}^{k-1} \bar{\xi}^h_\ell = \begin{cases} \big(f^0_{p'(h)} - f^0_{\lambda_h+1}\big) \lceil b_h \rceil & \text{if } p'(h) \le \lambda_h, \\ f^0_{p'(h)} \lceil b_h \rceil + \big(1 - f^0_{\lambda_h+1}\big)(\lceil b_h \rceil - 1) & \text{if } p'(h) > \lambda_h. \end{cases} \tag{5.108}$$

Assume $\bar{v}^i = 0$ for all indices $1 \le i \le n$. Lemma 5.12 can be used to show that two cases are possible: either $\bar{v}^0_0 = 0$ and $\lambda_h = [p'(h) - 1]$ (i.e. $r_h = k-1$), or $\bar{v}^0 = 1 - f'(b_h)$ and $\lambda_h = k - 1$. In the former case, the corresponding inequality (5.107) is

$$s + y_h \ge f^0_{p'(h)} \lceil b_h \rceil + (1 - f^0_{p'(h)})(\lceil b_h \rceil - 1), \tag{5.109}$$

while in the latter case it is

$$s + y_h \ge f^0_{p'(h)} \lceil b_h \rceil.$$

The above inequality can be discarded because it is dominated by (5.109), as $b_h \ge 0$. Recalling that $f^0_{p'(h)} = f'(b_h)$, inequality (5.109) is readily checked to be equivalent to $s + y_h \ge b_h$.

Now we assume that $\bar{v}^i > 0$ for at least one index $1 \le i \le n$. Let $i_1, \dots, i_{m-1}$ be the indices in $\{1, \dots, n\}$ such that $\bar{v}_{i_t} > 0$ and $p'(i_t) < p'(h)$ for $1 \le t \le m-1$. (Note that $m-1$ might be equal to zero.) Set $i_m := h$ and let $i_{m+1}, \dots, i_r$ be the indices in $\{1, \dots, n\}$ such that $\bar{v}_{i_t} > 0$ and $p'(i_t) > \lambda_h$ for $m+1 \le t \le r$. (Note that r might be equal to m). By Lemma 5.12 (iv), there does not exist an index i such that $p'(h) \le p'(i) \le \lambda_h$, thus $\{i_t : t \ne m\}$ is precisely the set of indices $i \ne 0$ such that $\bar{v}^i > 0$. Also note that there do not exist two distinct indices t, t', with $t \ne m \ne t'$, such that $p'(i_t) = p'(i_{t'})$, as otherwise the arcs $v^{i_t}, \xi^{i_t}_0, \xi^{i_{t'}}_0, v^{i_{t'}}$ would form a heavy cycle contained in the support of circulation $\big(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta}\big)$. We can then assume without loss of generality that $p'(i_1) < \cdots < p'(i_{m-1}) < p'(i_m) = p'(h) < p'(i_{m+1}) < \cdots < p'(i_r)$. We also define $i_{r+1} := n+1$ (thus $p'(i_{r+1}) = k$) and $f'(b_{n+1}) := 0$.

We now distinguish two cases.

1. Suppose first that $\bar{v}^0 > 0$. We claim that in this case $\bar{u}_0^0 = 0$. This follows immediately from Lemma 5.12 (v) if $\lambda_h = k-1$, so assume $\lambda_h < k-1$. Then if $\bar{u}_0^0 > 0$ the sequence of arcs $v^{i_r}, P^0(p'(i_r), 0), v^0$ would form a heavy cycle contained in the support of the circulation $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$.

   Thus $\bar{u}_0^0 = 0$. Also observe that $p'(h) \leq \lambda_h$, as otherwise Lemma 5.12 (iv) would be violated by index $i = 0$. The nonzero entries of $\bar{v}$ are (see Figure 5.9)

$$\bar{v}^{i_t} = \sum_{\ell=p'(i_t)}^{p'(i_{t+1})-1} \left(f_\ell^0 - f_{\ell+1}^0\right) = f_{p'(i_t)}^0 - f_{p'(i_{t+1})}^0 \quad \text{for } t \neq m,$$

$$\bar{v}^0 = \sum_{\ell=0}^{p'(i_1)-1} \left(f_\ell^0 - f_{\ell+1}^0\right) = 1 - f_{p'(i_1)}^0.$$

   The corresponding inequality (5.107) is then (also using equation (5.108))

$$s + y_h + \sum_{t \neq m} \left(f_{p'(i_t)}^0 - f_{p'(i_{t+1})}^0\right)(z_{i_t} - \lceil b_{i_t} \rceil) \geq \left(f_{p'(h)}^0 - f_{\lambda_h+1}^0\right) \lceil b_h \rceil.$$

   Recall that $i_m = h$ and observe that $p'(i_{m+1}) = \lambda_h + 1$. Then, after recalling that $f_{p'(i)}^0 = f'(b_i)$ for all indices $1 \leq i \leq n$, the above inequality reads

$$s + y_{i_m} + \sum_{t \neq m} \left(f'(b_{i_t}) - f'(b_{i_{t+1}})\right)(z_{i_t} - \lceil b_{i_t} \rceil) \geq \left(f'(b_{i_m}) - f'(b_{i_{m+1}}\right) \lceil b_{i_m} \rceil. \quad (5.110)$$

2. Now suppose $\bar{v}_0 = 0$. In this case the two alternatives $m < r$ and $m = r$ need to be considered separately.

   If $m < r$ then $p'(h) \leq \lambda_h$, as otherwise Lemma 5.12 (iv) would be violated by index $i = i_r$. The nonzero entries of $\bar{v}$ are (see Figure 5.10)

$$\bar{v}_{i_t} = \sum_{\ell=p'(i_t)}^{p'(i_{t+1})-1} \left(f_\ell^0 - f_{\ell+1}^0\right) = f_{p'(i_t)}^0 - f_{p'(i_{t+1})}^0 \quad \text{for } t \notin \{m, r\},$$

$$\bar{v}_{i_r} = \sum_{\ell=p'(i_r)}^{k-1} \left(f_\ell^0 - f_{\ell+1}^0\right) + \sum_{\ell=0}^{p'(i_1)-1} \left(f_\ell^0 - f_{\ell+1}^0\right) = f_{p'(i_r)}^0 + \left(1 - f_{p'(i_1)}^0\right),$$

   while $\bar{u}_0^0 = 1 - f_{p'(i_1)}^0$. The corresponding inequality (5.107) is then

$$s+y_h+\sum_{t \neq m} \left(f_{p'(i_t)}^0 - f_{p'(i_{t+1})}^0\right)(z_{i_t} - \lceil b_{i_t} \rceil) + \left(1 - f_{p'(i_1)}^0\right)(z_{i_r} - \lceil b_{i_r} \rceil + 1) \geq \left(f_{p'(h)}^0 - f_{\lambda_h+1}^0\right) \lceil b_h \rceil,$$

   which can be equivalently be written as

$$s + y_h + \sum_{t \neq m} \left(f'(b_{i_t}) - f'(b_{i_{t+1}})\right)(z_{i_t} - \lceil b_{i_t} \rceil)$$
$$+ \left(1 - f'(b_{i_1})\right)(z_{i_r} - \lceil b_{i_r} \rceil + 1) \geq \left(f'(b_{i_m}) - f'(b_{i_{m+1}}\right) \lceil b_{i_m} \rceil. \quad (5.111)$$

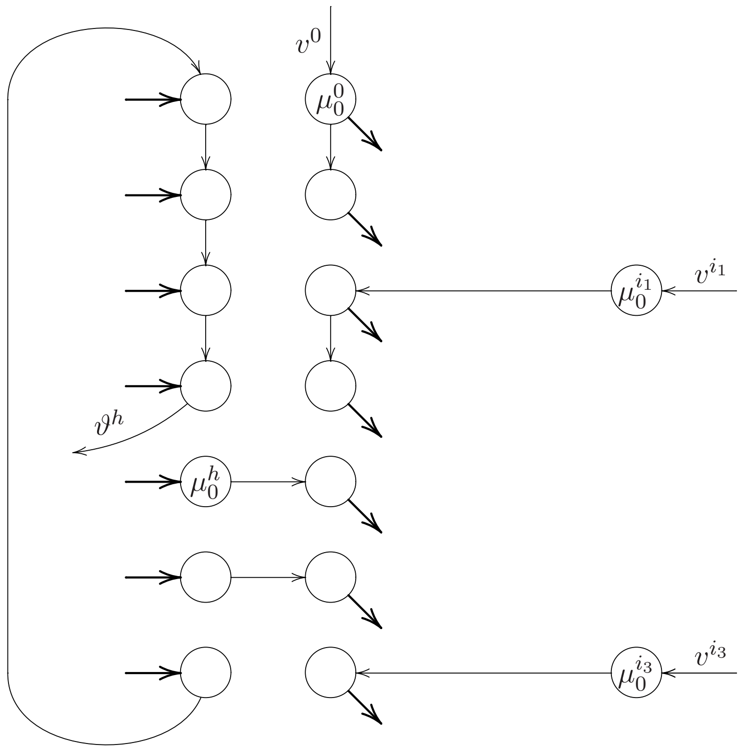


Figure 5.9: The case $\bar{v}^0 > 0$. Here $r = 3$ and $m = 2$.

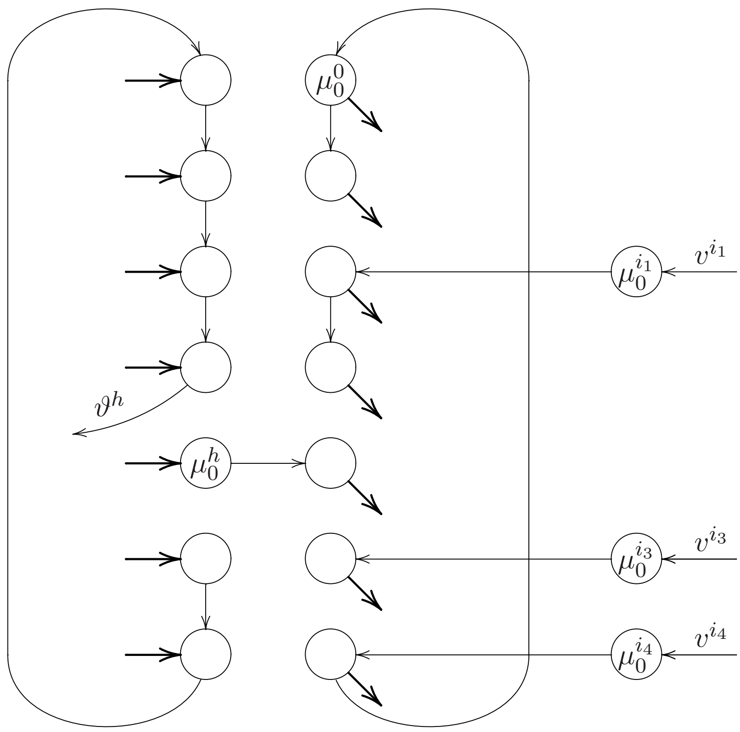


Figure 5.10: The case $\bar{v}^0 = 0$ and $m < r$. Here $r = 4$ and $m = 2$.

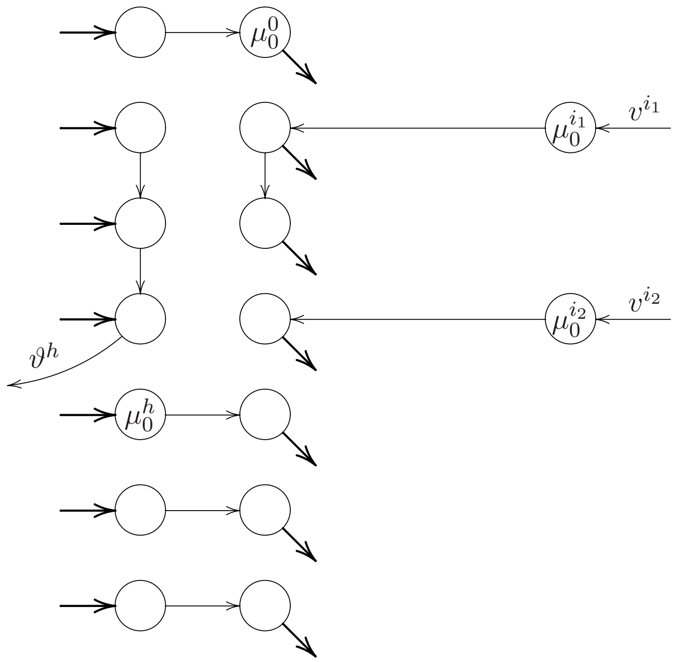


Figure 5.11: The case $\bar{v}^0 = 0$ and $m = r$. Here $m = r = 3$.

Now assume $m = r$. We claim that in this case $\bar{u}_0^0 = 0$. This follows immediately from Lemma 5.12 (v) if $\lambda_h = k - 1$, so assume $\lambda_h < k - 1$. Then, since $m = r$, we necessarily have $p'(h) > \lambda_h$. Lemma 5.12 (v) then implies $\bar{u}_0^0 = 0$.

The nonzero entries of $\bar{v}$ are (see Figure 5.11)

$$\bar{v}_{i_t} = \sum_{\ell = p'(i_t)}^{p'(i_{t+1})-1} \left(f_\ell^0 - f_{\ell+1}^0\right) = f_{p'(i_t)}^0 - f_{p'(i_{t+1})}^0 \quad \text{for } t \neq m.$$

The corresponding inequality (5.107) is then

$$s + y_h + \sum_{t \neq m} \left(f_{p'(i_t)}^0 - f_{p'(i_{t+1})}^0\right)(z_{i_t} - \lceil b_{i_t} \rceil) \geq f_{p'(h)}^0 \lceil b_h \rceil + \left(1 - f_{\lambda_h+1}^0\right)(\lceil b_h \rceil - 1),$$

which can be equivalently be written as

$$s + y_h + \sum_{t \neq r} \left(f'(b_{i_t}) - f'(b_{i_{t+1}})\right)(z_{i_t} - \lceil b_{i_t} \rceil) \geq$$
$$f'(b_{i_r}) \lceil b_{i_r} \rceil + \left(1 - f'(b_{i_{r+1}})\right)(\lceil b_{i_r} \rceil - 1). \quad (5.112)$$

This concludes the analysis of the case $J(\bar{w}) \neq \varnothing$.

The case $J(\bar{w}) = \varnothing$

We now consider the case $J(\bar{w}) = \varnothing$, that is $\bar{w}^i = 0$ for all $1 \le i \le n$. Since $\bar{w} \neq \mathbf{0}$ and $\bar{w}^0 \ge 0$, we can assume $\bar{w}^0 = 1$ without loss of generality.

Note that in this case $\bar{\vartheta}^i = 0$ for $1 \le i \le n$, as otherwise the support of $(\bar{u}, \bar{\xi}, \bar{v}, \bar{\vartheta})$ would contain the heavy cycle $v^i, u_1^i, \dots, u_{k-1}^i, \vartheta^i$. The same argument also shows that $\bar{v}^i = \bar{\xi}_0^i$ for $1 \le i \le n$. Inequality (5.87) can then be rewritten as

$$s + \sum_{i=1}^{n} \bar{v}^i (z_i - \lceil b_i \rceil) + \bar{u}_0^0 \ge 0. \tag{5.113}$$

The above considerations shows that the only arcs that can carry a positive flow (in a circulation that generates a non-redundant inequality) are the arcs of sector S^0 and arcs v^i, ξ_0^i for $1 \le i \le n$. Furthermore, for each $1 \le i \le n$ we can identify arcs v^i, ξ_0^i into a single arc. The network then reduces to that considered in Section 5.2 (Figure 5.1), where no arc enters the dummy node.

The acyclic circulations in such a network were shown in Section 5.2 (here we are clearly interested in the case $\bar{w} > 0$). It can be easily checked that the corresponding inequalities (5.113) are precisely $s \ge 0$ and the mixing inequalities listed in Section (5.2.3):

$$s + \sum_{t=1}^{r} \big(f'(b_{i_t}) - f'(b_{i_{t+1}})\big)(z_{i_t} - \lceil b_{i_t} \rceil) \ge 0, \tag{5.114}$$

$$s + \sum_{t=1}^{r} \big(f'(b_{i_t}) - f'(b_{i_{t+1}})\big)(z_{i_t} - \lceil b_{i_t} \rceil) + \big(1 - f'(b_{i_1})\big)\big(z_{i_r} - \lceil b_{i_r} \rceil - 1\big) \ge 0 \tag{5.115}$$

for all sequences of indices $i_1, \dots, i_r$ in $\{1, \dots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_r})$.

We have therefore proven the following result:

Proposition 5.13 *A linear inequality description of the convex hull of the mixing set with flows (5.53)–(5.56) in its original space is obtaining by adding to the original inequalities the following constraints:*

- (5.110) *for all sequences of indices $i_1, \dots, i_r$ in $\{1, \dots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_r})$ and all indices $1 \le h \le n$ and $1 \le m \le r$;*
- (5.111) *for all sequences of indices $i_1, \dots, i_r$ in $\{1, \dots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_r})$ and all indices $1 \le h \le n$ and $1 \le m < r$;*
- (5.112) *for all sequences of indices $i_1, \dots, i_r$ in $\{1, \dots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_r})$ and all indices $1 \le h \le n$;*
- (5.114)–(5.115) *for all sequences of indices $i_1, \dots, i_r$ in $\{1, \dots, n\}$ such that $f'(b_{i_1}) > \cdots > f'(b_{i_r})$.*

Conforti, Di Summa and Wolsey [13] obtained the linear inequality description of the mixing set with flows in a different form (see also Section 8.2).

Chapter 6

Dual network sets with a single integer variable

Recall that we denote by MIX^{2TU} any mixed-integer set of the form $\{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$, where A is a totally unimodular matrix with at most two nonzero entries per row and I is a nonempty subset of $N := \{1, \ldots, n\}$. In this chapter we consider problems of this type with $|I| = 1$, i.e. with a single integer variable. We give a linear inequality description of the convex hull of such sets in the original space. In contrast to Chapter 5, the convex hull is obtained here without constructing or projecting any extended formulation of the set.

In Section 6.1 we state the main result of the chapter, which provides a linear inequality formulation (in the original space) of the convex hull of an arbitrary dual network set with a single integer variable. By a result of Section 2.2 this also yields a formulation of a set MIX^{2TU} with a single integer variable.

The theorem stated in Section 6.1 is proven in Sections 6.2–6.3. More specifically, in Section 6.2 we prove the validity of the inequalities by showing that each of them is a simple MIR-inequality. In Section 6.3 we prove that the inequalities of the theorem are also sufficient to describe the convex hull of the set. This is done by following an idea that was applied in the study of sets defined by circular-ones matrices [24].

We conclude in Section 6.4 by discussing the Chvátal rank of a pure integer set that constitutes an equivalent formulation of a set MIX^{2TU} with a single integer variable. In particular, we show that there are very small and simple instances having Chvátal rank greater than one.

6.1 The convex hull in the original space

Let $X = \{x \in \mathbb{R}^n : Ax \geq b,\ x_i \text{ integer for } i \in I\}$ be a mixed-integer set of the type MIX^{2TU} with $|I| = 1$. We assume without loss of generality that the integer variable corresponds to the last column of A.

By Corollary 2.4, by multiplying by -1 a subset R of columns of A we can transform X into a set with dual network constraint matrix. Note that given a linear inequality description of the convex hull of the transformed set, a description of the original set is immediately obtained by changing again the sign of the variables with indices in R. Therefore we can assume without loss of generality that our set X is defined by a dual network matrix A, i.e X is a set of the type MIX^{DN}.

The linear system $Ax \geq b$ has then the form

$$x_i - x_j \geq l_{ij}, \quad (i,j) \in N^e, \tag{6.1}$$

$$x_i \geq l_i, \quad i \in N^l, \tag{6.2}$$

$$x_i \leq u_i, \quad i \in N^u, \tag{6.3}$$

where $N^e \subseteq N \times N$ and $N^l, N^u \subseteq N$. The set N^e does not contain any pair of the type (i,i) for $i \in N$. If the set of inequalities (6.2) does not include an explicit lower bound l_n on x_n, we set $l_n := -\infty$ (but we do not include the bound in the formulation). Similarly if no upper bound on x_n is included in the above system, we set $u_n := +\infty$. We also assume that the system $Ax \geq b$ is feasible.

Define $P := \text{conv}(X) = \text{conv}\{x \in \mathbb{R}^n : Ax \geq b,\ x_n \text{ integer}\}$. In order to give a linear inequality description of P in the x-space, we need to assume that l_n and u_n are tight bounds for x_n: that is, we assume that

$$l_n = \min\{x_n : x \in P\}, \quad u_n = \max\{x_n : x \in P\}. \tag{6.4}$$

If this is not the case, we can use the following easy result:

Lemma 6.1 *Define the values $m := \min\{x_n : Ax \geq b\}$ and $M := \max\{x_n : Ax \geq b\}$. If $\lceil m \rceil \leq \lfloor M \rfloor$,*[1] *then $\min\{x_n : x \in P\} = \lceil m \rceil$ and $\max\{x_n : x \in P\} = \lfloor M \rfloor$.*

Proof. We assume that both m and M are finite (the other cases are similar). Let x^1, x^2 be two points satisfying system $Ax \geq b$, with $x^1_n = m$ and $x^2_n = M$. All points in the segment $[x^1, x^2]$ satisfy $Ax \geq b$. Since $\lceil m \rceil \leq \lfloor M \rfloor$ by assumption, the segment $[x^1, x^2]$ contains points $\bar{x}^1, \bar{x}^2$ such that $\bar{x}^1_n = \lceil m \rceil$ and $\bar{x}^2_n = \lfloor M \rfloor$. This proves the result. □

If conditions (6.4) are not satisfied, we can compute the values m and M defined in the above lemma (this amounts to solving two linear programs). If $\lceil m \rceil = \lfloor M \rfloor + 1$ then $P = \varnothing$ (and we have found the convex hull of X). Otherwise $\lceil m \rceil \leq \lfloor M \rfloor$ and we can redefine $l_n := \lceil m \rceil$ and $u_n := \lfloor M \rfloor$. By the above lemma, conditions (6.4) are now satisfied.

We now prepare to present our result. Let $\mathcal{G} = (V, E)$ be the directed graph with vertex set $V := \{0, \dots, n-1\}$ and arc set E defined as follows:

(a) for each pair $(i,j) \in N^e$, where $i, j \neq n$, E contains an arc from node i to node j;

(b) for each pair $(i,n) \in N^e$, E contains an arc from node i to node 0; symmetrically, for each pair $(n,j) \in N^e$, E contains an arc from node 0 to node j;

(c) for each index $i \in N^l$ with $i \neq n$, E contains an arc from node i to node 0;

[1] Here $\lfloor +\infty \rfloor := +\infty$ and $\lceil -\infty \rceil := -\infty$.

(d) for each index $i \in N^u$ with $i \neq n$, E contains an arc from node 0 to node i.

Note that $\mathcal{G}$ may contain several pairs of parallel or opposite arcs.

Thus every inequality of the system $Ax \geq b$ (i.e. system (6.1)–(6.3)) gives rise to an arc of $\mathcal{G}$, except for the inequalities $l_n \leq x_n \leq u_n$ (if appearing in the system). We give weights to the arcs of $\mathcal{G}$ in the following very natural way: arcs arising from a pair $(i,j) \in N^e$ receive weight l_{ij}, arcs of type (c) receive weight l_i and arcs of type (d) weight $-u_i$. The weight of an arc $e \in E$ is denoted b_e. In other words b_e is the right-hand side of the inequality of (6.1)–(6.3) (written in the "$\geq$" form) corresponding to arc e.

Let $\mathcal{C}$ denote a sequence of arcs forming an undirected cycle in $\mathcal{G}$. Assume that the sequence of nodes and arcs in the cycle is $(i_0, e_0, i_1, \dots, i_k, e_k, i_{k+1})$, where $i_0 = i_{k+1}$. Let $E^+(\mathcal{C})$ be the set of arcs of $\mathcal{C}$ that are traversed accordingly to their orientation, i.e. $E^+(\mathcal{C}) := \{e_t : i_t \text{ is the tail of } e_t\}$. Symmetrically, let $E^-(\mathcal{C}) := \{e_t : i_t \text{ is the head of } e_t\}$ be the set of arcs of $\mathcal{C}$ that are traversed in the wrong direction. Let $T^+(\mathcal{C})$ (resp. $T^-(\mathcal{C})$) be the set of indices t such that e_t is in $E^+(\mathcal{C})$ (resp. $E^-(\mathcal{C})$). We define

$$b^+(\mathcal{C}) := \sum_{e \in E^+(\mathcal{C})} b_e, \quad b^-(\mathcal{C}) := \sum_{e \in E^-(\mathcal{C})} b_e.$$

We also define $d(\mathcal{C}) := b^+(\mathcal{C}) - b^-(\mathcal{C})$. Note that if one reverses the sequence of nodes and arcs forming $\mathcal{C}$, the values $b^+(\mathcal{C})$, $b^-(\mathcal{C})$ and $d(\mathcal{C})$ change sign. Thus, rather than just a cycle, $\mathcal{C}$ indicates in which order the arcs of that cycle are traversed.

We now present the main result of this chapter. As in the previous chapters, for a real number α we write $f(\alpha)$ to denote the fractional part of α, i.e. $f(\alpha) := \alpha - \lfloor \alpha \rfloor$.

Theorem 6.2 *A linear inequality description of P in its original space is given by the original system (6.1)–(6.3) plus all linear inequalities of the form*

$$\sum_{t \in T^+(\mathcal{C})} (x_{i_t} - x_{i_{t+1}}) + \big(\varepsilon(\mathcal{C}) + f(d(\mathcal{C}))\big) x_n \geq b^-(\mathcal{C}) + f(d(\mathcal{C})) \lfloor d(\mathcal{C}) + 1 \rfloor, \tag{6.5}$$

with the following meaning of notation:

- *$\mathcal{C} = (i_0, e_0, i_1, \dots, i_k, e_k, i_{k+1})$ is an undirected cycle in $\mathcal{G}$, with $k \geq 2$ and $i_0 = i_{k+1} = 0$. Arc e_0 is an arc of type (b) defined above, while e_k is not of type (b).*
- *Any occurrence of x_0 stands for a zero.*
- *The value $\varepsilon(\mathcal{C})$ is defined by* $\varepsilon(\mathcal{C}) := \begin{cases} 0 & \text{if } e_0 \in E^+(\mathcal{C}), \\ -1 & \text{otherwise.} \end{cases}$

In Section 6.2 we prove that inequalities (6.5) are valid for P, while in Section 6.3 we show that they suffice to describe P. We conclude in Section 6.4 by discussing the Chvátal rank of an equivalent pure integer formulation of P.

6.2 Validity of the inequalities

We show here that each of inequalities (6.5) can be obtained as a simple MIR-inequality (see Theorem 1.11) from an inequality that is implied by the original linear system (6.1)–(6.3). This proves that all inequalities (6.5) are valid for P.

Let $\mathcal{C}$ be an undirected cycle satisfying the conditions described in Theorem 6.2. Note that e_k is an arc of either type (c) or type (d). We now distinguish four possibilities. In all cases below, the following easy identity will be used:

$$\sum_{t\in T^+(\mathcal{C})}(x_{i_t}-x_{i_{t+1}})+\sum_{t\in T^-(\mathcal{C})}(x_{i_t}-x_{i_{t+1}})=0. \tag{6.6}$$

Case 1: $e_0\in E^+(\mathcal{C})$ and e_k is an arc of type (c).
First of all note that $e_k\in E^+(\mathcal{C})$, as e_k is an arc of type (c). We claim that the following inequalities are all included in the original system (6.1)–(6.3):

(i) inequalities (6.1) for $(i,j)=(i_t,i_{t+1})$ with $t\in T^+(\mathcal{C})\setminus\{0,k\}$;

(ii) inequality $x_n-x_{i_1}\geq l_{n,i_1}$;

(iii) inequality $x_{i_k}\geq l_{i_k}$.

The inequalities of group (i) belong to the original system because for each $t\in T^+(\mathcal{C})\setminus\{0,k\}$, arc e_t is necessarily of type (a). The inequality in (ii) is part of the original system as it corresponds to arc e_0, which is of type (b) by assumption. As to the last inequality, recall that we are assuming that e_k is an arc of type (c).

Summing up all the above inequalities gives (recall that $x_0=0$ and $0,k\in T^+(\mathcal{C})$)

$$\sum_{t\in T^+(\mathcal{C})}(x_{i_t}-x_{i_{t+1}})+x_n\geq b^+(\mathcal{C}), \tag{6.7}$$

which we rewrite as

$$\sum_{t\in T^+(\mathcal{C})}(x_{i_t}-x_{i_{t+1}})-b^-(\mathcal{C})+x_n\geq d(\mathcal{C}). \tag{6.8}$$

Similarly, all inequalities (6.1) for $(i,j)=(i_t,i_{t+1})$ with $t\in T^-(\mathcal{C})$ belong to the original system. Summing them up gives

$$\sum_{t\in T^-(\mathcal{C})}(x_{i_{t+1}}-x_{i_t})\geq b^-(\mathcal{C}). \tag{6.9}$$

Then if we set $s:=\sum_{t\in T^-(\mathcal{C})}(x_{i_t}-x_{i_{t+1}})-b^-(\mathcal{C})$, we have $s\geq 0$. Using equation (6.6), inequality (6.8) can now be written as $s+x_n\geq d(\mathcal{C})$. Since s is a nonnegative variable and x_n is an integer variable, the corresponding simple MIR-inequality is valid:

$$s+f(d(\mathcal{C}))x_n\geq f(d(\mathcal{C}))\lfloor d(\mathcal{C})+1\rfloor.$$

Substituting back for s, we obtain inequality (6.5).

Case 2: $e_0\in E^+(\mathcal{C})$ and e_k is an arc of type (d).
In this case $e_k\in E^-(\mathcal{C})$, as e_k is an arc of type (d). Similarly to the previous case, one can check that the following inequalities are all included in the original system (6.1)–(6.3):

(i) inequalities (6.1) for $(i,j) = (i_t, i_{t+1})$ with $t \in T^+(\mathcal{C}) \setminus \{0,k\}$;

(ii) inequality $x_n - x_{i_1} \geq l_{ni_1}$.

Summing up the above inequalities gives (recall that $x_0 = 0$ and $0 \in T^+(\mathcal{C})$)

$$\sum_{t \in T^+(\mathcal{C})} (x_{i_t} - x_{i_{t+1}}) + x_n \geq b^+(\mathcal{C}),$$

which we rewrite as

$$\sum_{t \in T^+(\mathcal{C})} (x_{i_t} - x_{i_{t+1}}) - b^-(\mathcal{C}) + x_n \geq d(\mathcal{C}). \tag{6.10}$$

The following inequalities are also part of the original system:

(i) inequalities (6.1) for $(i,j) = (i_t, i_{t+1})$ with $t \in T^-(\mathcal{C}) \setminus \{k\}$;

(ii) inequality $-x_{i_k} \geq -u_{i_k}$.

