Critical Point Theory and Nonlinear Problems

A. AMBROSETTI
SISSA, Trieste.

Padova, 16 Marzo 2007

Introduction

Let E be a Hilbert space and let $J \in C^1(E, \mathbb{R})$ be a functional.

A critical (or stationary) point of E is an $u \in E$ such that $dJ(u)[v] = 0$ for all $v \in E$,

or, using the gradient J' of J, $(J'(u) \mid v) = 0$ for all $v \in E$.

Critical points are weak solutions of (variational) differential equations.

Similarly, the critical points of a functional J constrained on a Hilbert manifold M give rise to solutions of eigenvalue variational problems.

For ex. if M is the unit sphere in the Hilbert space E, we find the solutions of $J'(u) = \lambda u$, with $\|u\| = 1$ (λ is nothing but the Lagrange multiplier).

Direct Methods in Calculus of Variations and Critical Point Theory

The Direct Methods (Hilbert, Tonelli, etc.), highlighted the fundamental role of the search of global minima of functionals in order to solve variational problems.

On the other hand, there are problems in which looking for minima of their Euler functional is not satisfactory. Roughly, there are:

- problems in which the minima give rise to $trivial$ solutions;
- problems in which one expects many solutions;
- problems in which the global minimum does not exist because the functional in not bounded from below.

For ex. a case in which the first difficulty arises is the search of closed geodesics on a compact smooth manifold $M \subset \mathbb{R}^n$.
A closed geodesic on M is a closed loop $\gamma(t)$ which makes stationary the length $\int |\dot{\gamma}|$.

Clearly the minimum is achieved at the trivial loop $\gamma(t) \equiv p \in M$.
The non-trivial closed geodesics are saddle points and have to be found with a procedure different than minimization.

Birkhoff in 1917 proved the following result:

Theorem. On any compact surface $M \subset \mathbb{R}^3$ which is C^3 diffeomorphic to the the standard sphere, there exists a non-trivial (i.e. non-constant) closed geodesic.

The geodesic is found by an appropriate min-max method.

Later, Lusternik and Schnirelman (1929) proved that there exist at least 3 geometrically distinct closed geodesics without self-intersections.

These results marked the beginning of the Critical Point theory.

Lusternik and Schnirelman developed a general abstract theory in which the number of critical points of a functional on a compact manifold is evaluated by means of the topological properties of M.

As a specific application, Lusternik and Schnirelman proved the following celebrated result

Theorem. Any smooth functional on the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ has at least n pairs of distinct critical points.

In the same years, M. Morse also developed an elegant theory (the Morse theory) in which the homological properties of M play their role. This theory gives rise to more precise results, provided J is smooth and all its critical points are non-degenerate.

This a-priori requirement is a severe restriction for applications to variational problems.

Extensions to infinite dimension

Critical point theory has been extended to infinite dimension.

Ref.: Vainberg, Krasnoselski, Palais, Smale, F.Browder.

The main new ingredient is a "compactness" condition.

Precisely, one assumes:

Every sequence u_n such that

(i) $J(u_n) \to c$,

(ii) $J'(u_n) \to 0$,

has a converging subsequence.

If $(i - ii)$ hold, we say that J satisfies the $(PS)_c$ condition (Palais-Smale condition at level c).
The sequences satisfying $(a) - (b)$ are called $(PS)_c$ sequences.

For example, if (PS) holds and J is bounded from below, then the steepest descent flow, namely the solutions of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\sigma & = -J'(\sigma) \\ \sigma(0) & = u \end{cases}$$

converges to a critical point of J as $t \to +\infty$. This could be false if (PS) does not hold.

If J is bounded from below and (PS) holds, then J the infimum is attained.

This could be false if (PS) does not hold.

A typical result is the following one.

Theorem. Let E be a separable infinite dimensional Hilbert space and suppose that $J \in C^1(E, \mathbb{R})$ is even, bounded from below on the sphere $S = \{u \in E : \|u\| = 1\}$, and satisfies the $(PS)_c$ condition for every $c < \sup_S J$.
Then J has infinitely many critical points on S.