If we sum them up and recall that $x_0 = 0$ and $k \in T^-(\mathcal{C})$, we find

$$\sum_{t \in T^-(\mathcal{C})} (x_{i_{t+1}} - x_{i_t}) \geq b^-(\mathcal{C}).$$

We can now set $s := \sum_{t \in T^-(\mathcal{C})}(x_{i_t} - x_{i_{t+1}}) - b^-(\mathcal{C})$ and proceed as in the previous case.

Case 3: $e_0 \in E^-(\mathcal{C})$ and e_k is an arc of type (c).
As in Case 1, $e_k \in E^+(\mathcal{C})$. Summing up all the inequalities corresponding to arcs e_t with $t \in T^+(\mathcal{C})$ and subtracting $b^-(\mathcal{C})$ from both sides gives

$$\sum_{t \in T^+(\mathcal{C})} (x_{i_t} - x_{i_{t+1}}) - b^-(\mathcal{C}) \geq d(\mathcal{C}). \tag{6.11}$$

The following inequalities are included in the original system:

(i) inequalities (6.1) for $(i,j) = (i_t, i_{t+1})$ with $t \in T^-(\mathcal{C}) \setminus \{k\}$;

(ii) inequality $x_{i_1} - x_n \geq l_{i_1,n}$.

Adding up all these inequalities and subtracting $b^-(\mathcal{C})$ from both sides gives

$$\sum_{t \in T^-(\mathcal{C})} (x_{i_t} - x_{i_{t+1}}) - x_n - b^-(\mathcal{C}) \geq 0.$$

If we define s to be the left-hand side of the above inequality, by equation (6.6) inequality (6.11) becomes $s + x_n \geq d(\mathcal{C})$. Applying the MIR-inequality and substituting back for s gives inequality (6.5).

Case 4: $e_0 \in E^-(\mathcal{C})$ and e_k is an arc of type (d).
This case is very similar to the previous one.

This concludes the proof of the validity of inequalities (6.5).

6.3 Sufficiency of the inequalities

We prove here that the original constraints (6.1)–(6.3) and all inequalities (6.5) are sufficient to describe $P = \mathrm{conv}(X)$. We use an idea appearing in a paper by Eisenbrand, Oriolo, Stauffer and Ventura [24]. We find useful to present here the approach used by the authors cited above, as we need to extend it to the case of a polyhedron that is not full-dimensional (the polyhedron studied in [24] is full-dimensional, and this property was implicitly used there).

6.3.1 Extending a slicing approach

The results presented in this subsection extend those appearing in [24] to the case of a polyhedron which is not full-dimensional. We remark that in this subsection we do not need any particular assumptions on X, except that X is a mixed-integer set in $\mathbb{R}^n$ with a single integer variable x_n, and that conditions (6.4) hold.

For each integer number α such that $l_n \le \alpha \le u_n$, we define the polyhedra $P^\alpha := \{x \in X : x_n = \alpha\} = \{x \in P : x_n = \alpha\}$ and $P^{\alpha,\alpha+1} := \mathrm{conv}\big(P^\alpha \cup P^{\alpha+1}\big)$. Clearly

$$P = \mathrm{conv}\big(\textstyle\bigcup_{\alpha=l_n}^{u_n} P^\alpha\big) = \mathrm{conv}\,\big(\textstyle\bigcup_{\alpha=l_n}^{u_n-1} P^{\alpha,\alpha+1}\big).$$

Moreover, the following simple result holds.

Lemma 6.3 *Given $\bar{x} \in \mathbb{R}^n$, $\bar{x} \in P$ if and only if $\bar{x} \in P^{\alpha,\alpha+1}$ for $\alpha = \lfloor \bar{x}_n \rfloor$.*

Proof. The 'if' part is obvious. To prove the 'only if' part, let $\bar{x}$ be a point in P. If $\bar{x}_n$ is an integer then $\bar{x} \in P^\alpha$ for $\alpha = \bar{x}_n$. So assume $\bar{x}_n \notin \mathbb{Z}$ and define $\alpha = \lfloor \bar{x}_n \rfloor$. By definition of P, $\bar{x}$ can be written as convex combination of two points $x^1, x^2 \in P$, where $x^1_n \le \alpha$ and $x^2_n \ge \alpha + 1$. Then the segment $\big[x^1, x^2\big]$, which is contained in P, intersects P^α and $P^{\alpha+1}$, thus showing that $\bar{x}$ is the convex combination of a point in P^α and a point in $P^{\alpha+1}$, i.e. $\bar{x} \in P^{\alpha,\alpha+1}$. □

Note that for each integer α satisfying $l_n \le \alpha \le u_n$, the polyhedron P^α is nonempty (this follows from conditions (6.4) and basic convexity). Then for $l_n \le \alpha < u_n$, the polyhedra P^α and $P^{\alpha+1}$ are nonempty faces of $P^{\alpha,\alpha+1}$ (induced by inequalities $x_n \ge \alpha$ and $x_n \le \alpha + 1$ respectively). Define $\mathcal{F}^{\alpha,\alpha+1}$ as a family of equations and inequalities that constitute a minimal description of $P^{\alpha,\alpha+1}$, except that we do not include in $\mathcal{F}^{\alpha,\alpha+1}$ any inequality defining face P^α or $P^{\alpha+1}$. We assume without loss of generality that all inequalities in $\mathcal{F}^{\alpha,\alpha+1}$ are of the "$\ge$" kind. We write $cx \sim \delta$ to denote a linear constraint that can be either inequality $cx \ge \delta$ or equation $cx = \delta$.

Lemma 6.4 *P is the set of points in $\mathbb{R}^n$ satisfying inequalities $l_n \le x_n \le u_n$ and all equations and inequalities in $\bigcup_{\alpha=l_n}^{u_n-1} \mathcal{F}^{\alpha,\alpha+1}$.*

Proof. Let Q be the set of points in $\mathbb{R}^n$ satisfying inequalities $l_n \le x_n \le u_n$ and all equations and inequalities in $\bigcup_{\alpha=l_n}^{u_n-1} \mathcal{F}^{\alpha,\alpha+1}$. We prove that $Q = P$.

If $\bar{x} \in Q$ then $l_n \le \bar{x}_n \le u_n$ and $\bar{x}$ satisfies all equations and inequalities in $\mathcal{F}^{\alpha,\alpha+1}$ where $\alpha = \lfloor \bar{x}_n \rfloor$. Since $\alpha \le \bar{x}_n \le \alpha + 1$ also holds, we have $\bar{x} \in P^{\alpha,\alpha+1}$, hence $\bar{x} \in P$. This shows that $Q \subseteq P$.

To prove the reverse inclusion, we show that for $l_n \leq \alpha < u_n$, every equation or inequality in $\mathcal{F}^{\alpha,\alpha+1}$ is valid for P. Assume that the contrary holds, i.e. there exist an integer α such that $l_n \leq \alpha < u_n$, an equation or inequality $cx \sim \delta$ in $\mathcal{F}^{\alpha,\alpha+1}$ and a point $\bar{x} \in P$ such that $c\bar{x} \not\sim \delta$. If $\alpha \leq \bar{x}_n \leq \alpha + 1$, Lemma 6.3 implies that $\bar{x} \in P^{\alpha,\alpha+1}$, thus $\bar{x}$ satisfies $cx \sim \delta$, a contradiction.

So we assume $\bar{x}_n \leq \alpha$ (the case $\bar{x}_n \geq \alpha + 1$ is similar). We now claim that there is a point $x^{\alpha+1} \in P^{\alpha+1}$ such that $cx^{\alpha+1} = \delta$. To prove this, we distinguish two cases.

1. Assume first that $cx \sim \delta$ is an inequality. Then inequality $cx \geq \delta$ defines a facet F of $P^{\alpha,\alpha+1}$ and we let k be the dimension of F. Since $P^{\alpha,\alpha+1} = \text{conv}\big(P^\alpha \cup P^{\alpha+1}\big)$, there exist $k+1$ affinely independent points in $F \cap \big(P^\alpha \cup P^{\alpha+1}\big)$ that satisfy equation $cx = \delta$. If all these $k+1$ points belonged to P^α, then inequality $cx \geq \delta$ would induce face[2] P^α, contradicting the fact that inequality $cx \geq \delta$ belongs to $\mathcal{F}^{\alpha,\alpha+1}$. Thus there is a point $x^{\alpha+1} \in P^{\alpha+1}$ such that $cx^{\alpha+1} = \delta$.

2. The other possibility is that $cx \sim \delta$ is an equation and thus $P^{\alpha,\alpha+1} \subseteq \{x \in \mathbb{R}^n : cx = \delta\}$. Since $P^{\alpha+1} \neq \varnothing$, there is a point $x^{\alpha+1} \in P^{\alpha+1} \subseteq \{x \in \mathbb{R}^n : cx = \delta\}$.

Thus in both cases there is a point $x^{\alpha+1} \in P^{\alpha+1}$ such that $cx^{\alpha+1} = \delta$. Since $c\bar{x} \not\sim \delta$, the segment $\big[\bar{x}, x^{\alpha+1}\big]$, which is contained in P, intersects P^α in a point x^α such that $cx^\alpha \not\sim \delta$. This is a contradiction, as the equation or inequality $cx \sim \delta$ is valid for P^α. □

Therefore, in order to find a linear inequality description of P, we have to find all equations and inequalities in the family $\mathcal{F}^{\alpha,\alpha+1}$ for $l_n \leq \alpha < u_n$.

In the following we write $A = [M \mid a_n]$, where M is the column submatrix constituted by the first $n-1$ columns of A and a_n is the n-th column of A. Similarly we decompose a point $x \in \mathbb{R}^n$ as $x = (x_M, x_n)$.

Lemma 6.5 *Fix a point $\bar{x} \in \mathbb{R}^n$ with $\mu(\bar{x}) := f(\bar{x}_n) > 0$ and an integer $l_n \leq \alpha < u_n$. Define*

$$b^\alpha := b - \alpha a_n, \quad b^{\alpha+1} := b - (\alpha+1)a_n. \tag{6.12}$$

Then $\bar{x} \in P^{\alpha,\alpha+1}$ if and only if the optimum value of the following linear program is zero:

$$\begin{aligned} \max \quad & -v^\alpha M\bar{x}_M + (1-\mu(\bar{x}))v^\alpha b^\alpha + \mu(\bar{x})v^{\alpha+1}b^{\alpha+1} && (6.13)\\ \textit{subject to} \quad & v^\alpha M - v^{\alpha+1}M = \mathbf{0}, && (6.14)\\ & v^\alpha, v^{\alpha+1} \geq \mathbf{0}. && (6.15) \end{aligned}$$

Proof. The point $\bar{x}$ belongs to $P^{\alpha,\alpha+1}$ if and only if there exist $x^\alpha \in P^\alpha$, $x^{\alpha+1} \in P^{\alpha+1}$ and $0 \leq \lambda \leq 1$ such that

$$\bar{x} = \lambda x^\alpha + (1-\lambda)x^{\alpha+1}. \tag{6.16}$$

By writing equation (6.16) for the n-th component, one finds $\mu(\bar{x}) = 1 - \lambda$. Then $\bar{x} \in P^{\alpha,\alpha+1}$ if and only if there exist $x^\alpha \in P^\alpha$ and $x^{\alpha+1} \in P^{\alpha+1}$ such that

$$\bar{x}_M = (1-\mu(\bar{x}))x^\alpha_M + \mu(\bar{x})x^{\alpha+1}_M.$$

[2] Actually facet in this case.

If we define b^α and $b^{\alpha+1}$ as in (6.12), the polyhedron P^α (resp. $P^{\alpha+1}$) is described by the conditions $x_n = \alpha$, $Mx_M \geq b^\alpha$ (resp. $x_n = \alpha + 1$, $Mx_M \geq b^{\alpha+1}$). Thus $\bar{x} \in P^{\alpha,\alpha+1}$ if and only if there exist $x^\alpha, x^{\alpha+1} \in \mathbb{R}^n$ such that

$$\bar{x}_M = (1 - \mu(\bar{x}))x^\alpha_M + \mu(\bar{x})x^{\alpha+1}_M, \quad Mx^\alpha_M \geq b^\alpha, \quad Mx^{\alpha+1}_M \geq b^{\alpha+1}.$$

After defining $y^\alpha := (1 - \mu(\bar{x}))x^\alpha_M$ and $y^{\alpha+1} := \mu(\bar{x})x^{\alpha+1}_M$, we obtain that $\bar{x} \in P^{\alpha,\alpha+1}$ if and only if the following linear system admits a feasible solution $(y^\alpha, y^{\alpha+1}) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$:

$$\begin{aligned} y^\alpha + y^{\alpha+1} &= \bar{x}_M, \\ My^\alpha &\geq (1 - \mu(\bar{x}))b^\alpha, \\ My^{\alpha+1} &\geq \mu(\bar{x})b^{\alpha+1}. \end{aligned}$$

By Farkas' lemma (Theorem 1.7), this happens if and only if all feasible solutions of the following linear program have non-positive cost:

$$\begin{aligned} \max \quad & u\bar{x}_M + (1 - \mu(\bar{x}))v^\alpha b^\alpha + \mu(\bar{x})v^{\alpha+1}b^{\alpha+1} \\ \text{subject to} \quad & u + v^\alpha M = \mathbf{0}, \\ & u + v^{\alpha+1}M = \mathbf{0}, \\ & v^\alpha, v^{\alpha+1} \geq \mathbf{0}. \end{aligned}$$

After eliminating variable u and observing that the all-zero solution is feasible, the proof is complete. □

Note that the feasible region (6.14)–(6.15) does not depend on $\bar{x}$.

Now fix $l_n \leq \alpha < u_n$, let $cx \sim \delta$ be an equation or inequality in $\mathcal{F}^{\alpha,\alpha+1}$ and call F the facet (or improper face) of $P^{\alpha,\alpha+1}$ that is induced by $cx \sim \delta$. Let $\bar{x}$ be a point in the relative interior of F (note that then $0 < \mu(\bar{x}) < 1$, as assumed in Lemma 6.5). Since $\bar{x} \in P^{\alpha,\alpha+1}$, the optimum value of the linear program (6.13)–(6.15) is zero. We call $\mathcal{Z}(\bar{x})$ the set of optimal (i.e. zero-cost) solutions of the linear program (6.13)–(6.15).

Lemma 6.6 *For each feasible vector* $(v^\alpha, v^{\alpha+1})$ *in* (6.14)–(6.15)*, the inequality*

$$v^\alpha Mx_M + \left(v^\alpha b^\alpha - v^{\alpha+1}b^{\alpha+1}\right)x_n \geq v^\alpha b^\alpha + \left(v^\alpha b^\alpha - v^{\alpha+1}b^{\alpha+1}\right)\alpha \tag{6.17}$$

is valid for $P^{\alpha,\alpha+1}$. *Furthermore, the equation or inequality* $cx \sim \delta$ *is implied by the family of inequalities* (6.17) *for* $(v^\alpha, v^{\alpha+1}) \in \mathcal{Z}(\bar{x})$.

Proof. For a point $x \in P^\alpha$, $x_n = \alpha$ holds and thus inequality (6.17) reduces to $v^\alpha Mx_M \geq v^\alpha b^\alpha$, which is valid for P^α (as it is a nonnegative combination of the inequalities of the system $Mx_M \geq b^\alpha$). Similarly, for a point $x \in P^{\alpha+1}$, $x_n = \alpha + 1$ holds and thus, recalling that $v^\alpha M = v^{\alpha+1}M$ by (6.14), inequality (6.17) reduces to $v^{\alpha+1}Mx_M \geq v^{\alpha+1}b^{\alpha+1}$, which is valid for $P^{\alpha+1}$.

Therefore inequality (6.17) is valid for $P^{\alpha,\alpha+1}$. Also, since $(v^\alpha, v^{\alpha+1})$ is a zero-cost solution of (6.13)–(6.15) and recalling that $\bar{x}_n = \alpha + \mu(\bar{x})$, it is straightforward to verify that inequality (6.17) is tight for $\bar{x}$.

To prove the second part of the lemma, let $\hat{x} \in \mathbb{R}^n$ be any point violating $cx \sim \delta$. We show that there exists a vector $\left(\bar{v}^\alpha, \bar{v}^{\alpha+1}\right) \in \mathcal{Z}(\bar{x})$ such that $\hat{x}$ violates the corresponding inequality (6.17).

Define $\gamma := \hat{x} - \bar{x}$ and $\hat{x}(\varepsilon) := \bar{x} + \varepsilon\gamma$ (thus the mapping $\varepsilon \mapsto \hat{x}(\varepsilon)$ for $\varepsilon \in [0,1]$ is a parameterization of the segment $[\bar{x}, \hat{x}]$). Since all inequalities (6.17) for $\left(v^\alpha, v^{\alpha+1}\right) \in \mathcal{Z}(\bar{x})$ are tight for $\bar{x}$, it is sufficient to prove the above claim for the points of the type $\hat{x}(\varepsilon)$ with $\varepsilon > 0$ small enough.

Since $\mu(\bar{x}) = f(\bar{x}_n) > 0$, for $\varepsilon > 0$ sufficiently small we have $\mu(\bar{x} + \varepsilon\gamma) = f(\bar{x}_n + \varepsilon\gamma_n) = f(\bar{x}_n) + \varepsilon\gamma_n = \mu(\bar{x}) + \varepsilon\gamma_n$, thus the objective function (6.13) corresponding to the point $\hat{x}(\varepsilon) = \bar{x} + \varepsilon\gamma$ is

$$\phi_\varepsilon\left(v^\alpha, v^{\alpha+1}\right) := -v^\alpha M(\bar{x}_M + \varepsilon\gamma_M) + (1 - \mu(\bar{x}) - \varepsilon\gamma_n)v^\alpha b^\alpha + (\mu(\bar{x}) + \varepsilon\gamma_n)v^{\alpha+1}b^{\alpha+1}.$$

Note that for $\varepsilon = 0$ we find exactly objective function (6.13).

Let R be the set of extreme rays of cone (6.14)–(6.15) with unit Euclidean norm. We partition R into two subsets $R_\varepsilon^+ := \left\{\left(v^\alpha, v^{\alpha+1}\right) \in R : \phi_\varepsilon\left(v^\alpha, v^{\alpha+1}\right) > 0\right\}$ and $R_\varepsilon^- := R \setminus R_\varepsilon^+$. Since $c\bar{x} \sim \delta$ whereas $c\hat{x} \not\sim \delta$, the point $\hat{x}(\varepsilon)$ violates constraint $cx \sim \delta$ for all $\varepsilon > 0$. Then $\hat{x}(\varepsilon) \notin P$ for all $\varepsilon > 0$. By Lemma 6.5 this implies that $R_\varepsilon^+ \neq \varnothing$ for all $\varepsilon > 0$.

Note that for a fixed vector $\left(v^\alpha, v^{\alpha+1}\right)$, the mapping $\varepsilon \mapsto \phi_\varepsilon\left(v^\alpha, v^{\alpha+1}\right)$ is linear. Also $\phi_0\left(v^\alpha, v^{\alpha+1}\right) \leq 0$ for all $\left(v^\alpha, v^{\alpha+1}\right) \in R$. These two observations imply that if $0 < \varepsilon < \varepsilon'$ then $R_\varepsilon^+ \subseteq R_{\varepsilon'}^+$ and $R_\varepsilon^- \supseteq R_{\varepsilon'}^-$. Since $R_\varepsilon^+ \neq \varnothing$ for all $\varepsilon > 0$ and since R is a finite set, this shows that there is a vector $\left(\bar{v}^\alpha, \bar{v}^{\alpha+1}\right)$ such that $\phi_0\left(\bar{v}^\alpha, \bar{v}^{\alpha+1}\right) = 0$ and $\phi_\varepsilon\left(\bar{v}^\alpha, \bar{v}^{\alpha+1}\right) > 0$ for $\varepsilon > 0$. It is readily checked that the inequality (6.17) corresponding to this vector is violated by $\hat{x}(\varepsilon)$ for $\varepsilon > 0$ sufficiently small.

Thus we have found a vector $\left(\bar{v}^\alpha, \bar{v}^{\alpha+1}\right) \in \mathcal{Z}(\bar{x})$ such that the corresponding inequality (6.17) is violated by $\hat{x}(\varepsilon)$ for $\varepsilon > 0$ sufficiently small. This concludes the proof of the lemma. □

Therefore in order to find the inequalities and equations in $\mathcal{F}^{\alpha,\alpha+1}$ we have to find the zero-cost solutions of problem (6.13)–(6.15). Note that we have not used any assumptions on the structure of the original system. Thus the above considerations yield a polynomial time separation algorithm for any mixed-integer set with a single integer variable: given $\bar{x}$, solve the linear program (6.13)–(6.15) with $\alpha = \lfloor \bar{x}_n \rfloor$; if there is a positive cost solution, then the corresponding inequality (6.17) separates $\bar{x}$ from P, otherwise $\bar{x} \in P$ (see also [24]).

6.3.2 Finding the inequalities

We now consider our mixed-integer set X with dual network constraint matrix and a single integer variable x_n. In the following we investigate the zero-cost solutions of (6.13)–(6.15).

First of all, if the linear program (6.13)–(6.15) has a zero-cost solution, then it has a zero-cost extreme ray. So we look for the extreme rays of the cone defined by (6.14)–(6.15).

Since M is a dual network matrix, the constraint matrix corresponding to system (6.14), i.e. matrix $\begin{bmatrix} M \\ -M \end{bmatrix}$, is totally unimodular. Then the extreme rays of (6.14)–(6.15) are 0-1 vectors.

Note that M may have some all-zero rows (corresponding to inequalities $x_n \geq l_n$ and/or $x_n \leq u_n$). Let us suppose that the t-th row is the all-zero vector. Then the vectors $(e_t, \mathbf{0})$ and $(\mathbf{0}, e_t)$ are extreme rays of (6.14)–(6.15). However, the corresponding inequalities (6.17) are scalar multiples of $x_n \geq \alpha$ and $x_n \leq \alpha + 1$. So in the following we only consider rays of (6.14)–(6.15) with $v_t^\alpha = v_t^{\alpha+1} = 0$.

Let $\mathcal{H}$ be the directed graph having $\begin{bmatrix} M \\ -M \end{bmatrix}$ as arc-node incidence matrix. Since some rows of M may contain a single nonzero entry, we include a dummy node 0 in the vertex set of $\mathcal{H}$ as explained in Section 5.1. Thus the vertex set of $\mathcal{H}$ is $\{0, \ldots, n-1\}$. Note that $\mathcal{H}$ and the graph $\mathcal{G}$ defined in Section 6.1 are define on the same vertex set. Furthermore there is a one-to-one correspondence between arcs in $\mathcal{G}$ and pairs of opposite arcs in $\mathcal{H}$. If an arc e of $\mathcal{G}$ corresponds to the pair of opposite arcs e^+, e^- in $\mathcal{H}$, we say that e is the arc *underlying* e^+ and e^-. Given any subset of arcs of $\mathcal{H}$, the underlying subset of arcs of $\mathcal{G}$ is defined similarly.

By Theorem 5.1, the 0-1 extreme rays $(v^\alpha, v^{\alpha+1})$ of (6.14)–(6.15) correspond to directed cycles in $\mathcal{H}$. Note however that not all directed cycles of $\mathcal{H}$ generate valid inequalities for X, as not all extreme rays of (6.14)–(6.15) are zero-cost solutions of (6.13)–(6.15) for some $\bar{x}$ belonging to the relative interior of a face defined by an inequality or an equation in $\mathcal{F}^{\alpha,\alpha+1}$. In the following, we detect which cycles need to be really considered. The simple lemma below will be useful.

Lemma 6.7 *For $l_n \leq \alpha < u_n$, let $(v^\alpha, v^{\alpha+1})$ be a feasible solution of (6.14)–(6.15) (not necessarily a zero-cost solution). If the corresponding inequality (6.17) belongs to $\mathcal{F}^{\alpha,\alpha+1}$ and is valid for $\{x \in \mathbb{R}^n : Ax \geq b,\ \alpha \leq x_n \leq \alpha+1\}$, then it is implied by the system $Ax \geq b$.*

Proof. Let $cx \geq \delta$ denote inequality (6.17). Assume that $cx \geq \delta$ is in $\mathcal{F}^{\alpha,\alpha+1}$ and is valid for $\{x \in \mathbb{R}^n : Ax \geq b,\ \alpha \leq x_n \leq \alpha+1\}$ but not for $\{x \in \mathbb{R}^n : Ax \geq b\}$. Then there exists a point $\hat{x}$ such that $A\hat{x} \geq b$, $c\hat{x} < \delta$ and either $\hat{x}_n < \alpha$ or $\hat{x}_n > \alpha + 1$. Since inequality $cx \geq \delta$ is in $\mathcal{F}^{\alpha,\alpha+1}$, there exist two points $x^\alpha \in P^\alpha$ and $x^{\alpha+1} \in P^{\alpha+1}$ such that $cx^\alpha = cx^{\alpha+1} = \delta$. If $\hat{x}_n < \alpha$ (the case $\hat{x}_n > \alpha+1$ is similar), the segment $[\hat{x}, x^{\alpha+1}]$ intersects P^α in a point y such that $cy < \delta$. However this is not possible, as all points in the segment $[\hat{x}, x^{\alpha+1}]$ satisfy $Ax \geq b$. □

Remark 6.8 *By Lemma 6.7, whenever we find an inequality $cx \geq \delta$ of the form (6.17) that is valid for $\{x \in \mathbb{R}^n : Ax \geq b,\ \alpha \leq x_n \leq \alpha+1\}$, we can ignore it, as one of the following two possibilities holds: either $cx \geq \delta$ is implied by the original constraints $Ax \geq b$, or it does not belong to $\mathcal{F}^{\alpha,\alpha+1}$.*[3]

For fixed $l_n \leq \alpha < u_n$, let $(v^\alpha, v^{\alpha+1})$ be an extreme ray of (6.14)–(6.15). Recall that the polyhedron P that we want to characterize is defined by inequalities of the form (6.17), which we rewrite here for convenience:

$$v^\alpha M x_M + \rho x_n \geq v^\alpha b^\alpha + \rho\alpha, \tag{6.18}$$

[3] This second alternative is possible because Lemma 6.7 does not require $(v^\alpha, v^{\alpha+1})$ to be a zero-cost solution.

where we use notation $\rho := v^\alpha b^\alpha - v^{\alpha+1} b^{\alpha+1}$. Since $v^\alpha M = v^{\alpha+1} M$, the above inequality can also be written this way:

$$v^{\alpha+1} M x_M + \rho x_n \geq v^{\alpha+1} b^{\alpha+1} + \rho(\alpha + 1). \tag{6.19}$$

We will use both versions of the inequality.

Let $\mathcal{D}$ be the directed cycle in $\mathcal{H}$ defined by ray $(v^\alpha, v^{\alpha+1})$. If $\mathcal{D}$ consists of a pair of opposite arcs that correspond to the same arc of $\mathcal{C}$, then $v^\alpha = v^{\alpha+1}$. Using (6.12) and equation $v^\alpha = v^{\alpha+1}$ one immediately obtains $\rho = a_n$. Then inequality (6.18) is equivalent to $v^\alpha[M \mid a_n]x \geq v^\alpha b^\alpha + \alpha a_n$, i.e. $v^\alpha A x \geq v^\alpha b$. This shows that inequality (6.18) is implied by the original system $Ax \geq b$.

Therefore from now on we assume that $\mathcal{D}$ is a directed cycle in $\mathcal{H}$ consisting of at least three arcs. Let $\mathcal{C}$ be the underlying undirected cycle in $\mathcal{G}$. We denote the sequence of nodes and arcs of $\mathcal{C}$ as follows: $(i_0, e_0, i_1, \ldots, i_k, e_k, i_{k+1})$ where $k \geq 2$ and $i_0 = i_{k+1}$.

The support of v^α corresponds to the arcs of $\mathcal{D}$ for which the underlying arcs of $\mathcal{C}$ are in $E^+(\mathcal{C})$. Symmetrically, the support of $v^{\alpha+1}$ corresponds to the arcs of $\mathcal{D}$ for which the underlying arcs of $\mathcal{C}$ are in $E^-(\mathcal{C})$. This implies

$$v^\alpha b = b^+(\mathcal{C}), \qquad v^{\alpha+1} b = b^-(\mathcal{C}). \tag{6.20}$$

Note that the support of column a_n corresponds to arcs of type (b) of $\mathcal{G}$. Then the value $v^\alpha a_n$ is the difference between the number of arcs of type (b) in $E^+(\mathcal{C})$ entering node 0 and the number of arcs of type (b) in $E^+(\mathcal{C})$ leaving node 0. Similarly, the value $v^{\alpha+1} a_n$ is the difference between the number of arcs of type (b) in $E^-(\mathcal{C})$ entering node 0 and the number of arcs of type (b) in $E^-(\mathcal{C})$ leaving node 0. It then follows that $v^\alpha a_n$ and $v^{\alpha+1} a_n$ can only take values in $\{0, \pm 1\}$. Furthermore, using the above interpretation one can check that the case $v^\alpha a_n = 1 = -v^{\alpha+1} a_n$ cannot hold. For convenience of notation we define $\delta := v^\alpha a_n$ and $\varepsilon := v^{\alpha+1} a_n$.

Using (6.12), one finds

$$v^\alpha b^\alpha = v^\alpha b - \delta\alpha, \qquad v^{\alpha+1} b^{\alpha+1} = v^{\alpha+1} b - \varepsilon(\alpha + 1), \tag{6.21}$$

$v^\alpha A = (v^\alpha M, \delta)$ and $v^{\alpha+1} A = (v^{\alpha+1} M, \varepsilon)$. This implies that inequalities $v^\alpha A x \geq v^\alpha b$ and $v^{\alpha+1} A x \geq v^{\alpha+1} b$ are equivalent respectively to

$$v^\alpha M x_M + \delta x_n \geq v^\alpha b^\alpha + \delta\alpha, \quad v^{\alpha+1} M x_M + \varepsilon x_n \geq v^{\alpha+1} b^{\alpha+1} + \varepsilon(\alpha + 1), \tag{6.22}$$

thus the above two inequalities are implied by the original system $Ax \geq b$.

We now distinguish three cases.

1. Assume $\rho \geq \delta$. If $x_n \geq \alpha$ holds, summing the first inequality in (6.22) and $(\rho - \delta) x_n \geq (\rho - \delta)\alpha$ gives inequality (6.18). This means that such an inequality is valid for all points in $\{x : Ax \geq b,\ x_n \geq \alpha\}$ and by Remark 6.8 we can ignore this case.

2. Now assume $\rho < \varepsilon$. If $x_n \leq \alpha + 1$ holds, summing the second inequality in (6.22) and $(\rho - \varepsilon)x_n \geq (\rho - \varepsilon)(\alpha + 1)$ gives inequality (6.19). This means that such an inequality is valid for all points in $\{x : Ax \geq b,\, x_n \leq \alpha + 1\}$ and this case can also be ignored.

3. Finally assume $\varepsilon \leq \rho < \delta$. This case is possible only if $\delta \geq \varepsilon + 1$. Since, as observed above, the case $\delta = 1, \varepsilon = -1$ cannot hold, we necessarily have $\delta = \varepsilon + 1$. Then, also using (6.21) and (6.20), we have

$$\rho = v^\alpha b^\alpha - v^{\alpha+1} b^{\alpha+1} = v^\alpha b - v^{\alpha+1} b - \alpha + \varepsilon = b^+(\mathcal{C}) - b^-(\mathcal{C}) - \alpha + \varepsilon = d(\mathcal{C}) - \alpha + \varepsilon,$$

which implies $\rho = f(d(\mathcal{C})) + \varepsilon$ and $\alpha = \lfloor d(\mathcal{C}) \rfloor$. We now show that $\mathcal{C}$ satisfies the conditions of Theorem 6.2 and inequality (6.18) is precisely inequality (6.5).

Since $\delta, \varepsilon \in \{0, \pm 1\}$ and $\delta = \varepsilon + 1$, we have either $\delta = 1$ and $\varepsilon = 0$, or $\delta = 0$ and $\varepsilon = -1$. Recalling the definition of δ and ε, one can verify that in both cases arc e_0 is of type (b) while e_k is not of type (b). Furthermore if $\delta = 1$ and $\varepsilon = 0$ then $e_0 \in E^+(\mathcal{C})$, while if $\delta = 0$ and $\varepsilon = -1$ then $e_0 \in E^-(\mathcal{C})$. Thus $\mathcal{C}$ satisfies the conditions of Theorem 6.2 and $\varepsilon = \varepsilon(\mathcal{C})$.

Since $v^\alpha b^\alpha = v^\alpha b - (\varepsilon + 1)\alpha = b^+(\mathcal{C}) - (\varepsilon + 1)\lfloor d(\mathcal{C}) \rfloor = b^-(\mathcal{C}) + d(\mathcal{C}) - (\varepsilon + 1)\lfloor d(\mathcal{C}) \rfloor$, one can check that the right-hand side of inequality (6.18) is

$$\begin{aligned} v^\alpha b^\alpha + \rho\alpha = b^-(\mathcal{C}) + d(\mathcal{C}) - (\varepsilon + 1)\lfloor d(\mathcal{C}) \rfloor + (f(d(\mathcal{C})) + \varepsilon)\lfloor d(\mathcal{C}) \rfloor \\ = b^-(\mathcal{C}) + f(d(\mathcal{C}))\lfloor d(\mathcal{C}) + 1 \rfloor, \end{aligned}$$

which is exactly the right-hand side of inequality (6.5).

One can also verify that $v^\alpha M x_M = \sum_{t \in T^+(\mathcal{C})} (x_{i_t} - x_{i_{t+1}})$, with the convention that $x_0 = 0$. Finally the coefficient of x_n in inequality (6.18) is $\rho = \varepsilon + f(d(\mathcal{C})) = \varepsilon(\mathcal{C}) + f(d(\mathcal{C}))$. Thus inequalities (6.18) and (6.5) coincide.

This concludes the proof of Theorem 6.2.

6.4 Chvátal rank

We proved in Section 6.3 that all inequalities (6.5) are simple MIR-inequalities (thus the split rank of the system (6.1)–(6.3) is one). We investigate here whether inequalities (6.5) can be obtained through Chvátal-Gomory rounding, when considering an equivalent pure integer formulation of P. That is, we discuss the Chvátal rank of such a formulation (see Section 1.3.1).

From now on we assume that all right-hand sides of the inequalities of the system $Ax \geq b$ (i.e. inequalities (6.1)–(6.3)) are rational number. Let K be the smallest positive integer such that $Kb_{ij} \in \mathbb{Z}$ for all $ij \in E$. Since the constraint matrix A of the system (6.1)–(6.3) is totally unimodular, Lemma 2.11 shows that for every vertex $\bar{x}$ of P, $K\bar{x}$ is an integral vector. This proves that the change of variables

$$y_i := Kx_i \text{ for } i \neq n, \;\; y_n := x_n \tag{6.23}$$

maps P into Q, where Q is the convex hull of the following pure integer set:

$$y_i - y_j \geq Kl_{ij}, \quad (i,j) \in N^e,\ i,j \neq n, \tag{6.24}$$
$$y_i - Ky_n \geq Kl_{in}, \quad (i,n) \in N^e, \tag{6.25}$$
$$Ky_n - y_i \geq Kl_{nj}, \quad (n,j) \in N^e, \tag{6.26}$$
$$y_i \geq Kl_i, \quad i \in N^l \setminus \{n\}, \tag{6.27}$$
$$y_i \leq Ku_i, \quad i \in N^u \setminus \{n\}, \tag{6.28}$$
$$l_n \leq y_n \leq u_n, \tag{6.29}$$
$$y_i \text{ integer}, \quad i \in N, \tag{6.30}$$

where the lower (resp. upper) bound in (6.29) appears if and only if $n \in N^l$ (resp. $n \in N^u$).

We prove here that if $K \leq 3$ then the Chvátal rank of the polyhedron (6.24)–(6.29) is one, while for every $K \geq 4$ it is possible to construct very simple instances with Chvátal rank greater than one.

For the case $K = 2$, a similar result was proven by Conforti, Gerards and Zambelli [15] for the set considered in Section 4.5.2 (with an arbitrary number of integer variables).

Lemma 6.9 *If $K \in \{2,3\}$, the polyhedron defined by (6.24)–(6.29) has Chvátal rank one.*

Proof. We prove that every inequality of the type (6.5) can be obtained by applying the Chvátal-Gomory procedure (Theorem 1.10) to the inequalities (6.24)–(6.29).

Let $\mathcal{C}$ be as in Theorem 6.2. We only consider the case $\varepsilon(\mathcal{C}) = 0$, the other case being analogous.

Recall from Section 6.2 that inequality (6.7) is valid for the original system $Ax \geq b$ whenever $\varepsilon(\mathcal{C}) = 0$. In the y-variables, this inequality reads

$$\sum_{t \in T^+(\mathcal{C})} (y_{i_t} - y_{i_{t+1}}) + Ky_n \geq Kb^+(\mathcal{C}). \tag{6.31}$$

Also recall that inequality (6.9) is valid for the original system whenever $\varepsilon(\mathcal{C}) = 0$. Using relation (6.6), this inequality in the y-variables reads

$$\sum_{t \in T^+(\mathcal{C})} (y_{i_t} - y_{i_{t+1}}) \geq Kb^-(\mathcal{C}). \tag{6.32}$$

We now combine inequalities (6.31) and (6.32) with coefficients $f(d(\mathcal{C}))$ and $1 - f(d(\mathcal{C}))$ respectively. The resulting inequality is

$$\sum_{t \in T^+(\mathcal{C})} (y_{i_t} - y_{i_{t+1}}) + Kf(d(\mathcal{C}))y_n \geq Kf(d(\mathcal{C}))b^+(\mathcal{C}) + K\big(1 - f(d(\mathcal{C}))\big)b^-(\mathcal{C}).$$

Using $d(\mathcal{C}) = b^+(\mathcal{C}) - b^-(\mathcal{C})$, we can rewrite the above inequality as follows:

$$\sum_{t \in T^+(\mathcal{C})} (y_{i_t} - y_{i_{t+1}}) + Kf(d(\mathcal{C}))y_n \geq Kb^-(\mathcal{C}) + Kf(d(\mathcal{C}))d(\mathcal{C}). \tag{6.33}$$

Clearly $Kf(d(\mathcal{C})) \in \{0, \ldots, K-1\}$. If $Kf(d(\mathcal{C})) = 0$, the right-hand side of inequality (6.33) is $Kb^-(\mathcal{C})$. Then in this case inequality (6.33) coincides with (6.5) under the change of variables (6.23).

If $Kf(d(\mathcal{C})) = 1$, the right-hand side of inequality (6.33) is $Kb^-(\mathcal{C}) + d(\mathcal{C})$. Since the left-hand side of the inequality is an integer while $d(\mathcal{C})$ is fractional, we can round the right-hand side to $Kb^-(\mathcal{C}) + \lfloor d(\mathcal{C}) + 1 \rfloor$. The resulting inequality coincides with (6.5) under the change of variables (6.23).

If $Kf(d(\mathcal{C})) = 2$ (and $K = 3$), the right-hand side of inequality (6.33) is $3b^-(\mathcal{C}) + 2d(\mathcal{C})$. Note that the fractional part of this number is $1/3$. Since the left-hand side of the inequality is an integer, we can round the right-hand side to

$$3b^-(\mathcal{C}) + 2d(\mathcal{C}) + 2/3 = 3b^-(\mathcal{C}) + 2(d(\mathcal{C}) + 1/3) = 3b^-(\mathcal{C}) + 2\lfloor d(\mathcal{C}) + 1 \rfloor.$$

The resulting inequality coincides with (6.5) under the change of variables (6.23). □

We remark that if $K = 4$, case $Kf(d(\mathcal{C})) = 2$ of the above proof fails, as in this case the right-hand side of inequality (6.33) is $4b^-(\mathcal{C}) + 2d(\mathcal{C})$. Since this number is now an integer, the rounding is not possible and we obtain an inequality which is weaker than (6.5).

In fact the result of the above lemma is best possible, as shown below.

Lemma 6.10 *For any $K \geq 4$ there exists a polyhedron of the type (6.24)–(6.29) with $n = 3$ having Chvátal rank greater than one.*

Proof. Consider the following dual network set:

$$-x_1 + x_2 \geq 1/K, \tag{6.34}$$
$$-x_1 + x_3 \geq 3/K, \tag{6.35}$$
$$x_2 \geq 0, \tag{6.36}$$
$$x_3 \text{ integer}. \tag{6.37}$$

Applying the change of variables (6.23), the pure integer reformulation of the type (6.24)–(6.30) is the following:

$$-y_1 + y_2 \geq 1, \tag{6.38}$$
$$-y_1 + Ky_3 \geq 3, \tag{6.39}$$
$$y_2 \geq 0, \tag{6.40}$$
$$y_1, y_2 \text{ integer}. \tag{6.41}$$

Define the graph $\mathcal{G}$ as explained in Section 6.1 and let $\mathcal{C}$ be the undirected cycle in $\mathcal{G}$ formed by the sequence of arcs $(0,1), (2,1), (2,0)$. The corresponding valid inequality (6.5) for (6.34)–(6.37) is $-x_1 + x_2 + \frac{2}{K}x_3 \geq \frac{3}{K}$, which in the y variables reads

$$-y_1 + y_2 + 2y_3 \geq 3. \tag{6.42}$$

We prove that this inequality is not a Chvátal-Gomory cutting plane for the polyhedron (6.38)–(6.40).

Any Chvátal-Gomory inequality for (6.38)–(6.40) is obtained by combining (6.38)–(6.40) with nonnegative coefficients and then rounding up the right hand side:

$$u(-y_1 + y_2) + v(-y_1 + Ky_3) + wy_2 \geq \lceil u + 3v \rceil,$$

where $u, v, w \geq 0$. Then (6.42) is a Chvátal-Gomory inequality if and only if the optimum value of the following linear program is greater than 2:

$$\max \quad u + 3v \tag{6.43}$$

$$\text{subject to} \quad -u - v = -1, \tag{6.44}$$

$$u + w = 1, \tag{6.45}$$

$$Kv = 2, \tag{6.46}$$

$$u, v, w \geq 0. \tag{6.47}$$

However conditions (6.46) and $K \geq 4$ imply $v \leq 1/2$. By (6.44) the objective function is then $u + 3v = 2v + 1 \leq 2$ and thus inequality (6.42) cannot be obtained via Chvátal-Gomory rounding. □

We can summarize the results of this section as follows:

Theorem 6.11 *The Chvátal rank of the polyhedron* (6.24)–(6.30) *is one if* $K \in \{2, 3\}$, *while it is (in general) greater than one for* $K \geq 4$.

If $K = 1$, the Chvátal rank of (6.24)–(6.30) is clearly equal to zero (i.e. the polyhedron is integral), as the constraint matrix is totally unimodular and the right-hand side is an integral vector.

Chapter 7

Extension to simple non dual network sets

In Chapters 2–5 we presented, discussed and demonstrated a technique to construct extended formulations for mixed-integer sets with dual network constraint matrix. Such a technique is based on the explicit enumeration of all the fractional parts taken by the continuous variables in the vertices of the convex hull of the set. It is natural to wonder whether this approach can be extended to other kinds of mixed-integer sets.

In this chapter we consider two examples of a mixed-integer set whose constraint matrix has a simple structure but is not totally unimodular (in fact, it is not even a $0, \pm 1$-matrix). Both sets are special cases of the following quite natural generalization of the mixing set (see Section 4.2):

$$s + C_i z_i \geq b_i, \quad 1 \leq i \leq n, \tag{7.1}$$
$$s \geq 0, \tag{7.2}$$
$$z_i \text{ integer}, \qquad 1 \leq i \leq n, \tag{7.3}$$

where $b_i, C_i \in \mathbb{R}$ for $1 \leq i \leq n$. Clearly the mixing set is the above set with $C_i = 1$ for all indices $1 \leq i \leq n$.

The motivation for the study of the above set is the same as that described for the mixing set in Section 4.2. In particular, the presence of more general coefficients C_i allows one to model lot-sizing problems with non-constant capacities (for this reason these coefficients are also called *capacities*). However, the above set is also interesting in its own right, as constraints (7.1) have a very simple form and thus a deep understanding of such a set would probably be useful to tackle more complicated mixed-integer sets. Unfortunately, it is still unknown whether linear optimization over a general set of the type (7.1)–(7.3) can be carried out in polynomial time.

In the next sections we show how the approach described in the previous chapters can be extended and how this yields extended formulations for the two sets that are analyzed here. However, we will point out that the success in finding such formulations relies upon the fact that each integer variable z_i appear in a single constraint (7.1).

In Section 7.1 we consider the set (7.1)–(7.3) where the capacities C_i satisfy a divisibility assumption, while in Section 7.2 we study the case of only two distinct capacities. For the former set the size of the extended formulation is polynomial in the size of the original description of the set, while for the latter we can only obtain a pseudo-polynomial description.

The results of Section 7.1 are joint work with Michele Conforti and Laurence A. Wolsey and are also summarized in [14].

7.1 The mixing set with divisible capacities

The *mixing set with divisible capacities* is a set of the type (7.1)–(7.3) where the coefficients (capacities) C_i for $1 \leq i \leq n$ can be ordered in such a way that they form a sequence of divisible numbers. Here we also allow arbitrary lower and upper bounds on the continuous variable. If we group together constraints of type (7.1) associated with the same value of C_i, the mixing set with divisible capacities can be described as follows:

$$s + C_k z_i \geq b_i, \quad i \in I_k,\ 0 \leq k \leq m, \tag{7.4}$$

$$b_l \leq s \leq b_u, \tag{7.5}$$

$$z_i \text{ integer}, \quad i \in I_0 \cup \cdots \cup I_m, \tag{7.6}$$

where C_k/C_{k-1} is an integer greater than one for $1 \leq i \leq m$ and $I_j \cap I_k = \varnothing$ for $j \neq k$. We assume that $l, u \notin I_0 \cup \cdots \cup I_m$ and all numbers C_k, b_i are rational. We denote by DIV the above mixed-integer set.

The assumption of divisibility of the coefficients was exploited by several authors to tackle integer sets that are otherwise untractable, such as integer knapsack problems. Under the divisibility assumption, Marcotte [42] gave a simple formulation of the integer knapsack set without upper bounds on the variables. Pochet and Wolsey [54] studied the same set where the knapsack inequality is of the "$\geq$" type. They gave both a formulation of the set in its original space (consisting of an exponential number of inequalities) and a compact formulation in an extended space. Pochet and Weismantel [51] provided a linear inequality description of the knapsack set where all variables are bounded. Other hard problems studied under the assumption of divisibility of the coefficients include network design [52] and lot-sizing problems [16].

The set (7.4)–(7.6) with just two distinct capacities (i.e. $m = 2$) and without upper bound on s was studied by Van Vyve in [63], where both a compact extended formulation and a linear inequality description of the set in its original space were given. The set DIV with general m and without upper bound on s was treated recently by Zhao and de Farias [72], who characterized the extreme points and extreme rays of the set and provided an $\mathcal{O}(n^4)$ algorithm for optimizing a rational linear function (such a running time can be improved to $\mathcal{O}(n^3)$ [20]). However, they did not give a linear inequality formulation of the set either in the original space or in an extended space.

We give here an extended formulation of the polyhedron conv(DIV) whose size is polynomial in the size of the original description (7.4)–(7.6). In Section 7.1.1 we introduce an expansion of a real number x:

$$x = \alpha_0(x) + \sum_{j=1}^{m+1} \alpha_j(x) C_{j-1},$$

where $0 \le \alpha_j(x) < \frac{C_j}{C_{j-1}}$ for $1 \le j \le m$, and $0 \le \alpha_0(x) < C_0$. Furthermore $\alpha_j(x)$ is an integer for $1 \le j \le m+1$. We show in Section 7.1.3 that for fixed j, the number of possible values that $\alpha_j(s)$ can take over the set of vertices of conv(DIV) is bounded by a linear function of the number of constraints (7.4). This property allows us to associate a binary variable with each of these possible values. These binary variables are the important additional variables of our compact extended formulation, which is construct in Sections 7.1.3–7.1.6. In contrast to Van Vyve's result [63] for the case $m = 2$, our formulation defines an integral polyhedron in the extended space. In Section 7.1.7 we briefly discuss how to formulate the polyhedron conv(DIV) when there are lower bounds on the integer variables. Finally, in Section 7.1.8 we point out some unsatisfactory aspects of our result.

7.1.1 Expansion of a number

The technique that we use here generalizes that adopted in Chapter 2 for mixed-integer sets with dual network constraint matrix. In that chapter, the continuous variables were decomposed into an integer part plus a fractional part. Here the presence of several distinct coefficients in constraints (7.4) leads us to iterate a decomposition of that type. This requires the introduction of some notation.

Our arguments are based on the following expansion of a real number x:

$$x = \alpha_0(x) + \sum_{j=1}^{m+1} \alpha_j(x) C_{j-1}, \tag{7.7}$$

where $0 \le \alpha_j(x) < \frac{C_j}{C_{j-1}}$ for $1 \le j \le m$, and $0 \le \alpha_0(x) < C_0$. Furthermore $\alpha_j(x)$ is an integer for $1 \le j \le m+1$ (this is not required for $\alpha_0(x)$). Note that the above expansion is unique. If we define

$$f_0(x) := \alpha_0(x), \quad f_k(x) := f_0(x) + \sum_{j=1}^{k} \alpha_j(x) C_{j-1} \text{ for } 1 \le k \le m, \tag{7.8}$$

we have that

$$x = f_k(x) + \sum_{j=k+1}^{m+1} \alpha_j(x) C_{j-1} \text{ for } 0 \le k \le m. \tag{7.9}$$

Therefore $f_k(x)$ is the remainder of the division of x by C_k and it can be checked that

$$\alpha_k(x) = \left\lfloor \frac{f_k(x)}{C_{k-1}} \right\rfloor = \frac{f_k(x) - f_{k-1}(x)}{C_{k-1}} \text{ for } 1 \le k \le m, \;\; \alpha_{m+1}(x) = \left\lfloor \frac{x}{C_m} \right\rfloor = \frac{x - f_m(x)}{C_m}.$$

We also define $\Delta_k(x)$ as the quotient of the division of x by C_k. That is,

$$\Delta_k(x) = \left\lfloor \frac{x}{C_k} \right\rfloor = \frac{x - f_k(x)}{C_k} = \sum_{j=k+1}^{m+1} \frac{C_{j-1}}{C_k} \alpha_j(x) \text{ for } 0 \le k \le m. \tag{7.10}$$

We remark that the above expression yields the following expansion of x:

$$x = C_k\Delta_k + f_k(x) \text{ for } 0 \le k \le m. \tag{7.11}$$

Note that if $C_k = 1$ then (7.11) is precisely the decomposition of a real number into an integer part plus a fractional part.

It is also useful to introduce the following notation: for $0 \le k \le m$, we define $J_k := I_k \cup I_{k+1} \cup \cdots \cup I_m \cup \{l, u\}$.

7.1.2 Assumptions on the upper bound

In this section we make some convenient assumptions on the value of b_u. As we now explain, this can be done without loss of generality.

If for any $\gamma \in \mathbb{R}$ we apply the mixed-integer linear mapping (see Section 4.1)

$$s' := s + \gamma, \;\; z'_i := z_i \text{ for } i \in I_0 \cup \cdots \cup I_m,$$

the mixed-integer set (7.4)–(7.6) becomes

$$s' + C_k z'_i \ge b'_i, \quad i \in I_k, \, 0 \le k \le m, \tag{7.12}$$
$$b'_l \le s' \le b'_u, \tag{7.13}$$
$$z_i \text{ integer}, \quad i \in I_0 \cup \cdots \cup I_m, \tag{7.14}$$

where $b'_i := b_i + \gamma$ for all $i \in J_0$. Since the above set is of the same type as (7.4)–(7.6), without loss of generality we can study the set (7.12)–(7.14) for a specific value of γ. We now choose a value of γ which will allow us to construct an extended formulation of the convex hull of the above set.

Lemma 7.1 *Define the set of indices $T := \{i \in J_0 \setminus \{u\} : \alpha_0(b_i) > \alpha_0(b_u)\}$ and the value*

$$\alpha^* := \begin{cases} \min_{i \in T} \alpha_0(b_i) & \text{if } T \neq \varnothing, \\ C_0 & \text{if } T = \varnothing. \end{cases}$$

If one sets $\gamma^* := C_0 - \alpha^*$, then $\alpha_0(b_u + \gamma^*) = \max_{i \in J_0} \alpha_0(b_i + \gamma^*)$.

Proof. First of all note that since $\alpha^* > \alpha_0(b_u)$, then $\alpha_0(b_u) + \gamma^* < C_0$. Thus $\alpha_0(b_u + \gamma^*) = \alpha_0(b_u) + \gamma^*$. Let i be any index in J_0. If $\alpha_0(b_i) + \gamma^* \ge C_0$ then

$$\alpha_0(b_i + \gamma^*) = \alpha_0(b_i) + \gamma^* - C_0 < \gamma^* \le \alpha_0(b_u) + \gamma^* = \alpha_0(b_u + \gamma^*).$$

We then assume $\alpha_0(b_i) + \gamma^* < C_0$, which is equivalent to $\alpha_0(b_i) < \alpha^*$. Then by definition of α^* we have $\alpha_0(b_i) \le \alpha_0(b_u)$, thus

$$\alpha_0(b_i + \gamma^*) = \alpha_0(b_i) + \gamma^* \le \alpha_0(b_u) + \gamma^* = \alpha_0(b_u + \gamma^*).$$

This concludes the proof of the lemma. $\square$

We choose γ to be any number such that $\alpha_0(b_u+\gamma)=\gamma^*$ and $\alpha_k(b_u+\gamma)=\frac{C_k}{C_{k-1}}-1$ for $1\le k\le m-1$. Note that condition $\alpha_0(b_u+\gamma)=\gamma^*$ and Lemma 7.1 together imply $\alpha_0(b'_u)=\max_{i\in J_0}\alpha_0(b'_i)$.

Without loss of generality, we assume directly that the above properties hold for our original set (7.4)–(7.6):

$$\alpha_0(b_u)=\max_{i\in J_0}\alpha_0(b_i),\;\;\alpha_k(b_u)=\frac{C_k}{C_{k-1}}-1 \text{ for } 1\le k\le m-1. \tag{7.15}$$

The above assumption, which will be useful in modeling the upper bound $s\le b_u$, will be discussed in Section 7.1.8.

7.1.3 Properties of the vertices

The technique described in Chapter 2 is based on the explicit enumeration of all the possible fractional parts taken by the continuous variables at a vertex of the convex hull of the set under consideration. More information is now needed to find an extended formulation of $\operatorname{conv}(DIV)$: in particular, for all $0\le k\le m$ we need to list all the possible values $\alpha_k(s)$ for the vertices (s,z) of $\operatorname{conv}(DIV)$.

This section described properties of the vertices of $\operatorname{conv}(DIV)$ that will be used to construct the extended formulation. The assumption described in Section 7.1.2 is not needed here.

Given a real number s and an index $0\le k\le m$, for $i\in J_m\setminus\{u\}$ we define

$$b_{i,k}(s)=\begin{cases} b_i+C_k & \text{if } f_k(b_i)>f_k(s),\\ b_i & \text{if } f_k(b_i)\le f_k(s),\end{cases}$$

while we set

$$b_{u,k}(s)=\begin{cases} b_u & \text{if } f_k(b_u)\ge f_k(s),\\ b_u-C_k & \text{if } f_k(b_u)< f_k(s).\end{cases}$$

We will see that the discrepancy in the above definitions reflects the fact that all constraints (7.4)–(7.5) are of the type "$\ge$", except $s\le b_u$.

Lemma 7.2 *Consider two indices $0\le k\le \ell$. Then for $i\in I_\ell$ the inequality*

$$\Delta_k(s)+\frac{C_\ell}{C_k}z_i\ge \Delta_k(b_{i,k}(s)) \tag{7.16}$$

is valid for (7.4)–(7.6) and implies inequality $s+C_\ell z_i\ge b_i$.

Proof. Expanding s and b_i as in (7.11), inequality $s+C_\ell z_i\ge b_i$ can be rewritten as

$$\Delta_k(s)+\frac{C_\ell}{C_k}z_i\ge \Delta_k(b_i)+\frac{f_k(b_i)-f_k(s)}{C_k}.$$

Since $\ell\ge k$, the left-hand side of the above inequality is an integer. Therefore the following inequality is valid for (7.4)–(7.6):

$$\Delta_k(s)+\frac{C_\ell}{C_k}z_i\ge \Delta_k(b_i)+\left\lceil\frac{f_k(b_i)-f_k(s)}{C_k}\right\rceil=\Delta_k(b_{i,k}(s)).$$

This also shows that inequality (7.16) implies the original inequality $s+C_\ell z_i\ge b_i$. $\square$

A similar argument can be used to prove the following lemma:

Lemma 7.3 *Consider an index $k \geq 0$. Then the inequalities*

$$\Delta_k(b_{l,k}(s)) \leq \Delta_k(s) \leq \Delta_k(b_{u,k}(s)) \tag{7.17}$$

are valid for (7.4)–(7.6) and imply inequalities $b_l \leq s \leq b_u$.

Proof. For the lower bound, the proof is essentially identical to that of Lemma 7.2. As to the upper bound, it is sufficient to make obvious changes to the above proof. □

Note that inequalities (7.16) and (7.17) involve the term $b_{i,k}(s)$ and thus are not linear inequalities. We will show in Section 7.1.4 how to linearize these constraints, using the fact that for fixed k, there are only two possible values for $b_{i,k}(s)$.

Lemma 7.4 *Let $(\bar{s}, \bar{z})$ be a point in* conv(DIV).

(i) *Given indices $1 \leq k \leq \ell$ and $i \in I_\ell$, if $\alpha_k(\bar{s}) \neq \alpha_k(b_{i,k-1}(\bar{s}))$ then $\bar{s} + C_\ell \bar{z}_i \geq b_i + C_{k-1}$.*

(ii) *Given an index $k \geq 1$, if $\alpha_k(\bar{s}) \neq \alpha_k(b_{l,k-1}(\bar{s}))$ then $\bar{s} \geq b_l + C_{k-1}$, and if $\alpha_k(\bar{s}) \neq \alpha_k(b_{u,k-1}(\bar{s}))$ then $\bar{s} \leq b_u - C_{k-1}$.*

Proof. We prove (i). By Lemma 7.2, $(\bar{s}, \bar{z})$ satisfies inequality (7.16) for the pair of indices $k-1, \ell$, that is,

$$\Delta_{k-1}(s) + \frac{C_\ell}{C_{k-1}} z_i \geq \Delta_{k-1}(b_{i,k-1}(s)).$$

By (7.10) the above inequality can be rewritten as

$$\sum_{j=k}^{m+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j(s) + \frac{C_\ell}{C_{k-1}} z_i \geq \sum_{j=k}^{m+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j(b_{i,k-1}(s)),$$

or equivalently as

$$\sum_{j=k+1}^{m+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j(s) + \frac{C_\ell}{C_{k-1}} z_i - \sum_{j=k+1}^{m+1} \frac{C_{j-1}}{C_{k-1}} \alpha_j(b_{i,k-1}(s)) \geq \alpha_k(b_{i,k-1}(s)) - \alpha_k(s). \tag{7.18}$$

Since $\left\{ \frac{C_{j-1}}{C_{k-1}} : k+1 \leq j \leq m+1 \right\}$ is a sequence of divisible integers and since $\ell \geq k$, the left-hand side of the above inequality is an integer multiple of C_k/C_{k-1}. Since the right-hand side is an integer satisfying $-C_k/C_{k-1} < \alpha_k(b_{i,k-1}(s)) - \alpha_k(s) < C_k/C_{k-1}$, this shows that if $\alpha_k(\bar{s}) \neq \alpha_k(b_{i,k-1}(\bar{s}))$, then inequality (7.18) cannot be tight for $(\bar{s}, \bar{z})$, thus

$$\Delta_{k-1}(\bar{s}) + \frac{C_\ell}{C_{k-1}} \bar{z}_i \geq \Delta_{k-1}(b_{i,k-1}(\bar{s})) + 1.$$

Since $b_{i,k-1}(\bar{s}) = b_i + C_{k-1}$ if $f_{k-1}(b_i) > f_{k-1}(\bar{s})$ and $b_{i,k-1}(\bar{s}) = b_i$ if $f_{k-1}(b_i) \leq f_{k-1}(\bar{s})$, this shows that in both cases

$$\frac{f_{k-1}(\bar{s})}{C_{k-1}} + \Delta_{k-1}(\bar{s}) + \frac{C_\ell}{C_{k-1}} \bar{z}_i \geq \Delta_{k-1}(b_i) + \frac{f_{k-1}(b_i)}{C_{k-1}} + 1.$$

Multiplying the above inequality by C_{k-1} gives $\bar{s} + C_\ell \bar{z}_i \geq b_i + C_{k-1}$.

The proof of (ii) is similar. □

The following result gives us the list of all possible values $\alpha_k(s)$ taken at the vertices of $\mathrm{conv}(DIV)$.

Lemma 7.5 *If* $(\bar{s}, \bar{z})$ *is a vertex of* $\mathrm{conv}(DIV)$*, then the following properties hold:*

(i) $\alpha_0(\bar{s}) = \alpha_0(b_i)$ *for some* $i \in J_0$.