Indefinite functionals: the Mountain-Pass Theorem

A functional $J : E \mapsto \mathbb{R}$ is said indefinite if

$$\inf_E J(u) = -\infty, \qquad \sup_E J(u) = +\infty$$

The Mountain-Pass theorem provides the existence of saddle points under suitable geometric assumptions + compactness.

Let $J \in C^1(E, \mathbb{R})$ satisfy the following two "geometric" assumptions:

$A1.$ J has a local strict minimum at, say, $u = 0$: there exist $r, \rho > 0$ such that $J(u) \geq \rho$ for all $u \in E$ with $\|u\| = r$.

$A2.$ $\exists\, v \in E$, $\|v\| > r$, such that $J(v) \leq 0 = J(0)$.

Let $J \in C^1(E, \mathbb{R})$ be a functional that satisfies the assumptions (A1-A2). Without loss of generality, we can also assume (to simplify notation) that $J(0) = 0$.

Consider the class of all paths joining $u = 0$ and $u = v$:

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = v\}$$

and set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Remark: $c \geq \rho > 0$

Theorem (Mountain-Pass) If $J \in C^1(E, \mathbb{R})$ satisfies (A1-A2) and $(PS)_c$ holds, then c is a positive critical level for J. Precisely, there exists $z \in E$ such that $J(z) = c > 0$ and $J'(z) = 0$. In particular $z \neq 0$ (and $z \neq v$).

Remarks. (a) J can be indefinite.

(b) The M-P critical point is a saddle point: if it is non-degenerate, then its Morse index is 1.

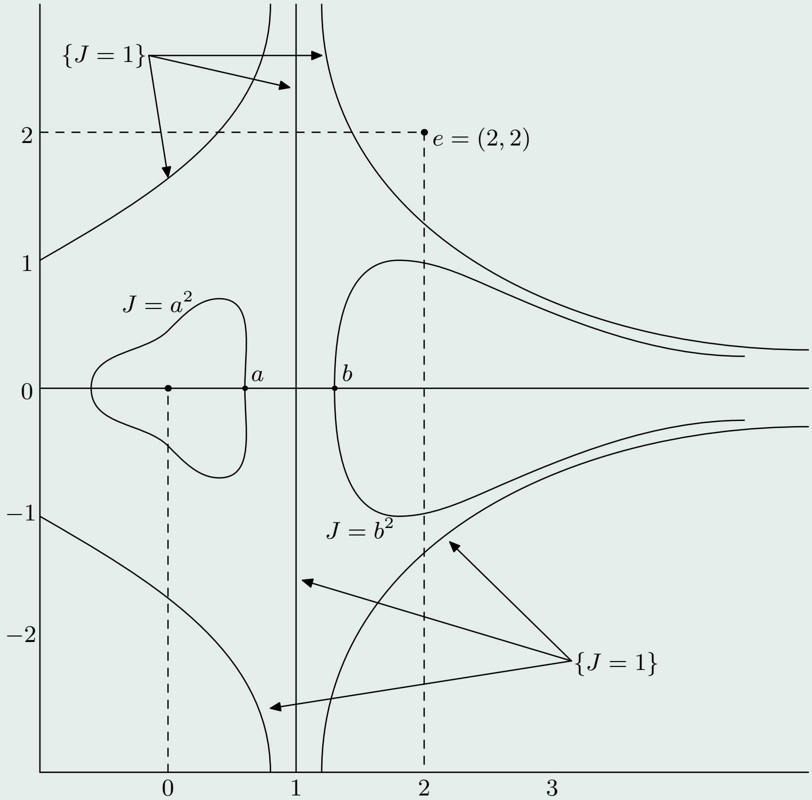
(c) The MP theorem can be extended to handle cases in which $u = 0$ is not a local minimum (Linking thms). The corresponding critical point will have Morse index greater than 1.