(ii) *For* $1 \le k \le m$, $\alpha_k(\bar{s}) = \alpha_k(b_{i,k-1}(\bar{s}))$ *for some* $i \in J_k$.

Proof. Let $(\bar{s}, \bar{z})$ be a vertex of $\mathrm{conv}(DIV)$. Since $\bar{z}$ is an integral vector, if (i) is violated then there exists a number $\varepsilon \neq 0$ such that $(\bar{s} \pm \varepsilon, \bar{z}) \in DIV$, a contradiction.

Assume that (ii) is violated, that is, there exists an index $1 \le k \le m$ such that $\alpha_k(\bar{s}) \neq \alpha_k(b_{i,k-1}(\bar{s}))$ for all $i \in J_k$. By Lemma 7.4 we have that $b_l + C_{k-1} \le \bar{s} \le b_u - C_{k-1}$ and $\bar{s} + C_\ell \bar{z}_i \ge b_i + C_{k-1}$ for all $i \in I_\ell$ with $\ell \ge k$. Consider the vector v whose components are defined as follows:

$$s = -C_{k-1}; \;\; z_i = \frac{C_{k-1}}{C_\ell} \text{ for } i \in I_\ell \text{ with } \ell \le k-1; \;\; z_i = 0 \text{ for } i \in I_\ell \text{ with } \ell \ge k.$$

Since both points $(\bar{s}, \bar{z}) \pm v$ belong to DIV, $(\bar{s}, \bar{z})$ is not a vertex of $\mathrm{conv}(DIV)$. □

We now introduce extra variables to model the possible values taken by s at a vertex of $\mathrm{conv}(DIV)$. The new variables are the following:

- Δ_0, $w_{0,i}$ for $i \in J_0$;
- Δ_k, $w^{\downarrow}_{k,i}, w^{\uparrow}_{k,i}$ for $1 \le k \le m$ and $i \in J_k$.

The role of the above variables is as follows:

- Variables Δ_k for $1 \le k \le m$ represent the quotients of the division of s by C_k. That is, $\Delta_k = \Delta_k(s)$ as defined in (7.10).
- Variables $w_{0,i}$ for $i \in J_0$ are binary variables. Exactly one of them is equal to 1: condition $w_{0,i} = 1$ indicates that $\alpha_0(s) = \alpha_0(b_i)$.
- For fixed $1 \le k \le m$, variables $w^{\downarrow}_{k,i}, w^{\uparrow}_{k,i}$ for $i \in J_k$ are binary variables. Exactly one of them is equal to one:
  (a) for $i \in J_k \setminus \{u\}$, condition $w^{\downarrow}_{k,i} = 1$ indicates that $\alpha_k(s) = \alpha_k(b_i)$, while condition $w^{\uparrow}_{k,i} = 1$ indicates that $\alpha_k(s) = \alpha_k(b_i + C_{k-1})$;
  (b) for $i = u$, condition $w^{\downarrow}_{k,u} = 1$ indicates that $\alpha_k(s) = \alpha_k(b_u - C_{k-1})$, while condition $w^{\uparrow}_{k,u} = 1$ indicates that $\alpha_k(s) = \alpha_k(b_u)$

In order to write the upcoming constraints in a compact form, we introduce the following simple notation: for $1 \le k \le m$ and $i \in J_k$, we define

$$b_{i,k} = \begin{cases} b_i & \text{if } i \neq u, \\ b_i - C_k & \text{if } i = u. \end{cases}$$

This definition allows us to unify (a) and (b) (see above) into the following:

(a)–(b) for all $i \in J_k$, condition $w^{\downarrow}_{k,i} = 1$ indicates that $\alpha_k(s) = \alpha_k(b_{i,k-1})$, while condition $w^{\uparrow}_{k,i} = 1$ indicates that $\alpha_k(s) = \alpha_k(b_{i,k-1} + C_{k-1})$.

Now consider the following conditions:

$$s = C_0\Delta_0 + \sum_{t\in J_0} \alpha_0(b_t)w_{0,t}, \tag{7.19}$$

$$\Delta_{k-1} = \frac{C_k}{C_{k-1}}\Delta_k + \sum_{t\in J_k} \left(\alpha_k(b_{t,k-1})w^{\downarrow}_{k,t} + \alpha_k(b_{t,k-1} + C_{k-1})w^{\uparrow}_{k,t}\right), \quad 1 \le k \le m, \tag{7.20}$$

$$w_{0,t} \ge 0,\ t \in J_0; \quad \sum_{t\in J_0} w_{0,t} = 1, \tag{7.21}$$

$$w^{\downarrow}_{k,t}, w^{\uparrow}_{k,t} \ge 0,\ t \in J_k; \quad \sum_{t\in J_k} (w^{\downarrow}_{k,t} + w^{\uparrow}_{k,t}) = 1, \quad 1 \le k \le m, \tag{7.22}$$

$$\sum_{\substack{t\in J_0:\\ \alpha_0(b_t)\ge\alpha_0(b_i)}} w_{0,t} \ge w^{\downarrow}_{1,i}, \quad i \in J_1, \tag{7.23}$$

$$\sum_{\substack{t\in J_k:\\ f_k(b_{t,k-1})\ge f_k(b_{i,k})}} w^{\downarrow}_{k,t} + \sum_{\substack{t\in J_k:\\ \alpha_k(b_{t,k-1}+C_{k-1})\ge\alpha_k(b_{i,k})+1}} w^{\uparrow}_{k,t} \ge w^{\downarrow}_{k+1,i}, \quad i \in J_{k+1},\ 1 \le k < m, \tag{7.24}$$

$$\Delta_k, w_{0,t}, w^{\downarrow}_{k,t}, w^{\uparrow}_{k,t} \text{ integer}, \quad t \in J_k,\ 0 \le k \le m. \tag{7.25}$$

Lemma 7.6 *Every vertex $(\bar{s}, \bar{z})$ of* conv(DIV) *can be completed to a vector $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ satisfying conditions (7.19)–(7.25).*

Proof. Given a vertex $(\bar{s}, \bar{z})$ of conv(DIV), let t_0 be any index in J_0 such that $\alpha_0(b_{t_0}) = \alpha_0(\bar{s})$ (t_0 exists by Lemma 7.5 (i)). Take $\bar{w}_{0,t_0} := 1$ and $\bar{w}_{0,t} := 0$ for $t \ne t_0$.

Now fix $k \ge 1$ and define

$$T_k(\bar{s}) := \{t \in J_k : \alpha_k(\bar{s}) = \alpha_k(b_{t,k-1}),\ f_{k-1}(\bar{s}) \ge f_{k-1}(b_{t,k-1})\}.$$

If $T_k(\bar{s}) \ne \varnothing$ then define t_k as any element in $T_k(\bar{s})$ such that $f_{k-1}(b_{t_k,k-1})$ is maximum and take $\bar{w}^{\downarrow}_{k,t_k} := 1$. Otherwise ($T_k(\bar{s}) = \varnothing$) define t_k as any index in J_k such that $\alpha_k(\bar{s}) = \alpha_k(b_{t_k,k-1} + C_{k-1})$ (t_k exists by Lemma 7.5 (ii)) and take $\bar{w}^{\uparrow}_{k,t_k} := 1$.

Finally take $\bar{\Delta}_k := \Delta_k(\bar{s})$ for $0 \le k \le m$.

We prove that the point thus constructed satisfies conditions (7.19)–(7.25). To see that (7.19) is satisfied, note that

$$C_0\bar{\Delta}_0 + \sum_{t\in J_0} \alpha_0(b_t)\bar{w}_{0,t} = C_0\Delta_0(\bar{s}) + \alpha_0(b_{t_0}) = C_0\Delta_0(\bar{s}) + f_0(b_{t_0}) = \bar{s}.$$

To prove the validity of (7.20), note that the following chain of equations holds:

$$\frac{C_k}{C_{k-1}}\bar{\Delta}_k + \sum_{t\in J_k} \left(\alpha_k(b_{t,k-1})\bar{w}^{\downarrow}_{k,t} + \alpha_k(b_{t,k-1} + C_{k-1})\bar{w}^{\uparrow}_{k,t}\right)$$
$$= \frac{C_k}{C_{k-1}}\Delta_k(\bar{s}) + \alpha_k(\bar{s}) = \Delta_{k-1}(\bar{s}) = \bar{\Delta}_{k-1}.$$

To see that (7.23) is verified, suppose that $\bar{w}^{\downarrow}_{1,i} = 1$ for the index $i \in J_1$. Then necessarily $i = t_1 \in T_1(\bar{s})$ and thus $f_0(\bar{s}) \geq f_0(b_{i,0}) = f_0(b_i)$, that is, $\alpha_0(\bar{s}) \geq \alpha_0(b_i)$. Then $\alpha_0(b_{t_0}) = \alpha_0(\bar{s}) \geq \alpha_0(b_i)$ and (7.23) is satisfied.

We now consider (7.24) for $k \geq 1$. Suppose that $w^{\downarrow}_{k+1,i} = 1$ for the index $i \in J_{k+1}$. Then necessarily $i = t_{k+1} \in T_{k+1}(\bar{s})$. Therefore $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_{i,k})$ and $f_k(\bar{s}) \geq f_k(b_{i,k})$. This implies $\alpha_k(\bar{s}) \geq \alpha_k(b_{i,k})$. We distinguish two cases.

1. Assume $\alpha_k(\bar{s}) \geq \alpha_k(b_{i,k}) + 1$. If $T_k(\bar{s}) \neq \varnothing$ then $\bar{w}^{\downarrow}_{k,t} = 1$ for an index $t \in J_k$ such that $\alpha_k(b_{t,k-1}) = \alpha_k(\bar{s}) \geq \alpha_k(b_{i,k}) + 1$ and thus $f_k(b_{t,k-1}) \geq f_k(b_{i,k})$. If $T_k(\bar{s}) = \varnothing$ then $\bar{w}^{\uparrow}_{k,t} = 1$ for an index $t \in J_k$ such that $\alpha_k(b_{t,k-1} + C_{k-1}) = \alpha_k(\bar{s}) \geq \alpha_k(b_{i,k}) + 1$. In both cases (7.24) is satisfied.

2. Now assume $\alpha_k(\bar{s}) = \alpha_k(b_{i,k})$. Then inequality $f_k(\bar{s}) \geq f_k(b_{i,k})$ implies $f_{k-1}(\bar{s}) \geq f_{k-1}(b_{i,k})$, thus $i \in T_k(\bar{s}) \neq \varnothing$. Then the choice of t_k shows that $\alpha_k(b_{t_k,k-1}) = \alpha_k(\bar{s}) = \alpha_k(b_{i,k})$ and $f_{k-1}(b_{t_k,k-1}) \geq f_{k-1}(b_{i,k})$, thus $f_k(b_{t_k,k-1}) \geq f_k(b_{i,k})$ and (7.24) is satisfied.

Constraints (7.21)–(7.22) and (7.25) are clearly satisfied. □

We say that $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is a *standard completion* of the vertex $(\bar{s}, \bar{z})$ of $\operatorname{conv}(DIV)$ if $\bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}$ are chosen as in the above proof. Then the above proof shows that every vertex of $\operatorname{conv}(DIV)$ has a standard completion satisfying (7.19)–(7.25).

Note that the final part of the proof of Lemma 7.6 also shows the following:

Lemma 7.7 *Fix $0 \leq k \leq m$ and $i \in I_k$. If $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is a standard completion of the vertex $(\bar{s}, \bar{z})$ of $\operatorname{conv}(DIV)$, where $f_k(\bar{s}) \geq f_k(b_i)$, then*

$$\sum_{\substack{t \in J_0: \\ \alpha_0(b_t) \geq \alpha_0(b_i)}} \bar{w}_{0,t} = 1 \qquad \text{if } k = 0, \tag{7.26}$$

$$\sum_{\substack{t \in J_k: \\ f_k(b_{t,k-1}) \geq f_k(b_{i,k})}} \bar{w}^{\downarrow}_{k,t} + \sum_{\substack{t \in J_k: \\ \alpha_k(b_{t,k-1} + C_{k-1}) \geq \alpha_k(b_{i,k}) + 1}} \bar{w}^{\uparrow}_{k,t} = 1 \quad \text{if } k \geq 1. \tag{7.27}$$

7.1.4 Linearizing the constraints

As already observed, constraints (7.16) and (7.17) are not linear inequalities. We show here how they can be linearized. For this purpose we need to prove a result which is stronger than the inverse of Lemma 7.7, as it holds not only for standard completions, but for all other vectors too.

Lemma 7.8 *Fix $0 \leq k \leq m$ and $i \in I_k$. If a point $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ satisfies conditions (7.19)–(7.25) along with equation (7.26) if $k = 0$ or (7.27) if $k \geq 1$, then $f_k(\bar{s}) \geq f_k(b_i)$.*

Proof. Assume that $k = 0$ and equation (7.26) is satisfied. If $t \in J_0$ is the index such that $\bar{w}_{0,t} = 1$ then, by (7.19) and (7.26), $f_0(\bar{s}) = \alpha_0(b_t) \geq \alpha_0(b_i) = f_0(b_i)$.

By induction, we now assume that the result holds for an index $0 \leq k < m$. We have to prove that if

$$\sum_{\substack{t\in J_{k+1}:\\ f_{k+1}(b_{t,k})\geq f_{k+1}(b_{i,k+1})}} \bar{w}^{\downarrow}_{k+1,t} + \sum_{\substack{t\in J_{k+1}:\\ \alpha_{k+1}(b_{t,k}+C_k)\geq \alpha_{k+1}(b_{i,k+1})+1}} \bar{w}^{\uparrow}_{k+1,t} = 1, \tag{7.28}$$

then $f_{k+1}(\bar{s}) \geq f_{k+1}(b_i)$.

If $\bar{w}^{\uparrow}_{k+1,t} = 1$ for some $t \in J_{k+1}$, then (7.20) and the above equation give $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_{t,k} + C_k) \geq \alpha_{k+1}(b_{i,k+1}) + 1$, thus $f_{k+1}(\bar{s}) \geq f_{k+1}(b_{i,k+1}) = f_{k+1}(b_i)$.

If $\bar{w}^{\downarrow}_{k+1,t} = 1$ for some $t \in J_{k+1}$, equation (7.28) implies that $f_{k+1}(b_{t,k}) \geq f_{k+1}(b_{i,k+1})$, thus $\alpha_{k+1}(b_{t,k}) \geq \alpha_{k+1}(b_{i,k+1})$. Assume first $\alpha_{k+1}(b_{t,k}) \geq \alpha_{k+1}(b_{i,k+1}) + 1$. Then $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_{t,k}) \geq \alpha_{k+1}(b_{i,k+1}) + 1$, thus $f_{k+1}(\bar{s}) \geq f_{k+1}(b_{i,k+1}) = f_{k+1}(b_i)$.

Finally assume that $\bar{w}^{\downarrow}_{k+1,t} = 1$ for some $t \in J_{k+1}$ such that $\alpha_{k+1}(b_{t,k}) = \alpha_{k+1}(b_{i,k+1})$. Since (7.28) implies $f_{k+1}(b_{t,k}) \geq f_{k+1}(b_{i,k+1})$, we then have $f_k(b_{t,k}) \geq f_k(b_{i,k+1})$. Inequality (7.24) for the index t implies that

$$\sum_{\substack{j\in J_k:\\ f_k(b_{j,k-1})\geq f_k(b_{t,k})}} \bar{w}^{\downarrow}_{k,j} + \sum_{\substack{j\in J_k:\\ \alpha_k(b_{j,k-1}+C_{k-1})\geq \alpha_k(b_{t,k})+1}} \bar{w}^{\uparrow}_{k,j} = 1.$$

Then, by induction, $f_k(\bar{s}) \geq f_k(b_t)$, which can also be written as $f_k(\bar{s}) \geq f_k(b_{t,k})$. This, together with inequality $f_k(b_{t,k}) \geq f_k(b_{i,k+1})$ proven above, shows that $f_k(\bar{s}) \geq f_k(b_{i,k+1})$. Using $\alpha_{k+1}(\bar{s}) = \alpha_{k+1}(b_{t,k}) = \alpha_{k+1}(b_{i,k+1})$, we conclude that $f_{k+1}(\bar{s}) \geq f_{k+1}(b_{i,k+1}) = f_{k+1}(b_i)$. □

The following result gives a linear version of inequality (7.16).

Lemma 7.9 *For $0 \leq k \leq m$ and $i \in I_k$, the following properties hold:*

(i) *Every vertex $(\bar{s}, \bar{z})$ of $\operatorname{conv}(DIV)$ can be completed to a point $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ that satisfies conditions (7.19)–(7.25) along with the linear inequality*

$$\Delta_0 + \sum_{\substack{t\in J_0:\\ \alpha_0(b_t)\geq \alpha_0(b_i)}} w_{0,t} + z_i \geq \left\lfloor \frac{b_i}{C_0} \right\rfloor + 1 \quad \text{if } k = 0, \tag{7.29}$$

$$\Delta_k + \sum_{\substack{t\in J_k:\\ f_k(b_{t,k-1})\geq f_k(b_{i,k})}} w^{\downarrow}_{k,t} + \sum_{\substack{t\in J_k:\\ \alpha_k(b_{t,k-1}+C_{k-1})\geq \alpha_k(b_{i,k})+1}} w^{\uparrow}_{k,t} + z_i \geq \left\lfloor \frac{b_i}{C_k} \right\rfloor + 1 \quad \text{if } k \geq 1. \tag{7.30}$$

(ii) *If a point $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ satisfies conditions (7.19)–(7.25) and inequality (7.29) if $k = 0$ or (7.30) if $k \geq 1$, then $\bar{s} + C_k\bar{z}_i \geq b_i$.*

Proof. (i) Let $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ be a standard completion of the vertex $(\bar{s}, \bar{z})$ of $\operatorname{conv}(DIV)$. By Lemma 7.2 (with $\ell = k$), $(\bar{s}, \bar{z})$ satisfies inequality

$$\Delta_k(s) + z_i \geq \begin{cases} \Delta_k(b_i) + 1 & \text{if } f_k(\bar{s}) < f_k(b_i), \\ \Delta_k(b_i) & \text{if } f_k(\bar{s}) \geq f_k(b_i). \end{cases} \tag{7.31}$$

After recalling that $\bar{\Delta}_k = \Delta_k(\bar{s})$ and $\Delta_k(b_i) = \lfloor b_i/C_k \rfloor$, the result follows form Lemma 7.7.

(ii) Note that for every point $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ satisfying (7.19)–(7.25), equation $\bar{\Delta} = \Delta(\bar{s})$ holds. Then, by Lemma 7.2 (with $\ell = k$), it is sufficient to prove that (7.31) is satisfied. This follows from Lemma 7.8. □

In the above proof we used Lemma 7.2 with $\ell = k$. In fact, the same lemma could be used to find different (but similar) linear versions of inequality (7.16). However, the choice $\ell = k$ is preferable as it leads to inequalities (7.29)–(7.30), which have a coefficient of 1 in variable z_i. This property will be crucial in the proof of Theorem 7.15 —the main result of this section.

We now show how to model the lower bound on s. As before, we present a linear inequality whose form will allow us to prove the main result of the section. Such a linear inequality involves variables $\Delta_m, w^{\downarrow}_{i,m}, w^{\uparrow}_{i,m}$ for $i \in J_m$. However, for each k the same technique allows one to write a similar inequality that uses variables $\Delta_k, w^{\downarrow}_{i,k}, w^{\uparrow}_{i,k}$ for $i \in J_k$.

Lemma 7.10 *The following properties hold:*

(i) *Every vertex* $(\bar{s}, \bar{z})$ *of* conv(DIV) *can be completed to a point* $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ *that satisfies conditions* (7.19)–(7.25) *along with the linear inequality*

$$\Delta_m + \sum_{\substack{t \in J_m: \\ f_m(b_{t,m-1}) \geq f_m(b_{i,m})}} w^{\downarrow}_{m,t} + \sum_{\substack{t \in J_m: \\ \alpha_m(b_{t,m-1}+C_{m-1}) \geq \alpha_m(b_{i,m})+1}} w^{\uparrow}_{m,t} \geq \left\lfloor \frac{b_l}{C_m} \right\rfloor + 1. \tag{7.32}$$

(ii) *If a point* $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ *satisfies conditions* (7.19)–(7.25) *and inequality* (7.32), *then* $\bar{s} \geq b_l$.

Proof. The proof is similar to that of Lemma 7.9 (Lemma 7.3 with $k = m$ is needed). □

We now turn to the upper bound constraint $s \leq b_u$. We would like to model this inequality in a way that is similar to what we did above. Without any specific assumptions on the value of b_u, the only simple way to do this seems to be the following (the proof is similar to that of the above lemma):

$$\Delta_0 + \sum_{\substack{i \in J_0: \\ \alpha_0(b_i) > \alpha_0(b_u)}} w_{0,i} \leq \left\lfloor \frac{b_u}{C_0} \right\rfloor. \tag{7.33}$$

However, such an inequality would not allow us to prove the main result of the section. We will reconsider this aspect in Section 7.1.8.

The non-restrictive assumption on the upper bound b_u made in Section 7.1.2 allows us to model the upper bound on s in a more convenient way.

Lemma 7.11 *The following properties hold:*

(i) *Every vertex* $(\bar{s}, \bar{z})$ *of* conv(DIV) *can be completed to a point* $\left(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow}\right)$ *that satisfies conditions* (7.19)–(7.25) *along with the linear inequality*

$$\Delta_m \leq \left\lfloor \frac{b_u}{C_m} \right\rfloor. \tag{7.34}$$

(ii) *If a point* $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ *satisfies conditions* (7.19)–(7.25) *and inequality* (7.34), *then* $\bar{s} \leq b_u$.

Proof. (i) Consider any completion $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ of $(\bar{s}, \bar{z})$ satisfying conditions (7.19)–(7.25). Inequality $s \leq b_u$ implies $\bar{\Delta}_m = \Delta_m(\bar{s}) \leq \Delta_m(b_u) = \lfloor b_u/C_m \rfloor$.

(ii) Assume that $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ satisfies conditions (7.19)–(7.25) along with inequality (7.34). By Lemma 7.3, it is sufficient to show that

$$\Delta_m(\bar{s}) \leq \begin{cases} \Delta_m(b_u) & \text{if } f_m(\bar{s}) \leq f_m(b_u), \\ \Delta_m(b_u) - 1 & \text{if } f_m(\bar{s}) > f_m(b_u). \end{cases}$$

By assumption (7.15), $\alpha_k(\bar{s}) \leq \alpha_k(b_u)$ for all $0 \leq k \leq m-1$, thus $f_m(\bar{s}) \leq f_m(b_u)$ by equation (7.8) (with $k = m$). The result now follows by (7.34). □

The same result holds if inequality (7.34) is replaced by $\Delta_k \leq \lfloor b_k/C_k \rfloor$ for any k, but the above is the most convenient form.

Let X be the mixed-integer set in the space of the variables $(s, z, \Delta, w, w^{\downarrow}, w^{\uparrow})$ defined by the following conditions:

- (7.19)–(7.25),
- (7.29) for $i \in J_0$,
- (7.30) for $i \in J_k$ with $k \geq 1$,
- (7.32) and (7.34).

Proposition 7.12 *The polyhedron* $\text{conv}(DIV)$ *is the projection of the polyhedron* $\text{conv}(X)$ *onto the space of the variables* (s, z).

Proof. Parts (ii) of Lemmas 7.9–7.11 show that $\text{proj}_{(s,z)}(X) \subseteq DIV$, thus $\text{proj}_{(s,z)}(\text{conv}(X)) \subseteq \text{conv}(DIV)$. Furthermore, parts (i) of the same lemmas show that every vertex of $\text{conv}(DIV)$ belongs to $\text{proj}_{(s,z)}(X)$. To conclude, we only need to prove that every extreme ray of $\text{conv}(DIV)$ is a ray of $\text{conv}(X)$.

Recall that since the values C_k and b_i are all rational numbers, by Theorem 1.8 the rays of $\text{conv}(X)$ are precisely the rays of the linear relaxation of X (that is, the polyhedron defined by inequalities (7.19)–(7.24), (7.29)–(7.30), (7.32) and (7.34)). It is easily checked that the extreme rays of $\text{conv}(DIV)$ are the vectors defined by setting $z_i := 1$ for some $i \in J_0 \setminus \{l, u\}$ and all other variables to zero. Each of these vectors can be completed to a feasible ray of $\text{conv}(X)$ by setting all other variables to zero. □

By the above proposition, in order to give an extended formulation of $\text{conv}(DIV)$ we have to find a linear inequality description of $\text{conv}(X)$.

7.1.5 Strengthening the constraints

Lemma 7.13 *The following inequalities are valid for X and dominate (7.23)–(7.24):*

$$\sum_{\substack{t\in J_0:\\ \alpha_0(b_t)\geq\alpha_0(b_i)}} w_{0,t} \geq \sum_{\substack{t\in J_1:\\ f_0(b_t)\geq f_0(b_i)}} w^{\downarrow}_{1,t}, \qquad i \in J_1, \tag{7.35}$$

$$\sum_{\substack{t\in J_k:\\ f_k(b_{t,k-1})\geq f_k(b_{i,k})}} w^{\downarrow}_{k,t} + \sum_{\substack{t\in J_k:\\ \alpha_k(b_{t,k-1}+C_{k-1})\geq\alpha_k(b_{i,k})+1}} w^{\uparrow}_{k,t} \geq \sum_{\substack{t\in J_{k+1}:\\ f_k(b_{t,k-1})\geq f_k(b_{i,k})}} w^{\downarrow}_{k+1,t}, \quad i \in J_{k+1},\ 1\leq k<m. \tag{7.36}$$

Proof. Fix $i \in J_{k+1}$ for $k \geq 1$ and define $L := \{t \in J_{k+1} : f_k(b_{t,k-1}) \geq f_k(b_{i,k})\}$. Inequality (7.36) can be derived by applying the Chvátal-Gomory procedure (Theorem 1.10) to the following $|L| + 1$ inequalities, which are all valid for X:

$$\sum_{\substack{t\in J_k:\\ f_k(b_{t,k-1})\geq f_k(b_{\ell,k})}} w^{\downarrow}_{k,t} + \sum_{\substack{t\in J_k:\\ \alpha_k(b_{t,k-1}+C_{k-1})\geq\alpha_k(b_{\ell,k})+1}} w^{\uparrow}_{k,t} \geq w^{\downarrow}_{k+1,\ell}, \quad \ell \in L, \tag{7.37}$$

$$1 \geq \sum_{\ell\in L} w^{\downarrow}_{k+1,\ell}, \tag{7.38}$$

with multipliers $1/|L|$ for each of inequalities (7.37) and $1 - 1/|L|$ for inequality (7.38).

The derivation of inequalities (7.35) is similar. □

7.1.6 The extended formulation

Let P be the polyhedron in the space of the variables $(s, z, \Delta, w, w^{\downarrow}, w^{\uparrow})$ defined by the following linear equations and inequalities:

- (7.19)–(7.22),
- (7.29) for $i \in J_0$,
- (7.30) for $i \in J_k$ with $k \geq 1$,
- (7.32) and (7.34),
- (7.35)–(7.36).

We denote by $Ax \sim b$ the linear system comprising the above equations and inequalities.

Lemma 7.14 *Let M be the submatrix of A indexed by the columns corresponding to variables $\Delta_m, w, w^{\downarrow}, w^{\uparrow}$ and the rows corresponding to constraints (7.21)–(7.22), (7.32) and (7.35)–(7.36). The matrix M is totally unimodular.*

Proof. We use the characterization of Ghouila-Houri [26] described in Section 1.3.2. We partition the rows of M into the submatrices $M_0, \ldots, M_m$ defined as follows:

- M_0 consists of the rows corresponding to equation (7.21) and inequalities (7.35) for $i \in J_1$;
- for $1 \leq k \leq m-1$, M_k consists of the rows corresponding to equation (7.22) and inequalities (7.36) for $i \in J_{k+1}$;
- M_m consists of the rows corresponding to equation (7.22) for $k = m$ and inequality (7.32).

For each odd k, we multiply by -1 the rows of M that belongs to M_k and the columns of M corresponding to variables $w^{\downarrow}_{k,i}, w^{\uparrow}_{k,i}$ for all $i \in J_k$. Then M becomes a 0-1 matrix.

For $1 \leq k \leq m-1$, we order the rows of M_k as follows: first equation (7.22), then inequalities (7.36) according to a non-decreasing order of the values $f_k(b_{i,k})$. The order of the rows of M_0 is analogous. The two rows of M_m are order as follows: first equation (7.22) and then inequality (7.32). Note that in every matrix M_k the support of any row, say the j-th row, contains that of the $(j+1)$-th row (in other words, the rows of M_k form a laminar family).

We can now give an equitable bicoloring of the rows of M: for k even (resp. odd), we give alternating colors to the rows of M_k starting with red (resp. blue). Since every submatrix of M has the same structure as M itself, this proves that every submatrix of M admits an equitable bicoloring of its rows and thus, by Theorem 1.14, M is totally unimodular. □

Theorem 7.15 *If $\bar{x} = (\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is a vertex of P, then $(\bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is an integral vector.*

Proof. Note that the columns of A corresponding to variables s and z_i for $i \in I_k$, $0 \leq k \leq m$, are unit columns (as s only appears in equation (7.19) and each variable z_i only appears in one of (7.29)–(7.30)).

Also note that in the subsystem of $Ax \sim b$ comprising inequalities (7.20)–(7.22), (7.32), (7.34) and (7.35)–(7.36) (i.e. with (7.19) and (7.29)–(7.30) removed) variables $\Delta_0, \ldots, \Delta_{m-1}$ appear with nonzero coefficient only in equations (7.20). Furthermore the submatrix of A indexed by the rows corresponding to (7.20) and the columns corresponding to variables $\Delta_0, \ldots, \Delta_{m-1}$ is an upper triangular matrix with 1 on the diagonal.

Let $Cx = d$ be a nonsingular subsystem of tight inequalities taken in $Ax \sim b$ that defines a vertex $\bar{x} = (\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ of P. The above observations show that (7.19)–(7.20) and (7.29)–(7.30) must be present in this subsystem. Furthermore let C' be the submatrix of C indexed by the columns corresponding to variables $\Delta_m, w, w^{\downarrow}, w^{\uparrow}$ and the rows that do not correspond to (7.19)–(7.20) or (7.29)–(7.30). Then the computation of a determinant with Laplace expansion shows that $|\det(C)| = |\det(C')| \neq 0$.

Since C' is a submatrix of the matrix M defined in Lemma 7.14 and C' is nonsingular, then $|\det(C)| = |\det(C')| = 1$. Since all entries of A (except those corresponding to equation (7.19)) are integer and the right-hand side vector b is integral, by Cramer's rule we have that $(\bar{z}, \bar{\Delta}, \bar{w}, \bar{w}^{\downarrow}, \bar{w}^{\uparrow})$ is an integral vector. □

Note that the proof of the above theorem strongly depends on the fact that each variable z_i appears in a single inequality of the system $Ax \sim b$. Even adding nonnegativity constraints on the integer variables would create serious problems (see Sections 7.1.7–7.1.8 below).

Corollary 7.16 *The linear inequalities of the system $Ax \sim b$ defining P constitute an extended formulation of* $\operatorname{conv}(DIV)$ *with* $\mathcal{O}(mn)$ *variables and constraints, where* $n := |I_0| + \cdots + |I_m|$.

Proof. Consider the set X defined in Section 7.1.4 and let $R(X)$ be its linear relaxation. By comparing the inequalities of the system $Ax \sim b$ (defining P) with those appearing in the definition of X, and using Lemma 7.13, one sees that $X \subseteq P \subseteq R(X)$. The above theorem then implies that $P = \operatorname{conv}(X)$. By Proposition 7.12, a linear inequality description of $\operatorname{conv}(X)$ is an extended formulation of $\operatorname{conv}(DIV)$, so the result follows. □

Observation 7.17 *If we drop the lower (resp. upper) bound from constraint* (7.4)–(7.6)*, an extended formulation is given by the same inequalities as above, except that constraint* (7.32) *(resp.* (7.34)*) must be removed.*

7.1.7 Lower bounds on the integer variables

We now consider the set DIV^+, the mixing set with divisible capacities and lower bounds on the integer variables. Without loss of generality such bounds can be assumed to be all equal to zero. The set DIV^+ is described by the following conditions:

$$s + C_k z_i \geq b_i, \quad i \in I_k,\ 0 \leq k \leq m, \tag{7.39}$$

$$b_l \leq s \leq b_u, \tag{7.40}$$

$$z_i \geq 0 \text{ integer}, \quad i \in I_0 \cup \cdots \cup I_m. \tag{7.41}$$

Di Summa [20] gave a polynomial time algorithm to optimize a linear function over DIV^+. We discuss the problem of finding a compact extended formulation of the polyhedron $\operatorname{conv}(DIV^+)$.

We do not know how to incorporate the bounds $z_i \geq 0$ in a formulation of the type given in Section 7.1.6, as the standard approach requires that the system $Ax \sim b$, purged of the equations defining s and Δ_k, be defined by a totally unimodular matrix (see for instance [11, 45, 53, 63, 65], as well as Chapter 2 of this thesis). However this is not the case, as discussed in Section 7.1.8. So we use an approach based on union of polyhedra, following an idea appearing in [2, 16].

Let $\{\beta_1, \dots, \beta_q\}$ be the set of distinct values in the set $\{b_i : i \in J_0,\ b_l < b_i < b_u\}$. Assume $\beta_1 < \cdots < \beta_q$ and define $\beta_0 := b_l$ and $\beta_{q+1} := b_u$. For each $0 \leq \ell \leq q$, let $DIV(\ell)$ be the following set:

$$s + C_k z_i \geq b_i, \quad i \in I_k : b_i > \beta_\ell,\ 0 \leq k \leq m, \tag{7.42}$$

$$\beta_\ell \leq s \leq \beta_{\ell+1}, \tag{7.43}$$

$$z_i \geq 0, \quad i \in I_k : b_i \leq \beta_\ell,\ 0 \leq k \leq m, \tag{7.44}$$

$$z_i \text{ integer}, \quad i \in I_0 \cup \cdots \cup I_m. \tag{7.45}$$

Lemma 7.18 $\operatorname{conv}(DIV^+) = \operatorname{conv}\left(\bigcup_{\ell=1}^{q} DIV(\ell)\right)$.

Proof. Fix $0 \leq \ell \leq q$ and assume that $(\bar{s}, \bar{z})$ is a feasible point in $DIV(\ell)$. If $i \in I_k$ is such that $b_i \leq \beta_\ell$, then $\bar{s} + C_k \bar{z}_i \geq \beta_\ell \geq b_i$. Thus $(\bar{s}, \bar{z})$ satisfies all inequalities (7.39). If $i \in I_k$ is such that $b_i > \beta_\ell$ (thus $b_i \geq \beta_{\ell+1}$), then $C_k z_i \geq b_i - \bar{s} \geq \beta_{\ell+1} - \bar{s} \geq 0$. Thus $(\bar{s}, \bar{z})$ satisfies all nonnegativity bounds on z. This shows that $\mathrm{conv}\big(\bigcup_{\ell=1}^{q} DIV(\ell)\big) \subseteq \mathrm{conv}(DIV^+)$. The reverse inclusion is obvious. □

Proposition 7.19 *The set* $\mathrm{conv}(DIV^+)$ *admits an extended formulation with* $\mathcal{O}(m^2 n)$ *variables and constraints, where* $n := |I_0| + \cdots + |I_m|$.

Proof. Fix an index $1 \leq \ell \leq q$. Note that the variables z_i appearing in inequalities (7.44) are not used by any other inequality of the system. This means that the above set is the cartesian product $X_1 \times X_2$ of the following two sets: X_1, which is defined by the conditions

$$\begin{aligned} s + C_k z_i \geq b_i, &\quad i \in I_k : b_i > \beta_\ell,\ 0 \leq k \leq m, \\ \beta_\ell \leq s \leq \beta_{\ell+1}, & \\ z_i \text{ integer}, &\quad i \in I_k : b_i > \beta_\ell,\ 0 \leq k \leq m, \end{aligned}$$

and X_2, which is described by the conditions

$$z_i \geq 0 \text{ integer}, \quad i \in I_k : b_i \leq \beta_\ell,\ 0 \leq k \leq m. \tag{7.46}$$

Relation $DIV(\ell) = X_1 \times X_2$ easily implies $\mathrm{conv}(DIV(\ell)) = \mathrm{conv}(X_1) \times \mathrm{conv}(X_2)$. The set X_1 is a mixing set with divisible capacities (without lower bounds on the integer variables), thus it admits an extended formulation with $\mathcal{O}(mn)$ variables and constraints, where $n := |I_0| + \cdots + |I_m|$ (Corollary 7.16). The convex hull of X_2 is clearly obtained by removing the integrality requirements from (7.46). Therefore there is an extended formulation of $\mathrm{conv}(DIV(\ell))$ that uses $\mathcal{O}(mn)$ variables and constraints.

The result now follows from Lemma 7.18 and Theorem 1.3. □

7.1.8 A different approach?

We conclude our study of the mixing set with divisible capacities by discussing two unsatisfactory aspects of the formulation that we constructed.

Upper bound

The first aspect concerns the assumption on the upper bound b_u made in Section 7.1.2. Even though such an assumption can be made without loss of generality, it would be interesting to understand whether our formulation really needs it.

As already pointed out in Section 7.1.4, the upper bound $s \leq b_u$ could be model by inequality (7.33) independently of the value of b_u. It is now clear that such a choice would have prevented us from proving Theorem 7.15, as in the proof of that result we used the fact that in the matrix obtained from A by removing the rows corresponding to (7.19) and (7.29)–(7.30), the column corresponding to variable Δ_0 is a unit vector.

In fact, examples can be constructed which show that if one uses inequality (7.33) to model constraints $s \le b_u$, the resulting formulation is not tight, in the sense that it contains points $(s, z, \Delta, w, w^{\downarrow}, w^{\uparrow})$ such that $(s, z) \notin \mathrm{conv}(DIV)$. An example of this type is now sketched.

Consider the following instance of DIV:

$$\begin{aligned} s + \quad z_0 &\ge 0.5, \\ s + 10z_1 &\ge 7.8, \\ 1.4 \le s &\le 15.6, \\ z_0, z_1 &\text{ integer.} \end{aligned}$$

Our formulation in the extended space, with inequality (7.33) instead of (7.34), is:

$$s = \Delta_0 + 0.5w_{0,0} + 0.8w_{0,1} + 0.4w_{0,l} + 0.6w_{0,u}, \tag{7.47}$$

$$w_{0,0}, w_{0,1}, w_{0,l}, w_{0,u} \ge 0,\ w_{0,0} + w_{0,1} + w_{0,l} + w_{0,u} = 1, \tag{7.48}$$

$$\Delta_0 = 10\Delta_1 + 7w^{\downarrow}_{1,1} + 8w^{\uparrow}_{1,1} + 1w^{\downarrow}_{1,l} + 2w^{\uparrow}_{1,l} + 4w^{\downarrow}_{1,u} + 5w^{\uparrow}_{1,u}, \tag{7.49}$$

$$w^{\downarrow}_{1,1}, w^{\uparrow}_{1,1}, w^{\downarrow}_{1,l}, w^{\uparrow}_{1,l}, w^{\downarrow}_{1,u}, w^{\uparrow}_{1,u} \ge 0, \tag{7.50}$$

$$w^{\downarrow}_{1,1} + w^{\uparrow}_{1,1} + w^{\downarrow}_{1,l} + w^{\uparrow}_{1,l} + w^{\downarrow}_{1,u} + w^{\uparrow}_{1,u} = 1, \tag{7.51}$$

$$w_{0,1} \ge w^{\downarrow}_{1,1},\ \ w_{0,1} + w_{0,u} \ge w^{\downarrow}_{1,1} + w^{\downarrow}_{1,u}, \tag{7.52}$$

$$w_{0,0} + w_{0,1} + w_{0,l} + w_{0,u} \ge w^{\downarrow}_{1,1} + w^{\downarrow}_{1,l} + w^{\downarrow}_{1,u}, \tag{7.53}$$

$$\Delta_0 + w_{0,0} + w_{0,1} + w_{0,u} + z_0 \ge 1, \tag{7.54}$$

$$\Delta_1 + w^{\downarrow}_{1,1} + w^{\uparrow}_{1,1} + z_1 \ge 1, \tag{7.55}$$

$$\Delta_0 + w_{0,0} + w_{0,1} + w_{0,l} + w_{0,u} \ge 2, \tag{7.56}$$

$$\Delta_0 + w_{0,1} \le 15. \tag{7.57}$$

The following point is a vertex of the above polyhedron:

$$s = 15.6,\ z_0 = -15,\ z_1 = -0.7,\ \Delta_0 = 15,\ w_{0,u} = 1,\ \Delta_1 = 0.7,\ w^{\uparrow}_{1,1} = 1. \tag{7.58}$$

(Apart form nonnegativity constraints, inequalities (7.53), (7.54) and (7.56) are the only non-tight inequalities.) The corresponding point in the original (s, z)-space does not belong to $\mathrm{conv}(DIV)$, as all points in DIV such that $s = 15.6$ satisfy $z_1 \ge 0$.

In order to make the proof of Theorem 7.15 work, constraint $s \le b_u$ should be modeled without using any of the variables $\Delta_0, \dots, \Delta_{m-1}$, thus one should use Δ_m. Without any assumptions on the value of b_u, this seems to be hard. The main reason for this is that the bound $s \le b_u$ is the only constraint of the type "$\le$", whereas our formulation (in particular conditions (7.23)–(7.24)) essentially fits the inequalities of the type "$\ge$".

For the above example, we could think of two (wrong) ways to model the upper bound using Δ_1. The first way is

$$\Delta_1 + w^{\downarrow}_{1,1} + w^{\uparrow}_{1,1} \le 1. \tag{7.59}$$

However, this is too weak, as the point

$$s = 15.8,\ z_1 = -15,\ z_2 = 0,\ \Delta_0 = 15,\ w_{0,1} = 1,\ \Delta_1 = 1,\ w^{\uparrow}_{1,u} = 1$$

would be feasible even though it violates inequality $s \leq 15.6$. The other way is

$$\Delta_1 + w^{\downarrow}_{1,1} + w^{\uparrow}_{1,1} + w^{\uparrow}_{1,u} \leq 1,$$

but this cut off the feasible point (7.58).

Total unimodularity

We now turn to the second unsatisfactory aspect. One might wonder whether it is possible to generalize the technique used for the set DIV to construct an extended formulation for DIV^+ without using Balas' result on the union of polyhedra. In other word, one could try to adapt the results of Lemma 7.5 to the set DIV^+.

However, for each $i \in J_0 \setminus \{l, u\}$, such an extended formulation would contain at least two inequalities with z_i in their support: inequality (7.30) (or (7.29)) and inequality $z_i \geq 0$. It follows that the technique used to prove Theorem 7.15 cannot be used in this case, thus to prove a result similar to that of Theorem 7.15 we should first show that the constraint matrix of the extended formulation is totally unimodular (ignoring equations (7.19)–(7.20)). However, in general this is false even for the set DIV, as the example below shows.

Consider the following instance of DIV (without upper bound on s):

$$\begin{aligned}
s + \quad\ z_1 &\geq 0.1,\\
s + \ 10z_2 &\geq 6.3,\\
s + 100z_3 &\geq 81.4,\\
s + 100z_4 &\geq 48.6,\\
s \qquad\quad &\geq 0,\\
z_1, \dots, z_4 &\ \text{integer}.
\end{aligned}$$

Note that $I_0 = \{1\}$, $I_1 = \{2\}$, $I_3 = \{3, 4\}$.

Among the constraints defining the extended formulation of the convex hull of the above set, we consider the following four inequalities:

$$\begin{gathered}
w^{\downarrow}_{1,2} + w^{\uparrow}_{1,2} + w^{\downarrow}_{1,3} + w^{\uparrow}_{1,3} + w^{\downarrow}_{1,4} + w^{\uparrow}_{1,4} \geq w^{\downarrow}_{2,3} + w^{\downarrow}_{2,4},\\
w_{0,3} + w_{0,4} \geq w^{\downarrow}_{1,3} + w^{\downarrow}_{1,4},\\
w^{\downarrow}_{1,4} + w^{\uparrow}_{1,4} \geq w^{\downarrow}_{2,4},\\
\Delta_1 + w^{\downarrow}_{1,2} + w^{\uparrow}_{1,2} + w^{\downarrow}_{1,4} + w^{\uparrow}_{1,4} + z_2 \geq 1,
\end{gathered}$$

which correspond respectively to inequality (7.36) for $k = 1$ and $i = 3$, inequality (7.35) for $i = 3$, inequality (7.36) for $k = 1$ and $i = 4$, and inequality (7.30) for $k = 1$ and $i = 2$.

The submatrix of the constraint matrix of the above four inequalities, restricted to variables $w^{\downarrow}_{1,4}, w^{\downarrow}_{1,3}, w^{\downarrow}_{2,4}, w^{\uparrow}_{1,2}$, is

$$\begin{bmatrix} 1 & 1 & -1 & 1\\ -1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & 1 \end{bmatrix},$$

which is not totally unimodular as its determinant is -2.

7.2 The mixing set with two capacities

The success in finding an extended formulation of the mixing set with divisible capacities (Section 7.1) strongly depends on the divisibility assumption. The study of a general set of the type (7.1)–(7.3) seems to be a much more difficult problem: it is not known whether linear optimization over a general set of this type can be carried out in polynomial time.

In this section we consider an instance of the mixed-integer set (7.1)–(7.3) where there are only two distinct capacities C_1, C_2. We denote such a set by $2CAP$:

$$\begin{aligned} s + C_1 z_i &\geq b_i, \quad i \in I_1, \\ s + C_2 z_i &\geq b_i, \quad i \in I_2, \\ s &\geq 0, \\ z_i \text{ integer}&, \quad i \in I_1 \cup I_2, \end{aligned}$$

where $0 < C_1 < C_2$, $I_1 \cap I_2 = \varnothing$. We assume without loss of generality that C_1 and C_2 are coprime integer numbers. We set $b_l := 0$, where $l \notin I_1 \cup I_2$, and define $C_0 := 1$.

We give an extended formulation of $\text{conv}(2CAP)$ with $\mathcal{O}(nC_1)$ variables and constraints, where $n := |I_1| + |I_2|$. Note that the formulation in non-compact, as its size depends on the value of C_1. However the size is independent of C_2, thus the formulation is compact whenever the value of the smallest coefficient C_1 is not "too large".

The formulation is obtained by adapting the technique used in the divisible case (in fact, if $C_1 = 1$ the two formulations coincide). However, complications will soon arise.

7.2.1 Notation

We first introduce some notation.

Given a real number x and an index $0 \leq k \leq 2$, we denote by $\Delta_k(x)$ and $f_k(x)$ respectively the quotient and the remainder of the division of x by C_k. Thus $x = C_k\Delta_k(x) + f_k(x)$. Similarly we define $\Delta_{12}(x)$ and $f_{12}(x)$ respectively as the quotient and the remainder of the division of x by C_1C_2.

Finally we set $J_1 := I_1 \cup I_2 \cup \{l\}$ and $J_2 := I_2 \cup \{l\}$.

7.2.2 Properties of the vertices

Lemma 7.20 *If* $(\bar{s}, \bar{z})$ *is a vertex of* $\text{conv}(2CAP)$ *then the following conditions hold:*

(i) $f_0(\bar{s}) = f_0(b_i)$ *for some* $i \in J_1$*.*

(ii) $f_2(C_1\Delta_1(\bar{s})) = f_2(\lfloor b_i \rfloor + \ell)$ *for some* $i \in J_2$ *and some integer* ℓ *such that:*

  (a) *either* $1 \leq \ell \leq C_1$*,*

  (b) *or* $-C_1 + 1 \leq \ell \leq 0$ *and* $f_1(\bar{s}) \geq -\ell + f_0(b_i)$*.*

Proof. If (i) is violated then, since $\bar{z}$ is an integral vector, there exists $\varepsilon \neq 0$ such that $(\bar{s} \pm \varepsilon, \bar{z}) \in 2CAP$, a contradiction.

To prove (ii) we first observe that there exists an index $i \in J_2$ such that $b_i \leq \bar{s} + C_2\bar{z}_i < b_i + C_1$: if not, after defining a vector v by setting

$$s := -C_1, \ \ z_i := 1 \text{ for } i \in I_1, \ \ z_i := 0 \text{ for } i \in I_2,$$

we would have that $(\bar{s}, \bar{z}) \pm v \in 2CAP$, a contradiction.

So we let $i \in J_2$ be such that $b_i \leq \bar{s} + C_2\bar{z}_i < b_i + C_1$. Then

$$f_2(\lfloor \bar{s} \rfloor) \in \{f_2(\lfloor b_i \rfloor), f_2(\lfloor b_i \rfloor + 1), \ldots, f_2(\lfloor b_i \rfloor + C_1)\}. \tag{7.60}$$

Since $C_1\Delta_1(\bar{s}) = \bar{s} - f_1(\bar{s}) = \lfloor \bar{s} \rfloor - f_1(\lfloor \bar{s} \rfloor)$ and $0 \leq f_1(\lfloor \bar{s} \rfloor) \leq C_1 - 1$, it follows by (7.60) that $f_2(C_1\Delta_1(\bar{s})) = f_2(\lfloor b_i \rfloor + \ell)$ for some integer ℓ such that $-C_1 + 1 \leq \ell \leq C_1$.

If $\ell \geq 1$ then (a) holds, so we assume $-C_1 + 1 \leq \ell \leq 0$. Suppose first that $\ell \leq C_1 - C_2$ and define $\ell' := C_2 + \ell$. Then $f_2(\lfloor b_i \rfloor + \ell) = f_2(\lfloor b_i \rfloor + \ell')$ and $1 \leq \ell' \leq C_1$, thus (a) holds with ℓ' in place of ℓ. So from now on we assume $C_1 - C_2 + 1 \leq \ell \leq 0$.

We now distinguish some cases.

1. If $f_2(\bar{s}) \geq f_2(b_i)$ and $f_2(\lfloor b_i \rfloor) + \ell \geq 0$, then $f_2(C_1\Delta_1(\bar{s})) = f_2(\lfloor b_i \rfloor + \ell) = f_2(\lfloor b_i \rfloor) + \ell$ and

$$\begin{aligned} f_1(\bar{s}) = f_2(f_1(\bar{s})) = f_2(\bar{s} - C_1\Delta_1(\bar{s})) &\geq f_2(\bar{s}) - f_2(C_1\Delta_1(\bar{s})) \\ &\geq f_2(b_i) - f_2(\lfloor b_i \rfloor) - \ell = f_0(b_i) - \ell, \end{aligned}$$

thus (a) holds.

2. Now assume $f_2(\bar{s}) \geq f_2(b_i)$ and $f_2(\lfloor b_i \rfloor) + \ell < 0$. Then $f_2(\lfloor b_i \rfloor + \ell) = f_2(\lfloor b_i \rfloor) + \ell + C_2$ and

$$f_2(\bar{s}) \leq f_2(b_i) + C_1 \leq f_2(b_i) + \ell + C_2 - 1 \leq f_2(\lfloor b_i \rfloor + \ell),$$

where the first inequality follows from (7.60) and the second one holds because $C_1 - C_2 + 1 \leq \ell$. This implies that $f_2(\bar{s} - (\lfloor b_i \rfloor + \ell)) = f_2(\bar{s}) - f_2(\lfloor b_i \rfloor + \ell) + C_2$, thus

$$\begin{aligned} f_1(\bar{s}) = f_2(f_1(\bar{s})) &= f_2(\bar{s} - C_1\Delta_1(\bar{s})) = f_2(\bar{s} - (\lfloor b_i \rfloor + \ell)) \\ &= f_2(\bar{s}) - f_2(\lfloor b_i \rfloor + \ell) + C_2 \geq f_2(b_i) - f_2(\lfloor b_i \rfloor) - \ell = f_0(b_i) - \ell \end{aligned}$$

and (a) holds.

3. We now consider the case $f_2(\bar{s}) < f_2(b_i)$. In this case inequalities $b_i \leq \bar{s} + C_2\bar{z}_i < b_i + C_1$ imply $f_2(b_i) > C_2 - C_1$. Then $f_2(\lfloor b_i \rfloor) + \ell > f_2(\lfloor b_i \rfloor) + C_1 - C_2 > 0$. This implies $f_2(\lfloor b_i \rfloor + \ell) = f_2(\lfloor b_i \rfloor) + \ell$. Furthermore,

$$f_2(\bar{s}) \leq f_2(b_i) + C_1 - C_2 \leq f_2(b_i) + \ell - 1 \leq f_2(\lfloor b_i \rfloor + \ell),$$

where the first inequality follows from $f_2(\bar{s}) < f_2(b_i)$ and (7.60). This implies

$$\begin{aligned} f_1(\bar{s}) = f_2(f_1(\bar{s})) &= f_2(\bar{s} - C_1\Delta_1(\bar{s})) = f_2(\bar{s} - (\lfloor b_i \rfloor + \ell)) \\ &= f_2(\bar{s}) - f_2(\lfloor b_i \rfloor + \ell) + C_2 \geq 0 - f_2(\lfloor b_i \rfloor) - \ell + C_2 \geq -\ell + 1 \end{aligned}$$

and (a) holds.

This concluses the proof of the lemma. □

For $i \in J_2$ and $-C_1+1 \le \ell \le C_1$, we define c_i^ℓ to be the unique integer number such that

$$0 \le c_i^\ell < C_1C_2, \ \ f_2(c_i^\ell) = f_2(\lfloor b_i \rfloor + \ell), \ \ f_1(c_i^\ell) = 0.$$

Existence and uniqueness of such a number follow from the Chinese remainder theorem (see e.g. [56] or any basic algebra book).

Remark 7.21 *Let i, ℓ be two indices as in part* (ii) *of Lemma 7.20. Then $f_{12}(C_1\Delta_1(\bar{s})) = c_i^\ell$, as the integer number $f_{12}(C_1\Delta_1(\bar{s}))$ satisfies the three conditions that define c_ℓ^i.*

We now introduce extra variables to model the possible values taken by s at a vertex of $\mathrm{conv}(2CAP)$. The new variables are the following:

- Δ, w_i^ℓ for $i \in J_1$ and $0 \le \ell \le C_1 - 1$;
- Γ, π_i^ℓ for $i \in J_2$ and $-C_1 + 1 \le \ell \le C_1$.

The role of the above variables is as follows:

- Variable Δ represents the quotient of the division of s by C_1. That is, $\Delta = \Delta(s)$ as defined in Section 7.2.1.
- Variable Γ represents the quotient of the division of s by C_1C_2. That is, $\Gamma = \Delta_{12}(s)$ as defined in Section 7.2.1.
- Variables w_i^ℓ for $i \in J_1$ and $0 \le \ell \le C_1 - 1$ are binary variables. Exactly one of them is equal to 1: condition $w_i^\ell = 1$ indicates that $f_0(s) = f_0(b_i)$ and $f_1(\lfloor s \rfloor) = \ell$, i.e. $f_1(s) = \ell + f_0(b_i)$.
- Variables π_i^ℓ for $i \in J_2$ and $-C_1 + 1 \le \ell \le C_1$ are binary variables. Exactly one of them is equal to one: condition $\pi_i^\ell = 1$ indicates that $f_{12}(C_1\Delta_1(\bar{s})) = c_i^\ell$.

Consider the following conditions:

$$s = C_1\Delta + \sum_{\ell=0}^{C_1-1} \sum_{t \in J_1} (\ell + f_0(b_t)) w_t^\ell, \tag{7.61}$$

$$C_1\Delta = C_1C_2\Gamma + \sum_{\ell=-C_1+1}^{C_1} \sum_{t \in J_2} c_t^\ell \pi_t^\ell, \tag{7.62}$$

$$w_t^\ell \ge 0,\ t \in J_1,\ 0 \le \ell \le C_1 - 1;\ \sum_{\ell=0}^{C_1-1} \sum_{t \in J_1} w_t^\ell = 1, \tag{7.63}$$

$$\pi_t^\ell \ge 0,\ t \in J_2,\ -C_1 + 1 \le \ell \le C_1;\ \sum_{\ell=-C_1+1}^{C_1} \sum_{t \in J_2} \pi_t^\ell = 1, \tag{7.64}$$

$$\sum_{j=-\ell+1}^{C_1-1} \sum_{t \in J_1} w_t^j + \sum_{\substack{t \in J_1: \\ f_0(b_t) \ge f_0(b_i)}} w_t^{-\ell} \ge \pi_i^\ell,\ \ i \in J_2,\ -C_1 + 1 \le \ell \le 0, \tag{7.65}$$

$$\Delta, w_t^\ell, \Gamma, \pi_t^\ell \text{ integer}. \tag{7.66}$$

Lemma 7.22 *Every vertex* $(\bar{s}, \bar{z})$ *of* $\mathrm{conv}(2CAP)$ *can be completed to a vector* $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ *satisfying* (7.61)–(7.66).

Proof. Lemma 7.20 and Remark 7.21 show that $f_{12}(C_1\Delta_1(\bar{s})) = c_i^\ell$ for some $i \in J_2$ and $-C_1 + 1 \le \ell \le C_1$. The vertex $(\bar{s}, \bar{z})$ can be completed as follows.

If there exist an index $i \in J_2$ and an integer $-C_1+1 \le \ell \le 0$, such that $f_{12}(C_1\Delta_1(\bar{s})) = c_i^\ell$ and $f_1(\bar{s}) \ge -\ell + f_0(b_i)$, then we set $\bar{\pi}_i^\ell = 1$. For convenience, if such a choice of ℓ is not unique, we choose ℓ as small as possible. If, after this, the choice of i is not unique, we choose i so that $f_0(b_i)$ is as large as possible. (Further ties can be broken arbitrarily.)

Otherwise there exist an index $i \in J_2$ and an integer $1 \le \ell \le C_1$ such that $f_{12}(C_1\Delta_1(\bar{s})) = c_i^\ell$, and we set $\bar{\pi}_i^\ell = 1$ for any such choice of i and ℓ.

By Lemma 7.20, there exist $t \in J_1$ and $0 \le h \le C_1 - 1$ such that $f_1(\bar{s}) = h + f_0(b_t)$. We then set $\bar{w}_t^h = 1$ for any such choice of t and h.

Finally we set $\bar{\Delta} = \Delta_1(\bar{s})$ and $\bar{\Gamma} = \Delta_{12}(\bar{s})$.

It is easily checked that the vertex thus constructed satisfies (7.61)–(7.64) and (7.66). To see that (7.65) is satisfied, suppose $\bar{\pi}_i^\ell = 1$ for some $i \in J_2$ and $-C_1+1 \le \ell \le 0$. Lemma 7.20 (ii) then implies that $h + f_0(b_t) = f_1(\bar{s}) \ge -\ell + f_0(b_i)$, that is, either $h \ge -\ell + 1$, or $h = -\ell$ and $f_0(b_t) \ge f_0(b_i)$. In both cases the left-hand side of (7.65) is equal to 1 and the inequality is satisfied. □

We say that $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ is a *standard completion* of the vertex $(\bar{s}, \bar{z})$ of $\mathrm{conv}(2CAP)$ if $\bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi}$ are chosen as described in the above proof. Then the above proof shows that every vertex of $\mathrm{conv}(2CAP)$ has a standard completion satisfying (7.61)–(7.66).

7.2.3 Modeling the constraints

Proposition 7.23 *For* $i \in I_1$, *a point* $(\bar{s}, \bar{z})$ *satisfies inequality* $s + C_1 z_i \ge b_i$ *if and only if every completion* $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ *of* $(\bar{s}, \bar{z})$ *fulfilling conditions* (7.61)–(7.66) *also satisfies inequality*

$$\Delta + \sum_{\ell=k+1}^{C_1-1} \sum_{t \in J_1} w_t^\ell + \sum_{\substack{t \in J_1: \\ f_0(b_t) \ge f_0(b_i)}} w_t^k + z_i \ge \left\lfloor \frac{b_i}{C_1} \right\rfloor + 1, \tag{7.67}$$

where $k := f_1(\lfloor b_i \rfloor)$.