(d) The following example shows that, even on $\mathbb{R}^n$, the geometric assumptions (A1-2) alone, without the (PS) condition, do not suffice for the existence of a M-P critical point.

Let $E = \mathbb{R}^2$ and $J(x, y) = x^2 + (1 - x)^3 y^2$. It is easy to see that $(0, 0)$ is a strict local minimum and that $J(2, 2) = J(0, 0) = 0$.

• The only critical point of J is $(0, 0)$.

• The M-P critical level is $c = 1$ and $(PS)_c$ does not hold for $c = 1$.



{J = 1}
e = (2, 2)
J = a²
a
b
J = b²
{J = 1}
2
1
0
−1
−2
0
1
2
3

As a typical application, one can show that the BVP

$$(1) \qquad \begin{cases} -\Delta u = \lambda u + F'(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

has a positive solution provided $F \simeq |u|^{p+1}$, with $2 < p+1 < 2^*$.

(2^* is the limit exponent for the embedding of $W_0^{1,2}(\Omega)$ in $L^q(\Omega)$: $2^* = 2n/(n-2)$ if $n \geq 3$)

One can also extend to indefinite functionals the Lusternik-Schnirelman theory. Roughly, if J has the MP geometry and is even, then there exist infinitely many critical points.
For ex. if F is even, (1) has infinitely many solutions.

Remark. If $\lambda \leq 0$, $F = |u|^{2^*}$ and Ω is star-shaped with respect to the origin, (1) has only the trivial solution $u \equiv 0$ (Pohozaev).

1. Elliptic problems on bounded domains, with critical exponent

$$(2) \qquad \begin{cases} -\Delta u = \lambda u + u^{2^*-1}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

H.Brezis and L.Nirenberg proved the following result.

Theorem. There exists $\lambda^* \in [0, \lambda_1[$ such that (2) has a positive solution provided $\lambda \in]0, \lambda_1[$. Furthermore, if $n \geq 4$ then $\lambda^* = 0$, while if $n = 3$, λ^* might be zero.

For ex. if Ω is the unit ball and $n = 3$, then $\lambda^* = \lambda_1/4$.

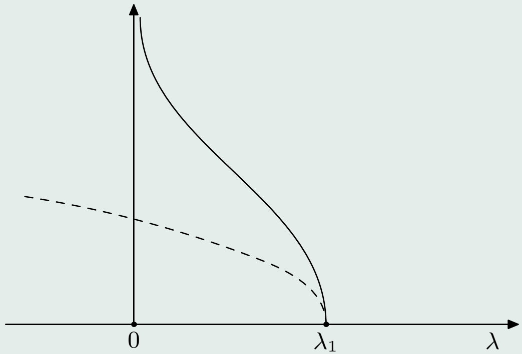


Figure 1: Bifurcation diagram of positive solutions of (2) when $\lambda^* = 0$ and Ω is star-shaped (solid curve). The dashed line represents the branch of positive solutions of the subcritical equation $-\Delta u = \lambda u + u^p$, $u \in H_0^1(\Omega)$, $1 < p < 2^* - 1$.

Roughly, one shows:

1) there exists a threshold $C^* = \frac{1}{n} S^{n/2}$ such that $(PS)_c$ holds for $c < C^*$.

2) the MP critical point c_λ is smaller than C^* provided $\lambda^* < \lambda < \lambda_1$.

2. Equations on $\mathbb{R}^n$

$$
(3) \qquad \begin{cases} -\Delta u + u = a(x)u^p & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to +\infty. \end{cases}
$$

We assume that $1 < p < \frac{n+2}{n-2}$, and that a is positive, smooth and such that

$$
\lim_{|x|\to+\infty} a(x) = \ell.
$$

Moreover, we suppose that either

$$
a(x) > \ell \qquad \text{for every } x \in \mathbb{R}^n; \quad or
$$

$$
\ell - Ce^{-\delta|x|} < a(x) < \ell \qquad \text{for } |x| \gg 1.
$$

If the above conditions hold, then (3) has a solution.

Ref.: A.Bahri, P.L.Lions, Y.Y.Li, etc.

3. Perturbation Problems

Let

$$J_\varepsilon(u) = J_0(u) + \varepsilon G(u)$$

We assume that there is a smooth d-dimensional manifold Z such that $J_0'(z) = 0$, for all $z \in Z$. Z is called critical manifold (of the *unperturbed functional* J_0).

Let $T_z Z$ denote the tangent space to Z at $z \in Z$. Since $J_0'(z) = 0$ for all $z \in Z$, differentiating along Z we get

$$(J_0''(z)[v] \mid \phi) = 0, \qquad \forall\, v \in T_z Z,\ \forall\, \phi \in E.$$

This shows that $T_z Z \subseteq Ker[J_0''(z)]$.

We say that Z is *Non-Degenerate (ND)* if

$$\text{(ND)} \qquad T_z Z = Ker[J_0''(z)]$$

We also suppose that $J_0''(z)$ is a $0-$Fredholm map.

If Z is ND, one can use a suitable finite dimensional reduction proving

Theorem. (AA- M.Badiale) Suppose that Z which is non-degenerate and let $\Gamma(z) = G(z)$. Then any "stable" critical point z_0 of Γ gives rise, for ε small enough, to a critical point u_ε of J_ε, with $u_\varepsilon \sim z_0$.

As an application, it is possible to show:

Theorem. (AA-Garcia Azorero-Peral) Suppose that $h \in L^\infty(\mathbb{R}^n)$ and $h \to 0$ as $|x| \to \infty$. Then the problem

$$-\Delta u + u = (1 + \varepsilon h(x))u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad 1 < p < \frac{n+2}{n-2},$$

has, for ε small enough, a positive solution.

Here,

$$J_0(u) = \tfrac{1}{2}\int (|\nabla u|^2 + u^2) - \tfrac{1}{p+1}\int |u|^{p+1},$$

$$G(u) = \tfrac{1}{p+1}\int h(x)|u|^{p+1}.$$

The unperturbed problem has a ND critical manifold given by

$$Z = \{U(x+\xi) : \xi \in \mathbb{R}^n\},$$

where U is the unique positive radially symmetric function in $W^{1,2}(\mathbb{R}^n)$ satisfying

$$-\Delta U + U = U^p$$

4. Semiclassical states of NLS

Consider

$$(NLS_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x) u = u^p, & \text{in } \mathbb{R}^n \\ u > 0, \quad u \in W^{1,2}(\mathbb{R}^n), \end{cases}$$

where $p > 1$ is subcritical and V is a smooth bounded function.

Problem (NLS_ε) arises in the study of the Nolinear Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \Delta \psi + \tilde{V}(x)\psi - |\psi|^{p-1}\psi \qquad \text{in } \mathbb{R}^n,$$

where $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is the *wave function*, $\tilde{V} : \mathbb{R}^n \to \mathbb{R}$ is the potential and $\hbar$ is the *Planck constant*.

Making the "Ansatz" $\psi(t,x) = e^{-i\frac{\omega t}{\hbar}} u(x)$, the function u satisfies (NLS_ε), with $V = \tilde{V} - \omega$ and $\varepsilon = \hbar$. Since $\varepsilon = \hbar$ is very small, one is interested is the asymptotic behavior of solutions in the limit $\varepsilon \to 0$, the so-called $semiclassical\ limit$.

We assume the following conditions on the potential V

(V1) $V \in C^2(\mathbb{R}^n)$, and $\|V\|_{C^2(\mathbb{R}^n)} < +\infty$;

(V2) $\inf_{\mathbb{R}^n} V > 0$.

Theorem. (Floer-Weinstein) Let $n = 1$, let $(V1)$ and $(V2)$ hold, and suppose x_0 is a non-degenerate max or min of V. Then there exists a solution $\bar{v}_\varepsilon$ of (NLS_ε) which concentrates at x_0 as $\varepsilon \longrightarrow 0$.