Proof. Using (7.61), inequality $s + C_1 z_i \ge b_i$ can be rewritten as

$$\Delta + \sum_{\ell=0}^{C_1-1} \sum_{t \in J_1} \frac{\ell + f_0(b_t)}{C_1} w_t^\ell + z_i \ge \frac{b_i}{C_1}. \tag{7.68}$$

Observe that $\frac{\ell + f_0(b_t)}{C_1} \ge f_0\left(\frac{b_i}{C_1}\right)$ if and only if $\ell + f_0(b_t) \ge f_1(b_i)$, that is, if and only if either $\ell \ge f_1(\lfloor b_i \rfloor) + 1$, or $\ell = f_1(\lfloor b_i \rfloor)$ and $f_0(b_t) \ge f_0(b_i)$. Inequality (7.67) can then be obtained by summing inequalities (7.68) and

$$-(f_0(b_i/C_1) - \varepsilon) \sum_{\ell=0}^{C_1-1} \sum_{t \in J_1} w_t^\ell \ge -(f_0(b_i/C_1) - \varepsilon)$$

for $\varepsilon > 0$ small enough and then applying Chvátal-Gomory rounding (see Theorem 1.10). $\square$

Proposition 7.24 *For $i \in I_2$, the following properties hold:*

(i) *Every vertex $(\bar{s}, \bar{z})$ of $\operatorname{conv}(2CAP)$ can be completed to a point $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ that satisfies conditions (7.61)–(7.66) along with the linear inequality*

$$C_1\Gamma + \sum_{\ell=-C_1+1}^{C_1} \sum_{t \in J_2} \beta_t^\ell \pi_t^\ell + z_i \geq \left\lfloor \frac{b_i}{C_2} \right\rfloor + 1, \tag{7.69}$$

where β_t^ℓ is defined as follows:

$$\beta_t^\ell := \begin{cases} \Delta_2(c_t^\ell) & \text{if } f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \leq \min\{0, \ell - 1\}, \\ \Delta_2(c_t^\ell) & \text{if } \ell = f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \leq 0 \text{ and } f_0(b_t) < f_0(b_i), \\ \Delta_2(c_t^\ell) + 1 & \text{if } \ell = f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \leq 0 \text{ and } f_0(b_t) \geq f_0(b_i), \\ \Delta_2(c_t^\ell) + 1 & \text{if } \ell < f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \leq 0, \\ \Delta_2(c_t^\ell) + 1 & \text{if } 0 < f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \leq C_2 - C_1, \\ \Delta_2(c_t^\ell) + 1 & \text{if } C_2 - C_1 < f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) < \ell + C_2, \\ \Delta_2(c_t^\ell) + 1 & \text{if } C_2 - C_1 < f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) = \ell + C_2 \text{ and } f_0(b_t) < f_0(b_i), \\ \Delta_2(c_t^\ell) + 2 & \text{if } C_2 - C_1 < f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) = \ell + C_2 \text{ and } f_0(b_t) \geq f_0(b_i), \\ \Delta_2(c_t^\ell) + 2 & \text{if } f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) > \max\{C_2 - C_1, \ell + C_2\}. \end{cases}$$

(ii) *If a point $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ satisfies conditions (7.61)–(7.66) along with inequality (7.69), then $\bar{s} + C_2 z_i \geq b_i$.*

Proof. (i) We show that every standard completion of a vertex of $\operatorname{conv}(2CAP)$ satisfies inequality (7.69).

Let $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ be a standard completion of the vertex $(\bar{s}, \bar{z})$ of $\operatorname{conv}(2CAP)$ and assume $\bar{\pi}_t^\ell = 1$. By (7.61)–(7.62),

$$\bar{s} = C_1 C_2 \bar{\Gamma} + c_t^\ell + f_1(\bar{s}) = C_1 C_2 \bar{\Gamma} + C_2 \Delta_2(c_t^\ell) + f_2(c_t^\ell) + f_1(\bar{s}). \tag{7.70}$$

Assume first $\beta_t^\ell = \Delta_2(c_t^\ell) + 2$. Then, using (7.70),

$$C_1\bar{\Gamma} + \sum_{\ell=-C_1+1}^{C_1} \sum_{t \in J_2} \beta_t^\ell \bar{\pi}_t^\ell + \bar{z}_i = C_1\bar{\Gamma} + \Delta_2(c_t^\ell) + 2 + \bar{z}_i = \frac{\bar{s} + C_2\bar{z}_i - f_2(c_t^\ell) - f_1(\bar{s}) + 2C_2}{C_2} > \frac{b_i}{C_2},$$

where the last inequality holds because $\bar{s} + C_2\bar{z}_i \geq b_i$ and $f_2(c_t^\ell) + f_1(\bar{s}) < C_2 + C_1 < 2C_2$. Thus inequality (7.69) is satisfied in this case.

Now assume $\beta_t^\ell = \Delta(c_t^\ell) + 1$ and $f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \leq C_2 - C_1$. Using (7.70),

$$C_1\bar{\Gamma} + \sum_{\ell=-C_1+1}^{C_1} \sum_{t \in J_2} \beta_t^\ell \bar{\pi}_t^\ell + \bar{z}_i = C_1\bar{\Gamma} + \Delta_2(c_t^\ell) + 1 + \bar{z}_i = \frac{\bar{s} + C_2\bar{z}_i - f_2(c_t^\ell) - f_1(\bar{s}) + C_2}{C_2} > \frac{b_i}{C_2},$$

where the last inequality holds because $\bar{s}+C_2\bar{z}_i \geq b_i$ and $-f_2\big(c_t^\ell\big)-f_1(\bar{s})+C_2 \geq C_1-f_2(\lfloor b_i \rfloor)-f_1(\bar{s}) > -f_2(b_i)$. Thus inequality (7.69) is satisfied in this case.

If $\beta_t^\ell = \Delta\big(c_t^\ell\big)+1$ and $f_2\big(c_t^\ell\big)-f_2(\lfloor b_i \rfloor) > C_2-C_1$, the proof is by contradiction: we assume that (7.69) is violated, that is (after multiplying by C_2),

$$C_1C_2\bar{\Gamma} + C_2\Delta_2\big(c_t^\ell\big) + C_2 + C_2\bar{z}_i \leq C_2\Delta_2(b_i). \tag{7.71}$$

Define $k := f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor) - C_2$. Since $f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor) > C_2 - C_1$, we have $k \geq -C_1 + 1$. Furthermore, $k \leq 0$. Since $f_2\big(c_t^\ell\big) = f_2(\lfloor b_i \rfloor) + k + C_2 = f_2\big(c_i^k\big)$, we see that $c_t^\ell = c_i^k$. In the following we show that $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ is not a standard completion of the vertex $(\bar{s}, \bar{z})$, as setting $\pi_i^k = 1$ would be a preferable choice.

By (7.70), $\bar{s} = C_1C_2\bar{\Gamma}+C_2\Delta_2\big(c_t^\ell\big)+f_2(\lfloor b_i \rfloor)+k+C_2+f_1(\bar{s})$. Then inequality $\bar{s}+C_2\bar{z}_i \geq b_i$ reads

$$C_1C_2\bar{\Gamma} + C_2\Delta_2\big(c_t^\ell\big) + f_2(\lfloor b_i \rfloor) + k + C_2 + f_1(\bar{s}) + C_2\bar{z}_i \geq C_2\Delta_2(b_i) + f_2(b_i).$$

By combining the above inequality with (7.71), we derive $f_1(\bar{s}) \geq -k + f_0(b_i)$.

On the other hand, conditions $\beta_t^\ell = \Delta\big(c_t^\ell\big) + 1$ and $f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor) > C_2 - C_1$, together with the definition of β_t^ℓ, show that necessarily $k \leq \ell$, and if $k = \ell$ then $f_0(b_i) > f_0(b_t)$. This, along with inequality $f_1(\bar{s}) \geq -k + f_0(b_i)$ derived above and the fact that $c_t^\ell = c_i^k$, shows that setting $\pi_i^k = 1$ would be a preferable choice for representing the vertex $(\bar{s}, \bar{z})$.

The above shows that inequalities (7.69) holds whenever $\beta_t^\ell > \Delta_2\big(c_t^\ell\big)$. We now assume $\beta_t^\ell = \Delta_2\big(c_t^\ell\big)$ and $f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor) > -C_1$. The proof is again by contradiction: we assume that (7.69) is violated, that is, $C_1C_2\bar{\Gamma} + C_2\Delta_2\big(c_t^\ell\big) + C_2\bar{z}_i \leq C_2\Delta_2(b_i)$. In this case we define $k := f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor)$ and proceed as in the previous case (note that $-C_1 + 1 \leq k \leq 0$).

Finally, assume $\beta_t^\ell = \Delta_2\big(c_t^\ell\big)$ and $f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor) \leq -C_1$. Using (7.70),

$$C_1\bar{\Gamma} + \sum_{\ell=-C_1+1}^{C_1} \sum_{t \in J_2} \beta_t^\ell \bar{\pi}_t^\ell + \bar{z}_i = C_1\bar{\Gamma} + \Delta_2\big(c_t^\ell\big) + \bar{z}_i = \frac{\bar{s} + C_2\bar{z}_i - f_2\big(c_t^\ell\big) - f_1(\bar{s})}{C_2} > \frac{b_i}{C_2},$$

where the last inequality holds because $\bar{s}+C_2\bar{z}_i \geq b_i$ and $f_2\big(c_t^\ell\big)+f_1(\bar{s}) < (f_2(\lfloor b_i \rfloor)-C_1)+C_1 \leq f_2(b_i)$. Thus inequality (7.69) is satisfied in this case.

(ii) We now show that if $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ satisfies (7.61)–(7.66) and (7.69), then $\bar{s}+C_2\bar{z}_i \geq b_i$.

Let $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ be a point satisfying (7.61)–(7.66) and (7.69), and assume $\bar{\pi}_t^\ell = 1$ and $\bar{w}_j^h = 1$. Note that (7.70) holds and $f_1(\bar{s}) = h + f_0(b_j)$.

If $\beta_t^\ell = \Delta_2\big(c_t^\ell\big)$ then, using (7.70) and (7.69),

$$\bar{s} + C_2\bar{z}_i = C_1C_2\bar{\Gamma} + C_2\Delta_2\big(c_t^\ell\big) + f_2\big(c_t^\ell\big) + f_1(\bar{s}) + C_2\bar{z}_i \geq C_2\Delta_2(b_i + C_2) + f_2\big(c_t^\ell\big) + f_1(\bar{s}) \geq b_i.$$

Now assume $\beta_t^\ell = \Delta_2\big(c_t^\ell\big) + 1$ and $f_2\big(c_t^\ell\big) - f_2(\lfloor b_i \rfloor) > 0$. Using (7.70) and (7.69),

$$\bar{s} + C_2\bar{z}_i = C_1C_2\bar{\Gamma} + C_2\Delta_2\big(c_t^\ell\big) + f_2\big(c_t^\ell\big) + f_1(\bar{s}) + C_2\bar{z}_i \geq C_2\Delta_2(b_i) + f_2\big(c_t^\ell\big) + f_1(\bar{s}) \geq b_i,$$

where the last inequality holds because $f_2(c_t^\ell) \ge f_2(\lfloor b_i \rfloor) + 1$.

If $\beta_t^\ell = \Delta(c_t^\ell) + 1$ and $f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \le 0$, the proof is by contradiction: we assume that $\bar{s} + C_2\bar{z}_i < b_i$, that is (again using (7.70)),

$$C_1C_2\bar{\Gamma} + C_2\Delta_2(c_t^\ell) + f_2(c_t^\ell) + f_1(\bar{s}) + C_2\bar{z}_i < b_i. \tag{7.72}$$

Summing the above inequality with inequality (7.69), which can be written as

$$-C_1C_2\bar{\Gamma} - C_2\Delta_2(c_t^\ell) - C_2 - C_2\bar{z}_i \le -C_2\Delta_2(b_i + C_2),$$

gives $f_2(c_t^\ell) + f_1(\bar{s}) < f_2(b_i)$. If we define $k := f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \le 0$, the latter inequality reads $f_1(\bar{s}) < -k + f_0(b_i)$. Since $f_1(\bar{s}) = h + f_0(b_j)$, this implies

$$h + f_0(b_j) < -k + f_0(b_i). \tag{7.73}$$

On the other hand, conditions $\beta_t^\ell = \Delta(c_t^\ell) + 1$ and $f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) \le 0$, together with the definition of β_t^ℓ, show that necessarily $\ell \le k \le 0$, and if $\ell = k$ then $f_0(b_t) \ge f_0(b_i)$. Then (7.73) implies $h + f_0(b_j) < -\ell + f_0(b_i)$. This implies that either $h < -\ell$, or $h = -\ell$ and $f_0(b_j) < f_0(b_i)$. In the former case, inequality (7.65) is clearly violated for the indices $t \in J_2$ and $\ell \le 0$. So we assume $h = -\ell$ and $f_0(b_j) < f_0(b_i)$.

By (7.73), $h \le -k$. This, together with $h = -\ell \ge -k$, shows that $\ell = k$. As seen above, this implies that $f_0(b_t) \ge f_0(b_i)$, thus $f_0(b_j) < f_0(b_t)$. This shows that inequality (7.65) is again violated for the indices $t \in J_2$ and $\ell \le 0$.

The above shows that $\bar{s} + C_2\bar{z}_i \ge b_i$ whenever $\beta_t^\ell \le \Delta_2(c_t^\ell) + 1$. We now assume $\beta_t^\ell = \Delta_2(c_t^\ell) + 2$. The proof is again by contradiction: we assume that (7.72) holds. In this case inequality (7.69) reads

$$-C_1C_2\bar{\Gamma} - C_2\Delta_2(c_t^\ell) - 2C_2 - C_2\bar{z}_i \le -C_2\Delta_2(b_i + C_2),$$

which together with (7.72) gives $f_2(c_t^\ell) + f_1(\bar{s}) < f_2(b_i) + C_2$. We then define $k := f_2(c_t^\ell) - f_2(\lfloor b_i \rfloor) - C_2 \le 0$ and continue as in the previous case. □

Proposition 7.25 *A point $(\bar{s}, \bar{z})$ satisfies inequality $s + C_1z_i \ge b_i$ if and only if every extension $(\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ of $(\bar{s}, \bar{z})$ fulfilling conditions (7.61)–(7.66) also satisfies inequality*

$$\Gamma \ge 0. \tag{7.74}$$

Proof. The result is obvious. □

Let X be the mixed-integer set in the space of the variables $(s, z, \Delta, w, \Gamma, \pi)$ defined by conditions

- (7.61)–(7.66),
- (7.67) for $i \in I_1$,
- (7.69) for $i \in I_2$,

- (7.74).

Proposition 7.26 *The polyhedron* conv$(2CAP)$ *is the projection of* conv(X) *onto the space of the variables* (s, z)*.*

Proof. The proof is similar to that given in Section 7.1 for the mixing set with divisible capacities (see Proposition 7.12). The only difference is that now there is one more extreme ray, namely the vector defined by setting $s := 1$, $z_i := 1/C_1$ for $i \in I_1$ and $z_i := 1/C_2$ for $i \in I_2$. □

By the above proposition, in order to give an extended formulation of conv$(2CAP)$ we have to find a linear inequality description of conv(X).

7.2.4 Strengthening the constraints

Lemma 7.27 *The following inequalities are valid for* (7.61)–(7.66) *and dominate* (7.65)*:*

$$\sum_{j=-\ell+1}^{C_1-1} \sum_{t\in J_1} w_t^j + \sum_{\substack{t\in J_1:\\ f_0(b_t)\geq f_0(b_i)}} w_t^{-\ell} \geq \sum_{j=-C_1+1}^{\ell-1} \sum_{t\in J_2} \pi_t^j + \sum_{\substack{t\in J_2:\\ f_0(b_t)\geq f_0(b_i)}} \pi_t^\ell, \;\; i \in J_2,\; -C_1+1 \leq \ell \leq 0. \tag{7.75}$$

Proof. Fix $-C_1 + 1 \leq \ell \leq 0$ and $i \in J_2$. Define

$$L := \{(\lambda, \tau) \in \{-C_1 - 1, \ldots, 0\} \times J_2 : \text{either } \lambda \leq \ell - 1, \text{ or } \lambda = \ell - 1 \text{ and } f_0(b_\tau) \geq f_0(b_i)\}.$$

Inequality (7.75) can be derived by applying the Chvátal-Gomory procedure to the following $|L| + 1$ inequalities, which are all valid for (7.61)–(7.66):

$$\sum_{j=-\lambda+1}^{C_1-1} \sum_{t\in J_1} w_t^j + \sum_{\substack{t\in J_1:\\ f_0(b_t)\geq f_0(b_\tau)}} w_t^{-\lambda} \geq \pi_\tau^\lambda, \quad (\lambda, \tau) \in L, \tag{7.76}$$

$$1 \geq \sum_{j=-C_1+1}^{\ell-1} \sum_{t\in J_2} \pi_t^j + \sum_{\substack{t\in J_2:\\ f_0(b_t)\geq f_0(b_i)}} \pi_t^\ell, \tag{7.77}$$

with multipliers $1/|L|$ for each of inequalities (7.76) and $1 - 1/|L|$ for inequality (7.77). □

7.2.5 The extended formulation

We show here the main result of the section. The proofs are similar to those of Section 7.1.6.

Let P be the polyhedron in the space of the variables $(s, z, \Delta, w, \Gamma, \pi)$ defined by the following linear equations and inequalities:

- (7.61)–(7.64),
- (7.67) for $i \in I_1$,

- (7.69) for $i \in I_2$,
- (7.74) and (7.75).

Note that if we divide equation (7.62) by C_1, all coefficients remain integer and the coefficient of Δ becomes 1. We denote by $Ax \sim b$ the linear system comprising the above equations and inequalities, where equation (7.62) has been divided by C_1.

Lemma 7.28 *Let M be the submatrix of A indexed by the columns corresponding to variables w, π and the rows corresponding to constraints (7.63)–(7.64) and (7.75). The matrix M is totally unimodular.*

Proof. We use the characterization of Ghouila-Houri [26] described in Section 1.3.2. We order the rows corresponding to inequalities (7.75) according firstly to a decreasing order of index ℓ and secondly to a non-decreasing order of $f_0(b_i)$. Note that with such an ordering, the support of any row, say the j-th row, contains that of the $(j+1)$-th row (in other words, the rows form a laminar family).

We now give an equitable bicoloring to the rows of M: we assign color red to the rows corresponding to equations (7.63)–(7.64) and then alternate the colors starting with blue. Since every submatrix of M has the same structure as M itself, this proves that every submatrix of M admits an equitable bicoloring of its rows and thus, by Theorem 1.14, M is totally unimodular. □

Theorem 7.29 *If $\bar{x} = (\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ is a vertex of P then $(\bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ is an integral vector.*

Proof. Note that the columns of A corresponding to variables s and z_i for $i \in I_1 \cup I_2$ are unit columns (as s only appears in equation (7.61) and each variable z_i only appears in one of (7.67), (7.69)).

In the subsystem of $Ax \sim b$ comprising inequalities (7.62)–(7.64), (7.74) and (7.75) (i.e. with (7.61), (7.67) and (7.69) removed) variables Δ, Γ appear with nonzero coefficient only in equations (7.62) and (7.74). Furthermore the submatrix of A indexed by the rows corresponding to (7.62), (7.74) and the columns corresponding to variables Δ, Γ is an upper triangular matrix with 1 on the diagonal.

Let $Cx = d$ be a nonsingular subsystem of tight inequalities taken in $Ax \sim b$ that defines a vertex $\bar{x} = (\bar{s}, \bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ of P. The above observations show that (7.61)–(7.62), (7.67), (7.69) or (7.74) must be present in this subsystem. Furthermore let C' be the submatrix of C indexed by the columns corresponding to variables w, π and the rows that do not correspond to (7.61)–(7.62), (7.67), (7.69) and (7.74). Then the computation of a determinant with Laplace expansion shows that $|\det(C)| = |\det(C')| \neq 0$.

Since C' is a submatrix of the matrix M defined in Lemma 7.28 and C' is nonsingular, then $|\det(C)| = |\det(C')| = 1$. Since all entries of A (except those corresponding to equation (7.61)) are integer and the right-hand side vector b is integral, by Cramer's rule we have that $(\bar{z}, \bar{\Delta}, \bar{w}, \bar{\Gamma}, \bar{\pi})$ is an integral vector. □

Corollary 7.30 *The linear inequalities of the system $Ax \sim b$ defining P constitute an extended formulation of* $\text{conv}(2CAP)$ *with $\mathcal{O}(nC_1)$ variables and constraints, where $n := |I_1| + |I_2|$.*

Proof. The proof is identical to that of Corollary 7.16. □

The extended formulation constructed here is only pseudo-polynomial, as it depends on the value C_1. Note however that the size of the formulation is independent of the value C_2. It was recently proven by Zhao and de Farias [73] that linear optimization over the set $2CAP$ can be carried out in polynomial time, but it is not known whether there exists a compact extended formulation of $\text{conv}(2CAP)$. Also, it seems hard to extend the above construction to the case of three capacities.

We finally remark that the proofs of both Theorem 7.29 above and Theorem 7.15 in Section 7.1.6 exploit the fact that each integer variable appears in a single constraint. We do not know how to deal with a more general mixed-integer set of the form

$$\begin{aligned} s_j + C_k z_i &\geq b_{ji}, & & 1 \leq j \leq q,\ i \in I_k,\ 0 \leq k \leq m, \\ b_{l_j} \leq s_j &\leq b_{u_j}, & & 1 \leq j \leq q, \\ z_i &\text{ integer}, & & i \in I_0 \cup \cdots \cup I_m, \end{aligned}$$

where either the capacities are divisible or take few values. In the case $C_0 = \cdots = C_m = 1$ the above is a set of the type MIX^{2TU} for which an extended formulation was given by Miller and Wolsey [45] (when there are no upper bounds b_{u_j}).

Chapter 8

A different technique

The approach to construct extended formulations introduced in Chapter 2 is based on the explicit enumeration of all the fractional parts that the variables take over the vertices of the convex hull of the set. The extension discussed in Chapter 7 is based on a refinement of the same technique, due to the presence of several distinct coefficients in the constraints that define the set.

We explore here another way of constructing a formulation of a mixed-integer set either in the original space or in an extended space. No explicit enumeration of fractional parts or other numbers is required (except possibly in the final phase of the process). We adopted this technique to formulate two specific sets, but we could not determine a class of mixed-integer sets for which this approach can be used.

Both mixed-integer sets considered here have been already discussed in this thesis: one is the mixing set with flows (Sections 4.2.2 and 5.3), the other is the continuous mixing set with flows (Section 4.2.1). We observed in Chapter 4 that each of these sets is equivalent to a dual network set and therefore admits an extended formulation of the type presented in Chapter 2. We also computed the projection of the extended formulation of the mixing set with flows, thus obtaining a linear inequality description in the original space (Chapter 5).

We reconsider here the above two sets and we give formulations for them by using a common approach, which is summarized below.

We first recall a well-known fact. Let X be a mixed-integer set. Suppose that there exist a mixed-integer set Z and a polyhedron P such that $X = Z \cap P$. It is easy to see that then

$$\operatorname{conv}(X) \subseteq \operatorname{conv}(Z) \cap P, \tag{8.1}$$

but equality does not hold in general.

To describe the common approach used for the two sets, we let X denote any of the two sets.

Step 1. The first step of our process is writing $X = Z \cap P$ for some mixed-integer set Z and some polyhedron P that is described by a simple linear system.

Step 2. Next we prove that for this particular choice of Z and P, equality holds in (8.1).

Step 3. We introduce another mixed-integer set Y and prove that the polyhedra $\text{conv}(Z)$ and $\text{conv}(Y)$ are in a one-to-one correspondence via an affine transformation.

Step 4. The final step is to give a formulation of $\text{conv}(Y)$ either in the original space or in an extended space. In the former case we immediately derive a formulation of $\text{conv}(Z)$, and thus of $\text{conv}(X) = \text{conv}(Z) \cap P$, in its original space; in the latter case an extended formulation is obtained.

The crucial point is proving that equality holds in (8.1). This will be done by using a polyhedral result that we introduce in Section 8.1.

The final step is different for the two sets. In the case of the mixing set with flows (Section 8.2) we give a formulation of Y both in the original space and in an extended space, thus both kinds of description are also obtained for X —the mixing set with flows itself. For the continuous mixing set with flows (Section 8.3) only extended formulations are given.

The results of this chapter are joint work with Michele Conforti and Laurence A. Wolsey, and also appear in [13, 12].

8.1 Some equivalences of polyhedra

Step 2 of the process described above will be possible thanks to a result on the equivalence of polyhedra that we introduce here.

For a nonempty polyhedron P in $\mathbb{R}^n$ and a vector $\alpha \in \mathbb{R}^n$, define $\mu_P(\alpha) := \min\{\alpha x : x \in P\}$ and let $M_P(\alpha)$ be the face $\{x \in P : \alpha x = \mu_P(\alpha)\}$, where $M_P(\alpha) = \varnothing$ whenever $\mu_P(\alpha) = -\infty$.

Lemma 8.1 *Let $P \subseteq Q$ be two nonempty polyhedra in $\mathbb{R}^n$ and let α be a nonzero vector in $\mathbb{R}^n$. Then the following conditions are equivalent:*

(i) $\mu_P(\alpha) = \mu_Q(\alpha)$;

(ii) $M_P(\alpha) \subseteq M_Q(\alpha)$.

Proof. Suppose $\mu_P(\alpha) = \mu_Q(\alpha)$. Since $P \subseteq Q$, every point in $M_P(\alpha)$ belongs to $M_Q(\alpha)$. So if (i) holds, then (ii) holds as well. The converse is obvious. □

Lemma 8.2 *Let $P \subseteq Q$ be two nonempty polyhedra in $\mathbb{R}^n$, where P is not an affine variety. Suppose that for every inequality $\alpha x \geq \beta$ that is facet-inducing for P, at least one of the following holds:*

(i) $\mu_P(\alpha) = \mu_Q(\alpha)$;

(ii) $M_P(\alpha) \subseteq M_Q(\alpha)$.

Then $P = Q$.

Proof. We prove that if $M_P(\alpha) \subseteq M_Q(\alpha)$ for every inequality $\alpha x \geq \beta$ that is facet-inducing for P, then every facet-inducing inequality for P is a valid inequality for Q and every hyperplane containing P also contains Q. This shows that $Q \subseteq P$ and therefore $P = Q$. By Lemma 8.1, the conditions $\mu_P(\alpha) = \mu_Q(\alpha)$ and $M_P(\alpha) \subseteq M_Q(\alpha)$ are equivalent and we are done.

Let $\alpha x \geq \beta$ be a facet-inducing inequality for P. Since $M_P(\alpha) \subseteq M_Q(\alpha)$, then $\beta = \mu_P(\alpha) = \mu_Q(\alpha)$ and $\alpha x \geq \beta$ is an inequality which is valid for Q.

Now let $\gamma x = \delta$ be a hyperplane containing P. If $Q \not\subseteq \{x : \gamma x = \delta\}$, then there exists $\bar{x} \in Q$ such that $\gamma \bar{x} \neq \delta$. We assume without loss of generality $\sigma = \gamma \bar{x} - \delta > 0$. Since P is not an affine variety, there exists an inequality $\alpha x \geq \beta$ which is facet-inducing for P (and so it is valid for Q). Then, for $\lambda > 0$ the inequality $(\lambda\alpha - \gamma)x \geq \lambda\beta - \delta$ is also facet-inducing for P, so it is valid for Q. Choosing $\lambda > 0$ such that $\lambda(\alpha\bar{x} - \beta) < \sigma$ gives a contradiction, as $(\lambda\alpha - \gamma)\bar{x} = \lambda\alpha\bar{x} - \gamma\bar{x} < \lambda\beta + \sigma - \gamma\bar{x} = \lambda\beta - \delta$. □

If P is not full-dimensional, for each facet F of P there are infinitely many distinct inequalities that define F (two inequalities are distinct if their associated half-spaces are distinct: that is, if one is not a positive multiple of the other). Observe that the hypotheses of the lemma must be verified for *all* distinct facet-defining inequalities (not just one facet-defining inequality for each facet), otherwise the result is false. For instance, consider the polyhedra

$$P = \{(x, y) : 0 \leq x \leq 1,\ y = 0\} \subset Q = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$$

The hypotheses of Lemma 8.2 are satisfied for the inequalities $x \geq 0$ and $x \leq 1$, which define all the facets of P.

Also note that the assumption that P is not an affine variety cannot be removed: indeed, in such a case P does not have proper faces, so the hypotheses of the lemma are trivially satisfied, even if $P \neq Q$.

Corollary 8.3 *Let $P \subseteq Q$ be two pointed polyhedra in $\mathbb{R}^n$, with the property that every vertex of Q belongs to P. Let $Cx \geq d$ be a system of inequalities that are valid for P such that for every inequality $\gamma x \geq \delta$ of the system, $P \not\subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$. If for every $\alpha \in \mathbb{R}^n$ such that $\mu_P(\alpha)$ is finite but $\mu_Q(\alpha) = -\infty$, $Cx \geq d$ contains an inequality $\gamma x \geq \delta$ such that $M_P(\alpha) \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$, then $P = Q \cap \{x \in \mathbb{R}^n : Cx \geq d\}$.*

Proof. We first show that $\dim(P) = \dim(Q)$. If not, there exists a hyperplane $\alpha x = \beta$ containing P but not Q. Without loss of generality we can assume that $\mu_Q(\alpha) < \beta = \mu_P(\alpha)$. So $\mu_Q(\alpha) = -\infty$, otherwise there would exist an α-optimal vertex $\bar{x}$ of Q such that $\alpha\bar{x} < \beta$, contradicting the fact that $\bar{x} \in P$. Now the system $Cx \geq d$ must contain an inequality $\gamma x \geq \delta$ such that $P = M_P(\alpha) \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$, a contradiction to the hypotheses of the corollary.

Let $Q' = Q \cap \{x \in \mathbb{R}^n : Cx \geq d\}$. Note that $P \subseteq Q' \subseteq Q$, thus $\dim(P) = \dim(Q') = \dim(Q)$. Let $\alpha x \geq \beta$ be a facet-inducing inequality for P. If $\mu_Q(\alpha)$ is finite, then Q contains an α-optimal vertex which is in P and therefore $\beta = \mu_P(\alpha) = \mu_{Q'}(\alpha) = \mu_Q(\alpha)$. If $\mu_Q(\alpha) = -\infty$, the system $Cx \geq d$ contains an inequality $\gamma x \geq \delta$ such that $M_P(\alpha) \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$ and $P \not\subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$. It follows that $\gamma x \geq \delta$ is a facet-inducing inequality for P and that it defines the same facet of P as $\alpha x \geq \beta$ (that is, $M_P(\alpha) = M_P(\gamma)$). This means that there exist $\nu > 0$, a vector λ and a system $Ax = b$ which is valid for P such that $\gamma = \nu\alpha + \lambda A$ and $\delta = \nu\beta + \lambda b$. Since $\dim(P) = \dim(Q')$ and $P \subseteq Q'$, the system $Ax = b$ is valid for Q', as well. As $\gamma x \geq \delta$ is also valid for Q', it follows that $\alpha x \geq \beta$ is valid for Q' (because $\alpha = \frac{1}{\nu}\gamma - \frac{\lambda}{\nu}A$ and $\beta = \frac{1}{\nu}\delta - \frac{\lambda}{\nu}b$). Therefore $\beta = \mu_P(\alpha) = \mu_{Q'}(\alpha)$.

Thus in all cases $\mu_P(\alpha) = \mu_{Q'}(\alpha)$. Now assume that P consists of a single point and $P \neq Q$. Then Q is a cone having P as apex. Given a ray α of Q, $\mu_P(\alpha)$ is finite while $\mu_Q(\alpha) = -\infty$, so the system $Cx \geq d$ contains an inequality $\gamma x \geq \delta$ such that $P \subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$, a contradiction. So we can assume that P is not a single point and thus P is not an affine variety, as it is pointed. Now we can conclude by applying Lemma 8.2 to the polyhedra P and Q'. □

We remark that in the statement of Corollary 8.3 the condition that the two polyhedra are pointed is not necessary: if we replace the property "every vertex of Q belongs to P" with "every minimal face of Q belongs to P", the proof needs a very slight modification to remain valid. (However, in this case we should assume that P is not an affine variety, so that we can apply Lemma 8.2 in the proof.)

We also observe that the condition "for every inequality $\gamma x \geq \delta$ of the system, $P \not\subseteq \{x \in \mathbb{R}^n : \gamma x = \delta\}$" is indeed necessary. For instance, consider the polyhedra

$$P = \{(x, y) : 0 \leq x \leq 1,\ y = 0\} \subset Q = \{(x, y) : x \geq 0,\ y = 0\}$$

and the system consisting of the single inequality $y \geq 0$.

8.2 The mixing set with flows

In this section we reconsider the mixing set with flows introduced in Section 4.2.2:

$$s + y_i \geq b_i, \quad 1 \leq i \leq n, \tag{8.2}$$

$$y_i \leq z_i, \quad 1 \leq i \leq n, \tag{8.3}$$

$$s \geq 0,\ y_i \geq 0, \quad 1 \leq i \leq n, \tag{8.4}$$

$$z_i \text{ integer}, \quad 1 \leq i \leq n, \tag{8.5}$$

where we assume without loss of generality $0 \leq b_1 \leq \cdots \leq b_n$. We denote the above set by X^{MF}.

The original motivation for studying X^{MF} was to generalize the mixing set X^{MIX}

$$\begin{aligned} s + z_i &\geq b_i, \quad 1 \leq i \leq n, \\ s &\geq 0, \\ z_i &\text{ integer}, \quad 1 \leq i \leq n, \end{aligned}$$

by introducing the continuous (flow) variables x (see also Section 4.2). However the mixing set with flows is also closely related to two lot-sizing models, as explained in Sections 4.2.1–4.2.2.

A linear inequality description of the convex hull of X^{MF} in its original space was computed in Section 5.3 by projecting an extended formulation of the set. In this section we obtain a linear inequality description of $\operatorname{conv}(X^{MF})$ both in the original space and in an extended space by using the approach summarized in Steps 1–4 above.

Steps 1–2 are performed in Section 8.2.1, while Steps 3–4 are the subject of Section 8.2.2. We conclude in Section 8.2.3 by studying a mixed-integer set that is closely related to X^{MF}.

8.2.1 A relaxation

We introduce a mixed-integer set Z which is the following relaxation of the set X^{MF}:

$$
\begin{aligned}
s + z_i \geq b_i, &\quad 1 \leq i \leq n, &(8.6)\\
s + y_j + z_i \geq b_i, &\quad 1 \leq j < i \leq n, &(8.7)\\
s + y_i \geq b_i, &\quad 1 \leq i \leq n, &(8.8)\\
s \geq 0,\ z_i \geq 0, &\quad 1 \leq i \leq n, &(8.9)\\
z_i \text{ integer}, &\quad 1 \leq i \leq n. &(8.10)
\end{aligned}
$$

Note that variables y_i are *not* required to take a nonnegative value in Z.

The following easy lemma constitutes Step 1 of the process.

Lemma 8.4 *Let X^{MF} and Z be defined on the same vector b. Then $X^{MF} = Z \cap \{(s, y, z) : \mathbf{0} \leq y \leq z\}$.*

Proof. Observe that for $(s, y, z) \in X^{MF}$, $s + z_i \geq s + y_i \geq b_i$ holds, so $s + z_i \geq b_i$ is a valid inequality for X^{MF}. Also, inequalities $s + z_i \geq b_i$ and $y_j \geq 0$ imply that $s + y_j + z_i \geq b_i$ is valid for X^{MF}. Inequalities $z_i \geq 0$ follow from $y_i \geq 0$ and $y_i \leq z_i$. This proves that Z is a relaxation of X^{MF}.

The only inequalities that define X^{MF} but do not appear in the definition of Z are the inequalities $\mathbf{0} \leq y \leq z$, thus $X^{MF} = Z \cap \{(s, y, z) : \mathbf{0} \leq y \leq z\}$. □

We prove here that $\operatorname{conv}(X^{MF}) = \operatorname{conv}(Z) \cap \{(s, y, z) : \mathbf{0} \leq y \leq z\}$ (Step 2). To do this, we need to establish some properties of the polyhedra $\operatorname{conv}(X^{MF})$ and $\operatorname{conv}(Z)$. We start by characterizing their extreme rays. In the following e_j denotes the n-dimensional vector with 1 in the first component and 0 elsewhere, while $\mathbf{1}$ is the n-dimensional all-one vector.

Lemma 8.5 *The extreme rays (s, y, z) of $\operatorname{conv}(X^{MF})$ are the following $2n + 1$ vectors:*

$$(1, \mathbf{0}, \mathbf{0}),\ \ (0, \mathbf{0}, e_j) \textit{ for } 1 \leq j \leq n,\ \ (0, e_j, e_j) \textit{ for } 1 \leq j \leq n.$$

The extreme rays (s, y, z) of $\operatorname{conv}(Z)$ are the following $2n + 1$ vectors:

$$(1, -\mathbf{1}, \mathbf{0}),\ \ (0, \mathbf{0}, e_j) \textit{ for } 1 \leq j \leq n,\ \ (0, e_j, \mathbf{0}) \textit{ for } 1 \leq j \leq n.$$

Proof. Since the left-hand sides of inequalities (8.2)–(8.5) and (8.6)–(8.10) have integer coefficients, the recession cones of X^{MF} and Z coincide with the recession cones of their linear relaxations (Theorem 1.8). One can check that the extreme rays of such relaxations are those listed above. □

Corollary 8.6 *The polyhedra* $\mathrm{conv}(X^{MF})$ *and* $\mathrm{conv}(Z)$ *are full-dimensional.*

Proof. One can check that the extreme rays of $\mathrm{conv}(X^{MF})$ (resp. $\mathrm{conv}(Z)$) listed above are linearly independent. This shows that the recession cone of $\mathrm{conv}(X^{MF})$ (resp. $\mathrm{conv}(Z)$) is full-dimensional and the conclusion follows. □

The following observation is easy.

Lemma 8.7 *Let* $(\bar{s}, \bar{y}, \bar{z})$ *be a vertex of* $\mathrm{conv}(Z)$ *and let* $1 \leq j \leq n$. *Then*

$$\bar{s} = \max \left\{ \begin{array}{ll} 0, & \\ b_i - \bar{z}_i, & 1 \leq i \leq n, \\ b_i - \bar{y}_i, & 1 \leq i \leq n, \\ b_i - \bar{z}_i - \bar{y}_j, & 1 \leq j < i \leq n \end{array} \right\}, \quad \bar{y}_j = \max \left\{ \begin{array}{ll} b_j - \bar{s}, & \\ b_i - \bar{s} - \bar{z}_i, & j < i \leq n \end{array} \right\}.$$

Proof. If $\bar{s}$ is not as above then there exists $\varepsilon \neq 0$ such that both points $(\bar{s} \pm \varepsilon, \bar{y}, \bar{z})$ satisfy conditions (8.6)–(8.10), which contradict the fact that $(\bar{s}, \bar{y}, \bar{z})$ is a vertex of $\mathrm{conv}(Z)$. For $\bar{y}_j$ the proof is similar. □

The following result is crucial for proving that $\mathrm{conv}(X^{MF}) = \mathrm{conv}(Z) \cap \{(s, y, z) : \mathbf{0} \leq y \leq z\}$.

Lemma 8.8 *Let* $(\bar{s}, \bar{y}, \bar{z})$ *be a vertex of* $\mathrm{conv}(Z)$. *Then* $\mathbf{0} \leq \bar{y} \leq \bar{z}$.

Proof. Assume $\bar{y}_k < 0$ for some index k. Then $\bar{s} > 0$, otherwise, if $\bar{s} = 0$, the constraints $s + y_k \geq b_k$ and $b_k \geq 0$ would imply $\bar{y}_k \geq 0$.

We now claim that there is an index $1 \leq i \leq n$ such that $\bar{s} = b_i - \bar{z}_i$. If not, $\bar{s} > b_i - \bar{z}_i$ for $1 \leq i \leq n$ and there exists $\varepsilon \neq 0$ such that $(\bar{s}, \bar{y}, \bar{z}) \pm \varepsilon(1, -\mathbf{1}, \mathbf{0})$ belong to $\mathrm{conv}(Z)$, a contradiction.

So there is an index $1 \leq i \leq n$ such that $\bar{s} = b_i - \bar{z}_i > 0$. Since $b_i - \bar{z}_i \geq b_i - \bar{z}_i - \bar{y}_j$ for $1 \leq j < i$, this implies $\bar{y}_j \geq 0$ for $1 \leq j < i$. Lemma 8.7 also implies $b_i - \bar{z}_i \geq b_j - \bar{y}_j$ for $1 \leq j \leq n$. Together with $\bar{z}_i \geq 0$ and $b_i \leq b_j$ for $j \geq i$, this implies $\bar{y}_j \geq \bar{z}_i \geq 0$ for $j \geq i$. This completes the proof that $\bar{y} \geq \mathbf{0}$.

Now assume $\bar{y}_j > \bar{z}_j$ for some index j. Then $\bar{z}_j \geq 0$ implies $\bar{y}_j > 0$. Assume $\bar{y}_j = b_j - \bar{s}$. Then inequality $\bar{s} + \bar{z}_j \geq b_j$ implies that $\bar{y}_j \leq \bar{z}_j$, a contradiction. Therefore by Lemma 8.7, $\bar{y}_j = b_i - \bar{s} - \bar{z}_i$ for some $i > j$. Since $\bar{y}_j > 0$, then $b_i - \bar{s} - \bar{z}_i > 0$, a contradiction to $\bar{s} + \bar{z}_i \geq b_i$. This shows that $\bar{y} \leq \bar{z}$. □

We can now prove the main theorem of this subsection:

Theorem 8.9 *Let* X^{MF} *and* Z *be defined on the same vector* b. *Then* $\mathrm{conv}(X^{MF}) = \mathrm{conv}(Z) \cap \{(s, y, z) : \mathbf{0} \leq y \leq z\}$.

Proof. We prove the result by applying Corollary 8.3 to the polyhedra $\mathrm{conv}(X^{MF})$ and $\mathrm{conv}(Z)$ and the system $\mathbf{0} \le y \le z$. To do this, we show that the hypotheses of that corollary are satisfied.

By Lemma 8.4, $\mathrm{conv}(X^{MF}) \subseteq \mathrm{conv}(Z)$. By Lemmas 8.8 and 8.4, every vertex of $\mathrm{conv}(Z)$ belongs to $\mathrm{conv}(X^{MF})$.

Let $\alpha = (h, p, q)$, with $h \in \mathbb{R}$, $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, be such that $\mu_{\mathrm{conv}(X^{MF})}(\alpha)$ is finite and $\mu_{\mathrm{conv}(Z)}(\alpha) = -\infty$. Since by Lemma 8.5, the extreme rays of $\mathrm{conv}(Z)$ that are not rays of $\mathrm{conv}(X^{MF})$ are $(0, e_j, \mathbf{0})$ for $1 \le j \le n$ and $(1, -\mathbf{1}, \mathbf{0})$, then either $p_j < 0$ for some index j or $h < \sum_{i=1}^n p_i$. Also note that $h \ge 0$, as otherwise $\mu_{\mathrm{conv}(X^{MF})}(\alpha) = -\infty$ because of ray $(1, \mathbf{0}, \mathbf{0})$.

If $p_j < 0$ for some index j, then $M_{\mathrm{conv}(X^{MF})}(\alpha) \subseteq \{(s, y, z) : y_j = z_j\}$.

If $h < \sum_{i=1}^n p_i$, let $N^+ := \{i : p_i > 0\}$. We can assume that $N^+ \neq \varnothing$: if not, either there is an index j such that $p_j < 0$ (and we are in the previous case) or $p_j = 0$ for all $1 \le j \le n$, in which case we have $h < 0$, contradicting our assumption $h \ge 0$. Thus $N^+ \neq \varnothing$ and we can safely define $j := \min\{i : i \in N^+\}$. We show that $M_{\mathrm{conv}(X^{MF})}(\alpha) \subseteq \{(s, y, z) : y_j = 0\}$. Suppose that $y_j > 0$ in some optimal solution. As the solution is optimal and $p_j > 0$, we cannot just decrease the variable y_j and remain feasible. Thus $s + y_j = b_j$, hence $s < b_j$. However this implies that for all $i \in N^+$, we have $y_i \ge b_i - s > b_i - b_j \ge 0$ as $i \ge j$. Now as $y_i > 0$ for all $i \in N^+$, we can increase s by $\varepsilon > 0$ and decrease y_i by ε for all $i \in N^+$. The new point is feasible in X^{MF} and has lower objective value, a contradiction.

Therefore we have shown that for every vector α such that $\mu_{\mathrm{conv}(X^{MF})}(\alpha)$ is finite and $\mu_{\mathrm{conv}(Z)}(\alpha) = -\infty$, the system $\mathbf{0} \le y \le z$ contains an inequality which is tight for the points in $M_{\mathrm{conv}(X^{MF})}(\alpha)$. To complete the proof, note that since $\mathrm{conv}(X^{MF})$ is full-dimensional (Corollary 8.6), the system $\mathbf{0} \le y \le z$ does not contain an inequality defining an improper face of $\mathrm{conv}(X^{MF})$. So we can now apply Corollary 8.3 to the polyhedra $\mathrm{conv}(X^{MF})$ and $\mathrm{conv}(Z)$ and the system $\mathbf{0} \le y \le z$. □

8.2.2 The intersection set

We now come to Step 3 of the process described at the beginning of the chapter. In this step a new mixed-integer set Y is introduced, which in our case is the *intersection set*:[1]

$$\sigma_j + z_i \ge b_i - b_j, \quad 0 \le j < i \le n, \tag{8.11}$$

$$\sigma_j \ge 0,\ z_i \ge 0, \quad 0 \le j \le n,\ 1 \le i \le n, \tag{8.12}$$

$$z_i \text{ integer}, \quad 1 \le i \le n, \tag{8.13}$$

where $0 := b_0 \le b_1 \le \ldots \le b_n$.

Note that Y is the intersection of the following $n+1$ mixing sets X_j^{MIX} (with nonnegativity bounds on the integer variables), each one associated with a distinct variable σ_j (in the constraints below j is a fixed index in $\{0, \ldots, n\}$):

$$\begin{aligned} \sigma_j + z_i \ge b_i - b_j, &\quad j < i \le n, \\ \sigma_j \ge 0,\ z_i \ge 0, &\quad j < i \le n, \\ z_i \text{ integer}, &\quad j < i \le n. \end{aligned}$$

[1] Note that this is *not* the same set as the intersection set defined in Section 4.3.

The theorem below shows that the polyhedra $\mathrm{conv}(Z)$ and $\mathrm{conv}(Y)$ are equivalent via an affine transformation (Step 3).

Theorem 8.10 *Let Z and Y be defined on the same vector b. The affine transformation*

$$\sigma_0 := s; \;\; \sigma_i := s + y_i - b_i, \;\; z_i := z_i \text{ for } 1 \leq i \leq n \tag{8.14}$$

maps $\mathrm{conv}(Z)$ *into* $\mathrm{conv}(Y)$.

Proof. It is straightforward to check that (8.14) transforms the inequalities in (8.6)–(8.10) into the inequalities in (8.11)–(8.13). Since this transformation is a mixed-integer linear mapping (see Section 4.1) plus a translation, the result follows. □

An immediate consequence is the following:

Corollary 8.11 *Let X^{MF} and Y be defined on the same vector b. The affine transformation (8.14) maps* $\mathrm{conv}(X^{MF})$ *into*

$$\mathrm{conv}(Y) \cap \{(\sigma, z) : 0 \leq \sigma_i - \sigma_0 + b_i \leq z_i \text{ for } 1 \leq i \leq n\}.$$

Proof. The result follows from Theorems 8.9 and 8.10. □

The above corollary shows that an external description of $\mathrm{conv}(X^{MF})$ can be obtained from an external description of $\mathrm{conv}(Y)$.

Recall that Y is the intersection of $n+1$ mixing sets defined on distinct continuous variables but sharing some of the integer variables. For the mixing set, both a compact extended formulation and a linear inequality description in the original space are known: the former was first obtained by Miller and Wolsey [45], the latter by Günlük and Pochet [31]. Both formulations were illustrated in Chapter 5.

The following result of Miller and Wolsey [45] shows that the convex hull of the intersection set Y is given by the intersection of the convex hulls of the single mixing sets.

Proposition 8.12 (Miller and Wolsey [45]) *For $1 \leq j \leq m$, let X_j^{MIX} be a mixing set. Assume that each set X_j^{MIX} is defined on a distinct continuous variable σ_j, while some or all integer variables are in common. Define $X^* := \bigcap_{j=1}^m X_j^{MIX}$. Then*

$$\mathrm{conv}(X^*) = \bigcap_{j=1}^{m} \mathrm{conv}(X_j^{MIX}).$$

It follows from Corollary 8.11 and Proposition 8.12 that an external description of the polyhedron $\mathrm{conv}(X^{MF})$ in its original space can be obtained by writing the external descriptions of all the polyhedra $\mathrm{conv}(X_j^{MIX})$ together with the inequalities $0 \leq \sigma_i - \sigma_0 + b_i \leq z_i$ for $1 \leq i \leq n$ and then applying the inverse of transformation (8.14). Similarly, a compact extended formulation of $\mathrm{conv}(X^{MF})$ can be obtained by writing the extended formulations of all the polyhedra $\mathrm{conv}(X_j^{MIX})$ together with the inequalities $0 \leq \sigma_i - \sigma_0 + b_i \leq z_i$ for $1 \leq i \leq n$ and then applying the inverse of transformation (8.14). The resulting extended formulation uses $\mathcal{O}(n^2)$ variables and constraints.

8.2.3 A variant

Here for the purpose of comparison we examine the convex hull of a set closely related to X^{MF}. Such a set is the relaxation obtained by dropping the nonnegativity constraints on the flow variables y. The *unrestricted mixing set with flows* X^{UMF} is the set:

$$\begin{aligned} s + y_i &\geq b_i, && 1 \leq i \leq n, \\ y_i &\leq z_i, && 1 \leq i \leq n, \\ s &\geq 0, && \\ z_i &\text{ integer}, && 1 \leq i \leq n. \end{aligned}$$

Its convex hull turns out to be much simpler and in fact the unrestricted mixing set with flows and the mixing set are closely related.

Proposition 8.13 *For an unrestricted mixing set with flows X^{UMF} and the mixing set X^{MIX} defined on the same vector b,*

$$\operatorname{conv}\left(X^{UMF}\right) = \left\{(s, y, z) : (s, z) \in \operatorname{conv}\left(X^{MIX}\right), \, b_i - s \leq y_i \leq z_i \textit{ for } 1 \leq i \leq n\right\}.$$

Proof. Let $P := \left\{(s, y, z) : (s, z) \in \operatorname{conv}\left(X^{MIX}\right), \, b_i - s \leq y_i \leq z_i \text{ for } 1 \leq i \leq n\right\}$. The inclusion $\operatorname{conv}(X^{UMF}) \subseteq P$ is obvious. In order to show that $P \subseteq \operatorname{conv}\left(X^{UMF}\right)$, we prove that the extreme rays (resp. vertices) of P are rays (resp. feasible points) of $\operatorname{conv}\left(X^{UMF}\right)$.

The cone $\{(s, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n_+ : -s \leq y_i \leq z_i, \ 1 \leq i \leq n\}$ is the recession cone of both P and $\operatorname{conv}\left(X^{UMF}\right)$, thus P and $\operatorname{conv}\left(X^{UMF}\right)$ have the same rays.

We now prove that if $(\bar{s}, \bar{y}, \bar{z})$ is a vertex of P, then $(\bar{s}, \bar{y}, \bar{z})$ belongs to $\operatorname{conv}\left(X^{UMF}\right)$. It is sufficient to show that $\bar{z}$ is integer. We do so by proving that $(\bar{s}, \bar{z})$ is a vertex of $\operatorname{conv}\left(X^{MIX}\right)$. If not, there exists a nonzero vector $(u, w) \in \mathbb{R} \times \mathbb{R}^n$ such that $(\bar{s}, \bar{z}) \pm (u, w) \in \operatorname{conv}\left(X^{MIX}\right)$ and $w_i = -u$ whenever $\bar{z}_i = b_i - \bar{s}$. Define a vector $v \in \mathbb{R}^n$ as follows: If $\bar{y}_i = b_i - \bar{s}$, set $v_i = -u$ and if $\bar{y}_i = \bar{z}_i$, set $v_i = w_i$. (Since $\bar{y}_i$ satisfies at least one of these two equations, this assignment is indeed possible). It is now easy to check that, for $\varepsilon > 0$ sufficiently small, $(\bar{s}, \bar{y}, \bar{z}) \pm \varepsilon(u, v, w) \in P$, a contradiction. Therefore $(\bar{s}, \bar{z})$ is a vertex of $\operatorname{conv}\left(X^{MIX}\right)$ and thus $(\bar{s}, \bar{z}) \in X^{MIX}$. Then $(\bar{s}, \bar{y}, \bar{z}) \in X^{UMF}$ and the result is proven. □

8.3 The continuous mixing set with flows

In this section we reconsider the continuous mixing set with flows introduced in Section 4.2.1:

$$\begin{aligned} s + r_i + y_i &\geq b_i, && 1 \leq i \leq n, \\ y_i &\leq z_i, && 1 \leq i \leq n, \\ s \geq 0, \ r_i \geq 0, \ y_i &\geq 0, && 1 \leq i \leq n, \\ z_i &\text{ integer}, && 1 \leq i \leq n, \end{aligned}$$

where we assume without loss of generality $0 \leq b_1 \leq \cdots \leq b_n$ (as all variables are nonnegative). We write X^{CMF} to denote this mixed-integer set.

The practical usefulness of X^{CMF} in lot-sizing problems was discussed in Section 4.2.1, where we showed that the convex hull of this set can be transformed into a dual network set and thus admits a compact extended formulation (Proposition 4.4). We propose here some different compact extended formulations of the polyhedron $\mathrm{conv}(X^{CMF})$ that are derived by using the approach sketched in Steps 1–4 at the beginning of this chapter.

For the set X^{CMF} studied here, the set Y of Steps 3–4 is an instance of the difference set defined in Section 4.3.1. We propose three compact extended formulations for the convex hull of this set and therefore we obtain three different compact extended formulations of the polyhedron $\mathrm{conv}(X^{CMF})$. All formulations derived here are less compact than that given in Section 4.2.1. However the existence of a compact extended formulation of $\mathrm{conv}(X^{CMF})$ was first proven by using the approach presented here, when the generality of the results of Chapter 2 was not clear.

Steps 1–2 of the process described at the beginning of the chapter are performed in Section 8.2.1, while Steps 3–4 are the subject of Section 8.2.2.

8.3.1 A relaxation

We introduce a mixed-integer set Z which is the following relaxation of the set X^{CMF}:

$$s + r_i + z_i \geq b_i, \qquad 1 \leq i \leq n, \tag{8.15}$$

$$s + r_j + y_j + r_i + z_i \geq b_i, \qquad 1 \leq j < i \leq n, \tag{8.16}$$

$$s + r_i + y_i \geq b_i, \qquad 1 \leq i \leq n, \tag{8.17}$$

$$s \geq 0,\ r_i \geq 0,\ z_i \geq 0, \qquad 1 \leq i \leq n, \tag{8.18}$$

$$z_i \text{ integer}, \qquad 1 \leq i \leq n. \tag{8.19}$$

Note that variables y_i are *not* required to take a nonnegative value in Z.

The following lemma constitutes Step 1 of the process:

Lemma 8.14 *Let X^{CMF} and Z be defined on the same vector b. Then $X^{CMF} = Z \cap \{(s,r,y,z) : \mathbf{0} \leq y \leq z\}$.*

Proof. Observe that for $(s,y,r,z) \in X^{MF}$, $s+r_i+z_i \geq s+r_i+y_i \geq b_i$ holds, so $s+r_i+z_i \geq b_i$ is a valid inequality for X^{CMF}. Also, inequalities $s + r_i + z_i \geq b_i$ and $y_j, r_j \geq 0$ imply that $s+r_j+y_j+r_i+z_i \geq b_i$ is valid for X^{CMF}. Inequalities $z_i \geq 0$ follow from $y_i \geq 0$ and $y_i \leq z_i$. This proves that Z is a relaxation of X^{MF}.

The only inequalities that define X^{CMF} but do not appear in the definition of Z are the inequalities $\mathbf{0} \leq y \leq z$, thus $X^{CMF} = Z \cap \{(s,r,y,z) : \mathbf{0} \leq y \leq z\}$. □

Similarly to what we did in Section 8.2.1, we prove here that $\mathrm{conv}(X^{CMF}) = \mathrm{conv}(Z) \cap \{(s,r,y,z) : \mathbf{0} \leq y \leq z\}$ (Step 2). To do this, we need to establish some properties of the polyhedra $\mathrm{conv}(X^{CMF})$ and $\mathrm{conv}(Z)$. We start by characterizing their extreme rays.

Lemma 8.15 *The extreme rays (s,r,y,z) of $\mathrm{conv}(X^{CMF})$ are the following $3n+1$ vectors:*

$$(1,\mathbf{0},\mathbf{0},\mathbf{0});\ \ (0,e_j,\mathbf{0},\mathbf{0}),\ \ (0,\mathbf{0},\mathbf{0},e_j),\ \ (0,\mathbf{0},e_j,e_j)\ \textit{for } 1 \leq j \leq n.$$

The extreme rays (s, r, y, z) *of* $\text{conv}(Z)$ *are the following* $3n+1$ *vectors:*

$$(1, \mathbf{0}, -\mathbf{1}, \mathbf{0});\ \ (0, \mathbf{0}, e_j, \mathbf{0}),\ \ (0, \mathbf{0}, \mathbf{0}, e_j),\ \ (0, e_j, -e_j, \mathbf{0})\ \ \textit{for}\ 1 \le j \le n.$$

Proof. The first part is easy. We characterize the extreme rays of $\text{conv}(Z)$. The recession cone C of $\text{conv}(Z)$ is defined by

$$\begin{aligned} s + r_j + y_j + r_i + z_i \ge 0, &\quad 1 \le j < i \le n, \\ s + r_i + y_i \ge 0, &\quad 1 \le i \le n, \\ s \ge 0,\ r_i \ge 0,\ z_i \ge 0, &\quad 1 \le i \le n. \end{aligned}$$

One can verify that the vectors $\rho := (1, \mathbf{0}, -\mathbf{1}, \mathbf{0})$, $u_j := (0, e_j, -e_j, \mathbf{0})$, $v_j := (0, \mathbf{0}, e_j, \mathbf{0})$, $w_j := (0, \mathbf{0}, \mathbf{0}, e_j)$ for $1 \le j \le n$ are extreme rays of $\text{conv}(Z)$ by checking that each of them satisfies at equality $3n$ linearly independent inequalities defining C.

Thus we only have to show that every vector in C can be expressed as conic combination of the above rays. Let $(\bar{s}, \bar{r}, \bar{y}, \bar{z})$ be in C. Note that

$$(\bar{s}, \bar{r}, \bar{y}, \bar{z}) = \bar{s}\rho + \sum_{j=1}^{n} \bar{r}_j u_j + \sum_{j=1}^{n} (\bar{s} + \bar{r}_j + \bar{y}_j) v_j + \sum_{j=1}^{n} \bar{z}_j w_j.$$

Since $(\bar{s}, \bar{r}, \bar{y}, \bar{z}) \in C$, all the coefficients appearing in the above combination are nonnegative. □

Corollary 8.16 *The polyhedra* $\text{conv}\big(X^{CMF}\big)$ *and* $\text{conv}(Z)$ *are full-dimensional.*

Proof. One can check that the extreme rays of $\text{conv}\big(X^{CMF}\big)$ (resp. $\text{conv}(Z)$) listed above are linearly independent. This shows that the recession cone of $\text{conv}\big(X^{CMF}\big)$ (resp. $\text{conv}(Z)$) is full-dimensional and the conclusion follows. □

Lemma 8.17 *Let* $(\bar{s}, \bar{r}, \bar{y}, \bar{z})$ *be a vertex of* $\text{conv}(Z)$ *and let* $1 \le j \le n$. *Then*

$$\begin{aligned} \bar{s} &= \max\{0;\ b_i - \bar{r}_i - \bar{z}_i : 1 \le i \le n\}, \\ \bar{y}_j &= \max\{b_j - \bar{s} - \bar{r}_j;\ b_i - \bar{s} - \bar{r}_j - \bar{r}_i - \bar{z}_i : 1 \le j < i \le n\}. \end{aligned}$$

Proof. Assume $\bar{s} > 0$ and $\bar{s} + \bar{r}_i + \bar{z}_i > b_i$ for $1 \le i \le n$. Then there exists $\varepsilon \ne 0$ such that $(\bar{s}, \bar{r}, \bar{y}, \bar{z}) \pm \varepsilon(1, \mathbf{0}, -\mathbf{1}, \mathbf{0})$ belong to $\text{conv}(Z)$, a contradiction. This proves the first statement. The second one is obvious. □

The following result is crucial for proving that $\text{conv}\big(X^{CMF}\big) = \text{conv}(Z) \cap \{(s, r, y, z) : \mathbf{0} \le y \le z\}$.

Lemma 8.18 *Let* $(\bar{s}, \bar{r}, \bar{y}, \bar{z})$ *be a vertex of* $\text{conv}(Z)$. *Then* $\mathbf{0} \le \bar{y} \le \bar{z}$.

Proof. Assume that $\{i : \bar{y}_i < 0\} \neq \varnothing$ and let $h = \min\{i : \bar{y}_i < 0\}$. Then $\bar{s} + \bar{r}_h > b_h \geq 0$ and together with $\bar{z}_h \geq 0$, this implies $\bar{s} + \bar{r}_h + \bar{z}_h > b_h$.

CLAIM: $\bar{r}_h > 0$.

PROOF. Assume $\bar{r}_h = 0$. Then $\bar{s} > b_h \geq 0$. By Lemma 8.17, $\bar{s} + \bar{r}_i + \bar{z}_i = b_i$ for some index i. It follows that $\bar{s} \leq b_i$, thus $i > h$ (as $b_h < \bar{s} \leq b_i$). Equation $\bar{s} + \bar{r}_i + \bar{z}_i = b_i$, together with $\bar{s} + \bar{r}_h + \bar{y}_h + \bar{r}_i + \bar{z}_i \geq b_i$, gives $\bar{r}_h + \bar{y}_h \geq 0$, thus $\bar{r}_h > 0$, as $\bar{y}_h < 0$, and this concludes the proof of the claim.