We say that a solution v_ε of (NLS_ε) *concentrates* at x_0 (as $\varepsilon \longrightarrow 0$) provided

$$(4) \ \ \forall\, \delta > 0, \quad \exists\, \varepsilon_0 > 0, \ R > 0 \ : v_\varepsilon(x) \le \delta, \ \forall\, |x - x_0| \ge \varepsilon\, R, \ \varepsilon < \varepsilon_0.$$

By the change of variable $x \mapsto \varepsilon x + x_0$, (NLS_ε) becomes

$$-\Delta v + V(\varepsilon x + x_0)v = v^p$$

which fits into the preceding perturbative setting with

$$J_0(v) = \frac{1}{2}\int (|\nabla v|^2 + V(x_0)v^2) - \frac{1}{p+1}\int |v|^{p+1}$$

while the perturbation is given by

$$G(\varepsilon, v) = \frac{1}{2}\int (V(\varepsilon x + x_0) - V(x_0))v^2 dx$$

Other results

- joint papers with:

- Badiale-Cingolani
- Malchiodi-W.M. Ni (concentration on a sphere in the radial case)
- Felli, Malchiodi, Ruiz (weakening of $(V2)$)
- Ruiz (systems of NLS, or systems of a NLS with a Poisson eq.)

- Further references: YY. Li, DelPino-Felmer, Malchiodi, etc.

5. Elliptic equations on $\mathbb{R}^n$ with critical exponent

$$(5) \qquad -\Delta u = (1+\varepsilon k(x))u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ u>0,\ n \geq 3.$$

Assumptions on $k(x)$. Let $Cr[k]$, denote the set of critical points of k.

$(k.0)$ $k \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$;

$(k.1)$ k is a Morse function and $\Delta k(x) \neq 0,\ \forall\, x \in Cr[k]$.

$(k.2)$ $\exists\, \rho > 0$ such that $\langle \nabla k(x), x\rangle < 0,\ \forall\, |x| \geq \rho$

$(k.3)$ $\langle \nabla k(x), x\rangle \in L^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \langle \nabla k(x), x\rangle dx < 0$;

For every $x \in Cr[k]$, $m(\nabla k, x)$ will denote the Morse index of x.

Theorem. (AA-Garcia Azorero-Peral) Let $(k.0-3)$ hold and suppose that

$$\sum_{x\in Cr[k],\,\Delta k(x)<0}(-1)^{m(\nabla k,x)}\neq(-1)^n. \tag{6}$$

Then (5) has at least a solution, provided $|\varepsilon|\ll 1$.

Solutions of (5) are the critical points of

$$J_\varepsilon(u)=\tfrac{1}{2}\|u\|^2-\frac{1}{2^*}\int|u|^{2^*}-\frac{\varepsilon}{2^*}\int k(x)|u|^{2^*},$$

where $u\in\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $\|u\|^2=\int|\nabla u|^2$.

Once more, the lack of compactness is bypassed by taking advantage of the fact that the problem is perturbative.

For $\varepsilon = 0$ the unperturbed problem $-\Delta u = u^{\frac{n+2}{n-2}}$ has a ND critical manifold (in $\mathcal{D}^{1,2}(\mathbb{R}^n)$) given by

$$Z = \{z_{\mu,\xi}(x) = \mu^{-(n-2)/2} U\left(\frac{x-\xi}{\mu}\right) : \mu > 0, \xi \in \mathbb{R}^n\},$$

where (up to an unifluent constant)

$$U(x) = \left(\frac{1}{1+|x|^2}\right)^{(n-2)/2}.$$

According to the previous abstract result we are lead to find the "stable" critical points of

$$\Gamma(\mu,\xi) = \int k(x)|z_{\mu,\xi}|^{2^*} dx, \qquad \mu > 0,\ \xi \in \mathbb{R}^n$$

One has:

$$\Gamma(\mu,\xi) = \mu^{-n}\int_{\mathbb{R}^n} k(x)U^{2^*}\left(\frac{x-\xi}{\mu}\right)dx = \int_{\mathbb{R}^n} k(\mu y+\xi)U^{2^*}(y)dy.$$

As a consequence, we can extend Γ to all of $\mathbb{R}^n$ by setting $\widetilde{\Gamma}(0,\xi) = a_0 k(\xi)$ and $\widetilde{\Gamma}(\mu,\xi) = \Gamma(-\mu,\xi)$ if $\mu < 0$. The extended function is of class C^1 and

$$\xi \in Cr[k] \quad \Longleftrightarrow \quad (0,\xi) \in Cr[\widetilde{\Gamma}],$$

Moreover, we find

$$D^2_{\mu\mu}\widetilde{\Gamma}(0,\xi) = a_1 \Delta k(\xi), \quad D^2_{\mu\xi_i}\widetilde{\Gamma}(0,\xi) = 0, \qquad i = 1,\ldots,n.$$

Thus the Hessian matrix $\widetilde{\Gamma}''(0,\xi)$ at any $\xi \in \mathbb{R}^n$ has the form

$$\widetilde{\Gamma}''(0,\xi) \;=\; \begin{pmatrix} a_0 D^2 k(\xi) & 0 \\ & \\ 0 & a_1\,\Delta k(\xi) \end{pmatrix}.$$

Since the critical points of k are non-degenerate and $\Delta k(\xi) \neq 0$, it follows that $(0, \xi)$ is a non-degenerate critical point of $\tilde{\Gamma}$.

Moreover, one has:

$$m(\nabla\widetilde{\Gamma}, (0,\xi)) = \begin{cases} m(\nabla k, \xi) & if \;\; \Delta k(\xi) > 0 \\ m(\nabla k, \xi) + 1 & if \;\; \Delta k(\xi) < 0 \end{cases}$$

Using also $(k.2-3)$ one finds that there exists $R > 0$ such that

$$deg(\nabla\widetilde{\Gamma}, B_R^{n+1}, 0) = (-1)^{n+1}$$

If, by contradiction, $\widetilde{\Gamma}$ has no other critical points but $(0, \xi)$, then

$$deg(\nabla\widetilde{\Gamma}, B_R^{n+1}, 0) = (-1)^{n+1} = \sum_{\xi \in Cr[k]} (-1)^{m(\nabla\widetilde{\Gamma},(0,\xi))}.$$

$$
\begin{aligned}
(-1)^{n+1} &= \sum_{\xi\in Cr[k]} (-1)^{m(\nabla\widetilde{\Gamma},(0,\xi))} \\
&= \sum_{\Delta k(\xi)>0} (-1)^{m(\nabla\widetilde{\Gamma},(0,\xi))} + \sum_{\Delta k(\xi)<0} (-1)^{m(\nabla\widetilde{\Gamma},(0,\xi))} \\
&= \sum_{\Delta k(\xi)>0} (-1)^{m(\nabla k,\xi)} - \sum_{\Delta k(\xi)<0} (-1)^{m(\nabla k,\xi)}.
\end{aligned}
$$

On the other hand, from $(k.2)$ it immediately follows that $deg(\nabla k, B^n_R, 0) = (-1)^n$ and hence

$$
\sum_{\xi\in Cr[k]} (-1)^{m(\nabla k,\xi)} = \sum_{\Delta k(\xi)>0} (-1)^{m(\nabla k,\xi)} + \sum_{\Delta k(\xi)<0} (-1)^{m(\nabla k,\xi)} = (-1)^n
$$

This and the preceding equation yield

$$
\sum_{\xi\in Cr[k],\Delta k(\xi)<0} (-1)^{m(\nabla k,\xi)} = (-1)^n,
$$

a contradiction.

6. The scalar curvature problem

Let (S^n, g_0) denote the n-dimensional sphere endowed with the standard metric. Given a function R, the scalar curvature problem amounts to finding a metric g conformally equivalent to g_0 such that the scalar curvature of (S^n, g) is R.