Inequalities $\bar{s} + \bar{r}_h + \bar{z}_h > b_h$ and $\bar{r}_j + \bar{y}_j \geq 0$ for $1 \leq j < h$ imply $\bar{s} + \bar{r}_j + \bar{y}_j + \bar{r}_h + \bar{z}_h > b_h$ for $1 \leq j < h$.

All these observations show the existence of an $\varepsilon \neq 0$ such that both points $(\bar{s}, \bar{r}, \bar{y}, \bar{z}) \pm \varepsilon(0, e_h, -e_h, \mathbf{0})$ belong to $\mathrm{conv}(Z)$, a contradiction to the fact that the point $(\bar{s}, \bar{r}, \bar{y}, \bar{z})$ is a vertex of $\mathrm{conv}(Z)$. Thus $\bar{y} \geq \mathbf{0}$.

Suppose now that there exists h such that $\bar{y}_h > \bar{z}_h$. Then constraint $s + r_h + z_h \geq b_h$ gives $\bar{s} + \bar{r}_h + \bar{y}_h > b_h$. Lemma 8.17 then implies that $\bar{s} + \bar{r}_h + \bar{y}_h + \bar{r}_i + \bar{z}_i = b_i$ for some $i > h$. This is not possible, as inequalities $\bar{y}_h > \bar{z}_h \geq 0$, $\bar{r}_h \geq 0$ and $\bar{s} + \bar{r}_i + \bar{z}_i \geq b_i$ imply $\bar{s} + \bar{r}_h + \bar{y}_h + \bar{r}_i + \bar{z}_i > b_i$. Thus $\bar{y} \leq \bar{z}$. □

We can now prove the main theorem of this subsection:

Theorem 8.19 *Let X^{CMF} and Z be defined on the same vector b. Then* $\mathrm{conv}(X^{CMF}) = \mathrm{conv}(Z) \cap \{(s, r, y, z) : \mathbf{0} \leq y \leq z\}$.

Proof. We prove the result by applying Corollary 8.3 to the polyhedra $\mathrm{conv}(X^{CMF})$ and $\mathrm{conv}(Z)$ and the system $\mathbf{0} \leq y \leq z$. To do this, we show that the hypotheses of that corollary are satisfied.

By Lemma 8.14, $\mathrm{conv}(X^{CMF}) \subseteq \mathrm{conv}(Z)$. By Lemmas 8.18 and 8.14, every vertex of $\mathrm{conv}(Z)$ belongs to $\mathrm{conv}(X^{CMF})$.

Let $\alpha = (h, d, p, q)$, with $h \in \mathbb{R}$, $d \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, be such that $\mu_{\mathrm{conv}(X^{CMF})}(\alpha)$ is finite and $\mu_{\mathrm{conv}(Z)}(\alpha) = -\infty$. Since by Lemma 8.15, the extreme rays of $\mathrm{conv}(Z)$ that are not rays of $\mathrm{conv}(X^{CMF})$ are the vectors $(0, \mathbf{0}, e_j, \mathbf{0})$ for $1 \leq j \leq n$, $(0, e_j, -e_j, \mathbf{0})$ for $1 \leq j \leq n$ and $(1, \mathbf{0}, -\mathbf{1}, \mathbf{0})$, then either $p_j < 0$ for some index j, or $d_j < p_j$ for some index j, or $h < \sum_{i=1}^{n} p_i$. Also note that $h \geq 0$, as otherwise $\mu_{\mathrm{conv}(X^{MF})}(\alpha) = -\infty$ because of ray $(1, \mathbf{0}, \mathbf{0}, \mathbf{0})$.

If $p_j < 0$ for some index j, then $M_{\mathrm{conv}(X^{CMF})}(\alpha) \subseteq \{(s, r, y, z) : y_j = z_j\}$.

If $d_j < p_j$ for some index j, then $M_{\mathrm{conv}(X^{CMF})}(\alpha) \subseteq \{(s, r, y, z) : y_j = 0\}$, otherwise, given an optimal solution with $y_j > 0$, we could increase r_j by a small $\varepsilon > 0$ and decrease y_j by ε, thus obtaining a feasible point with lower objective value.

If $h < \sum_{i=1}^{n} p_i$, let $N^+ := \{i : p_i > 0\}$. We can assume that $N^+ \neq \varnothing$: if not, either there is an index j such that $p_j < 0$ (and we are in the first case above) or $p_j = 0$ for all $1 \leq j \leq n$, in which case we have $h < 0$, contradicting our assumption $h \geq 0$. Thus $N^+ \neq \varnothing$ and we can safely define $j := \min\{i : i \in N^+\}$. We show that $M_{\mathrm{conv}(X^{CMF})}(\alpha) \subseteq \{(s, r, y, z) : y_j = 0\}$. Suppose that $y_j > 0$ in some optimal solution. As the solution is optimal and $p_j > 0$, we cannot just decrease y_j and remain feasible. Thus $s + r_j + y_j = b_j$, which implies that $s < b_j$. Then for all $i \in N^+$ we have $r_i + y_i \geq b_i - s > b_i - b_j \geq 0$, as $i \geq j$. Since we can assume

$d_i \geq p_i$ for every i (otherwise we are in the previous case), $r_i = 0$ for every i: if not, chosen an index i such that $r_i > 0$, one can decrease r_i by a small $\varepsilon > 0$ and increase y_i by ε, thus obtaining a feasible point with lower objective value, a contradiction. So $r_i = 0$ for every i and thus, since $r_i + y_i > 0$ for all $i \in N^+$, we have $y_i > 0$ for all $i \in N^+$. Then we can increase s by a small $\varepsilon > 0$ and decrease y_i by ε for all $i \in N^+$. The new point is feasible in X^{CMF} and has lower objective value, a contradiction.

Therefore we have shown that for every vector α such that $\mu_{\mathrm{conv}(X^{CMF})}(\alpha)$ is finite and $\mu_{\mathrm{conv}(Z)}(\alpha) = -\infty$, the system $\mathbf{0} \leq y \leq z$ contains an inequality which is tight for the points in $M_{\mathrm{conv}(X^{CMF})}(\alpha)$. To complete the proof, since $\mathrm{conv}\big(X^{CMF}\big)$ is full-dimensional (Corollary 8.16), the system $\mathbf{0} \leq y \leq z$ does not contain an inequality defining an improper face of $\mathrm{conv}\big(X^{CMF}\big)$. So we can now apply Corollary 8.3 to the polyhedra $\mathrm{conv}\big(X^{CMF}\big)$ and $\mathrm{conv}(Z)$ and the system $\mathbf{0} \leq y \leq z$. □

8.3.2 The difference set

We now arrive to Step 3 of the process, where a new mixed-integer set Y is introduced. In our case Y is the *difference set*, which was also discussed in Section 4.3:

$$\sigma_j + r_i + z_i \geq b_i - b_j, \quad 0 \leq j < i \leq n, \tag{8.20}$$

$$\sigma_j \geq 0,\ r_i \geq 0,\ z_i \geq 0, \quad 0 \leq j \leq n,\ 1 \leq i \leq n, \tag{8.21}$$

$$z_i \text{ integer}, \quad 1 \leq i \leq n. \tag{8.22}$$

where $0 = b_0 \leq b_1 \leq \ldots \leq b_n$. Note that this definition is equivalent to that given in Section 4.3, because for $j \geq i$ the constraint $\sigma_j + r_i + z_i \geq b_i - b_j$ is redundant (as $b_j \geq b_i$ and all variables are nonnegative).

The theorem below shows that the polyhedra $\mathrm{conv}(Z)$ and $\mathrm{conv}(Y)$ are equivalent via an affine transformation (Step 3).

Theorem 8.20 *Let Z and Y be defined on the same vector b. The affine transformation*

$$\sigma_0 := s;\ \ \sigma_i := s + r_i + y_i - b_i,\ \ z_i := z_i \ \textit{for}\ 1 \leq i \leq n \tag{8.23}$$

maps $\mathrm{conv}(Z)$ *into* $\mathrm{conv}(Y)$.

Proof. It is straightforward to check that (8.23) transforms the inequalities in (8.15)–(8.19) into the inequalities in (8.20)–(8.22). Since this transformation is a mixed-integer linear mapping (see Section 4.1) plus a translation, the result follows. □

An immediate consequence is the following:

Corollary 8.21 *Let X^{CMF} and Y be defined on the same vector b. The affine transformation* (8.23) *maps* $\mathrm{conv}\big(X^{CMF}\big)$ *into*

$$\mathrm{conv}(Y) \cap \{(\sigma, r, z) : 0 \leq \sigma_i - \sigma_0 - r_i + b_i \leq z_j \ \textit{for}\ 1 \leq i \leq n\}.$$

Proof. The result follows from Theorems 8.19 and 8.20. □

The above corollary shows that an external description of $\operatorname{conv}(X^{CMF})$ can be obtained from an external description of $\operatorname{conv}(Y)$. Unfortunately, the convex hull of a set of the type Y in its space of definition is not known. However there are several ways of giving a compact extended formulation of $\operatorname{conv}(Y)$ (Step 4).

First approach: transforming Y into a dual network set

Recall that in Section 4.3 we showed that $\operatorname{conv}(Y)$ admits a compact extended formulation, as it can be transformed into a dual network set having a short complete list of fractional parts. Thus that extended formulation yields a compact extended formulation for $\operatorname{conv}(X^{CMF})$. This approach might appear quite odd, as the set X^{CMF} itself can be transformed into a dual network set, thus it seems more convenient to write the corresponding extended formulation directly for such set. Nonetheless this approach was adopted by Conforti, Di Summa and Wolsey [12] to provide the first compact extended formulation for $\operatorname{conv}(X^{CMF})$, when the generality of the results of Chapter 2 was not completely clear.

Second approach: formulating conv(Y) as a union of polyhedra

A second possible way of constructing a compact extended formulation of $\operatorname{conv}(Y)$ consists in using the approach sketched in Section 1.5.4, which exploits Balas' result on the union of polyhedra (Theorem 1.3). Such a technique was used by Atamtürk [2] to model a simple set and was discussed and demonstrated in a paper by Conforti and Wolsey [16].

Enumeration of fractional parts is still present in this formulation. However, the fractional parts are listed in a way that is different from that considered in Chapter 2. To explain this, let us consider the σ-variables. Instead of giving a list of values containing all the fractional parts taken by the σ-variables over the set of vertices of $\operatorname{conv}(Y)$, we provide a list of $(n+1)$-dimensional vectors $\mathcal{F} = \{f^1, \dots, f^k\}$ such that each vertex $(\bar\sigma, \bar r, \bar z)$ of $\operatorname{conv}(Y)$ satisfies $\big(f(\bar\sigma_0), \dots, f(\bar\sigma_n)\big) \in \mathcal{F}$.

Such a list is given by the following result:

Proposition 8.22 *Let $(\bar\sigma, \bar r, \bar z)$ be a vertex of $\operatorname{conv}(Y)$. Then there exist two indices $0 \le h \le \ell \le n$ such that $f(\bar\sigma_j) = 0$ for $h \le j \le n$ and $f(\bar\sigma_j) = f(b_\ell - b_j)$ for $0 \le j < h$.*

Proof. Let $(\bar\sigma, \bar r, \bar z)$ be a vertex of $\operatorname{conv}(Y)$, define $\alpha := \max_{1 \le i \le n}\{b_i - \bar r_i - \bar z_i\}$ and let $T_\alpha \subseteq \{1, \dots, n\}$ be the subset of indices for which this maximum is achieved.

CLAIM 1: *For each $1 \le j \le n$, $\bar\sigma_j = \max\{0, \alpha - b_j\}$.*

PROOF. The inequalities that define Y show that $\bar\sigma_j \ge \max\{0, \alpha - b_j\}$. If $\bar\sigma_j > \max\{0, \alpha - b_j\}$, then there is an $\varepsilon > 0$ such that $(\bar\sigma, \bar r, \bar z) \pm \varepsilon(e_j, \mathbf{0}, \mathbf{0})$ are both in $\operatorname{conv}(Y)$, a contradiction to the fact that $(\bar\sigma, \bar r, \bar z)$ is a vertex of $\operatorname{conv}(Y)$. This concludes the proof of the claim.

Define $h := \min\{j : \alpha - b_j \le 0\}$. (This minimum is well defined: since the only inequality involving σ_n is $\sigma_n \ge 0$, certainly $\bar\sigma_n = 0$; then, by Claim 1, $\alpha - b_n \le 0$.) Since $0 = b_0 \le b_1 \le \cdots \le b_n$, Claim 1 shows that $\bar\sigma_j > 0$ for $j < h$ and $\bar\sigma_j = 0$ for $j \ge h$ and this proves part of the proposition. Furthermore $\bar\sigma_j + \bar r_i + \bar z_i = b_i - b_j$ for all $j < h$ and $i \in T_\alpha$.

Claim 2: *Either* $\bar{r}_i = 0$ *for some* $i \in T_\alpha$*, or* $f(r_i) = f(b_i - b_h)$ *for every* $i \in T_\alpha$*.*
Proof. We use the fact that $(\bar{\sigma}, \bar{r})$ is a vertex of the polyhedron:

$$Q := \left\{(\sigma, r) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^n_+ : \sigma_j + r_i \geq b_i - b_j - \bar{z}_i \text{ for } 0 \leq j < i \leq n\right\}.$$

We now consider the following two cases:
Case 1: $\alpha - b_h < 0$.
For $j \geq h$, the only inequality that is tight for $(\bar{\sigma}, \bar{r})$ and contains σ_j in its support is $\sigma_j \geq 0$. For $j < h$, the only inequalities that are tight for $(\bar{\sigma}, \bar{r})$ and contain σ_j in their support are $\sigma_j + r_i \geq b_i - b_j - \bar{z}_i$ for $i \in T_\alpha$.
Let e_H be the $(n+1)$-vector having the first h components equal to 1 and the others to 0, let e_{T_α} be the incidence vector of T_α and assume that $\bar{r}_i > 0$ for all $i \in T_\alpha$. Then the vectors $(\bar{\sigma}, \bar{r}) \pm \varepsilon(e_H, -e_{T_\alpha})$ for some $\varepsilon > 0$ are both in Q, contradicting the fact that $(\bar{\sigma}, \bar{r})$ is a vertex of Q. So $\bar{r}_i = 0$ for some $i \in T_\alpha$.
Case 2: $\alpha - b_h = 0$.
Then $(\bar{\sigma}, \bar{r}, \bar{z})$ satisfies $\bar{\sigma}_h + \bar{r}_i + \bar{z}_i = b_i - b_h$ for all $i \in T_\alpha$. Since $\bar{\sigma}_h = 0$ and $\bar{z}_i$ is integer, then $f(\bar{r}_i) = f(b_i - b_h)$ for all $i \in T_\alpha$ and this completes the proof of Claim 2.

Assume $\bar{r}_i = 0$ for some $i \in T_\alpha$. Since $\bar{\sigma}_j + \bar{r}_i + \bar{z}_i = b_i - b_j$ for all $j < h$ and $\bar{z}_i$ is an integer, then $f(\bar{\sigma}_j) = f(b_i - b_j)$ for all $j < h$. Note that if $i < h$ then $\alpha - b_h > 0$ and thus (recalling that $i \in T_\alpha$) $b_i - \bar{r}_i - \bar{z}_i - b_h > 0$, which is not possible as $b_i \leq b_h$ and $\bar{r}_i, \bar{z}_i \geq 0$. Thus $i \geq h$ and the result holds with $\ell = i$.

If $f(\bar{r}_i) = f(b_i - b_h)$ for all $i \in T_\alpha$, since $\bar{\sigma}_j + \bar{r}_i + \bar{z}_i = b_i - b_j$ for all $i \in T_\alpha$ and for all $j < h$ and since $\bar{z}$ is an integral vector, then $f(\bar{\sigma}_j) = f(b_h - b_j)$ for all $j < h$. Then the result holds with $\ell = h$. □

A similar result can be proven for the variables r_t:

Proposition 8.23 *Let* $(\bar{\sigma}, \bar{r}, \bar{z})$ *be a vertex of* $\text{conv}(Y)$*. Then there exist two indices* $0 \leq \ell' \leq h' \leq n$ *such that* $f(\bar{r}_i) = 0$ *for* $1 \leq i \leq h'$ *and* $f(\bar{r}_i) = f(b_i - b_{\ell'})$ *for* $h' < i \leq n$*.*

Proof. We omit the proof, which is symmetric to that of Proposition 8.22. We only remark that throughout the proof, the role of a variable σ_j is now played by the sum $r_i + z_i$: for instance, one defines $\alpha' := \max_{0 \leq j \leq n}\{-b_j - \bar{\sigma}_j\}$ and then proves that for each $0 \leq i \leq n$, $\bar{r}_i + \bar{z}_i = \max\{0, \alpha' + b_i\}$. □

Let T be the set of quadruples of indices $\tau = (h, \ell, h', \ell')$ with $0 \leq h \leq \ell \leq n$ and $0 \leq \ell' \leq h' \leq n$. For each $\tau \in T$, let Y^τ be the set of points $(\sigma, r, z) \in Y$ for which the values $f(\sigma_j), f(r_i)$ satisfy the properties of Propositions 8.22–8.23. Note that every vertex of $\text{conv}(Y)$ belongs to Y^τ for some $\tau \in T$. Furthermore, it can be checked that the recession cone of each polyhedron $\text{conv}(Y^\tau)$ coincides with that of $\text{conv}(Y)$. This is sufficient to see that $\text{conv}(Y) = \text{conv}\left(\bigcup_{\tau \in T} Y^\tau\right)$. Then, if we give a formulation of $\text{conv}(Y^\tau)$ for each $\tau \in T$, Balas' result (Theorem 1.3) will provide an extended formulation for $\text{conv}(Y)$.

Fix $\tau = (h, \ell, h', \ell') \in T$. Since the fractional part of each continuous variable is fixed in Y^τ, we can model the continuous variables as shown below:

$$\sigma_j = \mu_j + f(b_\ell - b_j), \quad 0 \le j \le h, \tag{8.24}$$
$$\sigma_j = \mu_j, \quad h < j \le n, \tag{8.25}$$
$$r_i = \nu_i, \quad 1 \le i \le h', \tag{8.26}$$
$$r_i = \nu_i + f(b_i - b_{\ell'}), \quad h' < i \le n, \tag{8.27}$$
$$\mu_j, \nu_i \text{ integer}, \quad 0 \le j \le n,\ 1 \le i \le n. \tag{8.28}$$

Under the above conditions, inequalities (8.20)–(8.22) can be rewritten as follows:

$$\begin{aligned}
&\mu_j + \nu_i \ge b_i - b_j - f(b_\ell - b_j) - f(b_i - b_{\ell'}), && 0 \le j \le h,\ h' < i \le n,\ j < i,\\
&\mu_j + \nu_i \ge b_i - b_j - f(b_\ell - b_j), && 0 \le j \le h,\ 1 \le i \le h',\ j < i,\\
&\mu_j + \nu_i \ge b_i - b_j - f(b_i - b_{\ell'}), && h < j \le n,\ h' < i \le n,\ j < i,\\
&\mu_j + \nu_i \ge b_i - b_j, && h < j \le n,\ 1 \le i \le h',\ j < i,\\
&\mu_j \ge 0,\ \nu_i \ge 0,\ z_i \ge 0, && 0 \le j \le n,\ 1 \le i \le n,\\
&z_i \text{ integer}, && 1 \le i \le n.
\end{aligned}$$

Since the constraint matrix of the above system is totally unimodular and all variables are integer, the convex hull is obtained by rounding up the right-hand sides and removing the integrality restrictions. The resulting linear system, together with equations (8.24)–(8.27) (which define the original variables) is an extended formulation of $\operatorname{conv}(Y^\tau)$. By applying Balas'result (Theorem 1.3) we obtain an extended formulation of $\operatorname{conv}(Y)$.

Third approach: a mixture of the above methods

When discussing the first approach to formulate $\operatorname{conv}(Y)$, we pointed out that Conforti, Di Summa and Wolsey used that technique in [12], where the first compact extended formulation of $\operatorname{conv}(X^{CMF})$ was given. In fact that paper describes two compact extended formulations of $\operatorname{conv}(X^{CMF})$. The other formulation was given by using in a sense a mixture of the two approaches illustrated above, as we now explain.

The first part of the process is as in the second approach above, except that only Proposition 8.22 is used. More specifically, let T be the set of pairs of indices $\tau = (h, \ell)$ with $0 \le h \le \ell \le n$. For each $\tau \in T$, let Y^τ be the set of points $(\sigma, r, z) \in Y$ for which the values $f(\sigma_j)$ for $0 \le j \le n$ satisfy the properties of Proposition 8.22. As above, one can prove that $\operatorname{conv}(Y) = \operatorname{conv}\left(\bigcup_{\tau \in T} Y^\tau\right)$. Then, if we give a formulation of $\operatorname{conv}(Y^\tau)$ for each $\tau \in T$, Balas' result (Theorem 1.3) will provide an extended formulation for $\operatorname{conv}(Y)$.

Fix $\tau = (h, \ell) \in T$. Since the fractional parts of variables σ_j are fixed in Y^τ, we can model these variables as shown below:

$$\sigma_j = \mu_j + f(b_\ell - b_j), \quad 0 \le j \le h, \tag{8.29}$$
$$\sigma_j = \mu_j, \quad h < j \le n, \tag{8.30}$$
$$\mu_j \text{ integer}, \quad 0 \le j \le n. \tag{8.31}$$

Under the above conditions, inequalities (8.20)–(8.22) can be rewritten as follows:

$$\mu_j + r_i + z_i \geq b_i - b_j - f(b_\ell - b_j), \quad 0 \leq j \leq h,\ 0 \leq j < i \leq n, \tag{8.32}$$
$$\mu_j + r_i + z_i \geq b_i - b_j, \quad h < j \leq n,\ 0 \leq j < i \leq n, \tag{8.33}$$
$$\mu_j \geq 0,\ r_i \geq 0,\ z_i \geq 0, \quad 0 \leq j \leq n,\ 1 \leq i \leq n \tag{8.34}$$
$$\mu_j, z_i \text{ integer}, \quad 0 \leq j \leq n,\ 1 \leq i \leq n. \tag{8.35}$$

In [12] the above system is strengthened in a way that is similar to that discussed in Chapter 2:[2] for each $1 \leq i \leq n$, a list $\mathcal{F}_i$ is given that contains all the fractional parts taken by variable r_i over the vertices of the convex hull of (8.32)–(8.35). In other words, $\mathcal{F}_i$ is complete for the above mixed-integer set with respect to variable r_i.

Lemma 8.24 *The list of fractional parts $\mathcal{F}_i := \{0, f(b_i - b_\ell)\} \cup \{f(b_i - b_j) : 0 \leq j < i\}$ is complete for (8.32)–(8.35) with respect to variable r_i.*

Proof. First of all note that the fractional part of the right-hand side of inequality (8.32) is $f(b_i - b_\ell)$. Let $(\bar{\mu}, \bar{r}, \bar{z})$ be a vertex of the convex hull of (8.32)–(8.35). Since $\bar{\mu}$ and $\bar{z}$ are integral vectors, if $f(\bar{r}_i)$ were not in the list $\mathcal{F}_i$ defined above then both points $(\bar{\mu}, \bar{r} \pm \varepsilon e_i, \bar{z})$ would satisfy (8.32)–(8.35) for some $\varepsilon \neq 0$. This contradicts the assumption that $(\bar{\mu}, \bar{r}, \bar{z})$ is a vertex. □

For each index $1 \leq i \leq n$, define $f_i^j := f(b_i - b_j)$ for $0 \leq j \leq i$ and $f_i^{i+1} := f(b_i - b_\ell)$, so that $\mathcal{F}_i = \{f_i^0, \ldots, f_i^{i+1}\}$. We model the r-variables as follows:

$$r_i = \nu_i + \textstyle\sum_{t=0}^{i+1} f_i^t \delta_i^t, \quad 1 \leq i \leq n, \tag{8.36}$$
$$\textstyle\sum_{t=0}^{i+1} \delta_i^t = 1,\ \delta_i^t \geq 0, \quad 1 \leq i \leq n,\ 0 \leq t \leq i+1, \tag{8.37}$$
$$\nu_i, \delta_i^t \text{ integer}, \quad 1 \leq i \leq n,\ 0 \leq t \leq i+1. \tag{8.38}$$

Under the above conditions and using Chvátal-Gomory rounding similarly to what we did in the proof of Lemma 2.5, inequalities (8.32)–(8.33) become

$$\mu_j + \nu_i + \sum_{t: f_i^t \geq f(b_i - b_\ell)} \delta_i^t + z_i \geq \lfloor b_i - b_j - f(b_\ell - b_j) \rfloor + 1, \quad 0 \leq j \leq h,\ 0 \leq j < i \leq n, \tag{8.39}$$
$$\mu_j + \nu_i + \sum_{t: f_i^t \geq f(b_i - b_j)} \delta_i^t + z_i \geq \lfloor b_i - b_j \rfloor + 1, \quad h < j \leq n,\ 0 \leq j < i \leq n. \tag{8.40}$$

Therefore the set Y^τ is described by conditions (8.29)–(8.31), (8.36)–(8.38) and (8.39)–(8.40).

Proposition 8.25 *The constraint matrix of the system comprising inequalities (8.37) and (8.39)–(8.40) is totally unimodular.*

[2] In fact the set defined by (8.32)–(8.35) could be mapped into a dual network set. However we present the result as in [12].

Proof. Let A be the constraint matrix of the system comprising inequalities (8.37) and (8.39)–(8.40). Order the columns of A according to the following ordering of the variables:

$$\mu_0, \dots, \mu_n;\ z_1, \nu_1, \delta_1^1, \delta_1^2;\ z_2, \nu_2, \delta_2^1, \delta_2^2, \delta_2^3;\ \dots\ ;\ z_i, \nu_i, \delta_i^1, \dots, \delta_i^{i+1}\ ;\ \dots\ ;\ z_n, \nu_n, \delta_n^1, \dots, \delta_n^{n+1}.$$

For each row of A, the 1's that appear in a block $\left[z_i, \nu_i, \delta_i^1, \dots, \delta_i^{i+1}\right]$ are consecutive and start from the first position. Furthermore, for each row of A only one of these blocks contains nonzero elements.

Consider an arbitrary column submatrix of A. We give color red to all the μ_j (if any) and then, for each of the blocks $\left[z_i, \nu_i, \delta_i^1, \dots, \delta_i^{i+1}\right]$, we give alternating colors, always starting with blue, to the columns of this block that appear in the submatrix. Since this is an equitable bicoloring, the result of Ghouila-Houri (Theorem 1.14) shows that A is totally unimodular. □

Since each variable σ_j, r_i is defined by the corresponding equation in (8.29)–(8.30) or (8.36), and does not appear in any other constraint, the above proposition implies that the integrality requirements can be dropped. Thus inequalities (8.29)–(8.30), (8.36)–(8.37) and (8.39)–(8.40) form an extended formulation of $\mathrm{conv}(Y^\tau)$. By applying Balas'result we obtain an extended formulation of $\mathrm{conv}(Y)$.

To conclude, we point out that each of the three extended formulations of $\mathrm{conv}\left(X^{CMF}\right)$ discussed here is less compact than that given in Section 4.2.1. In particular the formulation obtained here by using the second approach is very large, as it uses $\mathcal{O}\left(n^6\right)$ variables and constraints.

Chapter 9

Open problems

We conclude this dissertation by addressing some questions that remain unanswered.

In Chapter 2 we introduced a technique to construct extended formulations for mixed-integer sets MIX^{2TU} whose constraint matrix is totally unimodular and contains at most two nonzero entries per row. The technique is based on the explicit enumeration of all possible fractional parts that the variables take at the vertices of $\text{conv}(MIX^{2TU})$. As shown in Chapter 3, since there exist sets of the type MIX^{2TU} that do not admit a complete list of fractional parts whose size is compact, a formulation of this type might have exponential size.

A first natural question is then the following: Is it possible to modify our approach so that a compact extended formulation is obtained even if no complete list for the set is compact? A first failed attempt was briefly discussed in Section 2.4.2, but the answer to the above question is not known.

We remark that even if no complete list for the set is compact, still we do have an extended formulation for the set, as Lemma 2.11 provides us with a list which is always complete. Thus we can weaken the above question to the following: Is it possible to use our extended formulation to optimize in polynomial time even if no complete list for the set is compact?

The inequalities constituting our formulation are explicitly given. The fact that the number of these inequalities might be exponential is probably a minor issue, thanks to the equivalence between separation and optimization (Theorem 1.6). The major problem is the fact that the number of variables can be exponentially large with respect to the original description of the set. Nonetheless there is much structure in our extended formulation, so there may be a hope to handle this problem.

This thesis contains no computational experiment. However this is also an aspect that should be explored. As pointed out for instance in [70], complicated mixed-integer sets can be effectively tackled by constructing relaxations that have a simpler structure and then tightening or reformulating such relaxations. As shown for instance in Chapter 4, there are several well studied simple-structured mixed-integer sets that are of the type MIX^{2TU}, and many others can probably arise in other contexts. It would be interesting to understand how effective an extended formulation of our type can be when used to tighten a substructure of a more complicated mixed-integer set. Also, it is not obvious how such a formulation should be used: one could for instance add all or only some of the inequalities of the formulation to the original set, or use the extended formulation to separate.

Another interesting aspect is the following. Note that even if all complete list of fractional parts for a set are non-compact, one can consider a short sublist and constructing the corresponding extended formulation. By doing so, one obtains the description of a subset (not a relaxation) of the convex hull of the original set. Can this idea be used to effectively approximate a mixed-integer set of our family?

A question that arises naturally is about projections. It is probably hard to compute the projection of our extended formulation onto the original space of variables in the general case. Still, since such formulations have a common structure, there is a hope that the extended formulations can be used to find some general properties of the facet-defining inequalities in the original space. (However information about the convex hull in the original space can also be found without using extended formulations or projections, as demonstrated in Chapters 6 and 8.)

Another question that we address concerns the possible generalizations of the approach presented in Chapter 2. In Chapter 7 we considered two variants of a specific set MIX^{2TU} (namely the mixing set) obtained by multiplying the columns of the constraint matrix by some constants. Under the assumption of divisibility, we could (non-trivially) extend the approach presented in the previous chapters. It would be nice to understand whether a generalization of this type is only possible for those specific sets, or the idea underlying our extension can be pushed further.

Recall that we pointed out in Section 7.1.8 that for the formulations of Chapter 7 the constraint matrix is not (in general) totally unimodular. In fact, the construction of integral extended formulations was possible because of the presence of a single constraint for each integer variable. It would be useful to remove this strong limitation.

Finally, we observe that the approach illustrated in Chapter 8 is somehow mysterious. First, it is not clear to which class of sets it can be applied. Second, even restricting ourselves to the cases studied in that chapter (i.e. the mixing set with flow and the continuous mixing set with flows), it is difficult to see a rational criterion for choosing that relaxation Z rather than another one (except for the *a posteriori* consideration that such a choice works!).

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