A necessary condition for the existence is that $\max_{S^n} R > 0$. Other integral necessary conditions have been found by Kasdan and Warner.

Consider, for simplicity, the case $n = 3$.

Theorem. Suppose that $R > 0$ is a C^2 Morse function such that $\Delta_{g_0} R(y) \neq 0$ for all $y \in Cr(R)$. Furthermore, let us assume that

$$\sum_{y \in Cr[R],\, \Delta_{g_0} R(y) < 0} (-1)^{m(\nabla R, y)} \neq -1.$$

Then the SC problem has a solution.

Setting $g = u^{4/(n-2)} g_0$ $(n \geq 3)$, and using stereographic co-ordinates with north pole the point of minimum of R on S^n, one finds that u satisfies the equation

$$-\Delta u = K(x) u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ u > 0,$$

where $K(x) = R \circ \pi^{-1}$ and $\pi : S^n \mapsto \mathbb{R}^n$ denotes the ster. proj.

First, one considers the case $K = 1 + \varepsilon k$. It is easy to check that k satisfies the assumptions made before and one applies the previous theorem.

Next, one performs an homotopy between $\varepsilon \sim 0$ and $\varepsilon = 1$. This involves difficult a-priori estimates. In dimension $n > 3$ this last step requires further restrictions.

Ref.: Moser, A.Chang-P.Yang, Gursky, Y.Y.Li, Bahri-Coron, A.A-YY.Li-Malchiodi, etc.

Recent trends: prescribing the Paneitz Curvature (a 4th order invariant) on S^4 (A.Chang-P.Yang, A.Malchiodi)

7. Vortex Rings in an Ideal Fluid

Consider a perfect fluid filling all of $\mathbb{R}^3$ and suppose it is cylindrically symmetric.
In terms of the stream function $\Psi(r, z)$ defined on $\Pi = \{(r, z) : r > 0,\ z \in \mathbb{R}\}$, the velocity is given by

$$q = \left(-\frac{\Psi_z}{r}, 0\,, \frac{\Psi_r}{r} \right).$$

A *vortex* is a toroidal region $\mathcal{R}$, such that $curl(q) \neq 0$ if and only if $(r, z) \in \mathcal{R}$.Let A, the *vortex core*, denote the cross section of the vortex $\mathcal{R}$. One is lead to the look for a bounded set $A \subset \Pi$ and Ψ such that

$$-L\Psi = -\left(r \left(\frac{1}{r} \Psi_r \right)_r + \Psi_{zz} \right) = \begin{cases} r^2\Psi & \text{in } A \\ 0 & \text{in } \Pi \setminus \overline{A}. \end{cases}$$

where

$$curl(q) = (0, -L\Psi/r\, 0).$$

The preceding equation is completed by suitable boundary conditions.

We require

$$\Psi(0,z) = -k,\ \forall\, z \in \mathbb{R}, \quad \text{and} \quad \Psi(r,z) = 0,\ \forall\,(r,z) \in \partial A,$$

and

$$\mathbf{q} \to (0,0,-W), \qquad \text{as } r^2 + z^2 \to +\infty.$$

The first condition prescribes the amount $k \geq 0$ of fluid flowing between the stream surfaces $r = 0$ and ∂A.
The second boundary condition demands that $\Psi \to -\frac{1}{2}Wr^2 - k$ at infinity.

Introducing the Heaviside function H and setting

$$\psi(r,z) = \Psi(r,z) + \tfrac{1}{2}Wr^2 + k,$$

the preceding problem becomes

$$(7) \qquad \begin{cases} -L\,\psi &= r^2 H(\psi - \frac{1}{2}Wr^2 - k) & (r,z) \in \Pi, \\ \psi &= 0 & \text{on } r = 0, \\ \psi &\to 0 & \text{as } r^2 + z^2 \to +\infty, \\ |\nabla\psi|/r &\to 0 & \text{as } r^2 + z^2 \to +\infty. \end{cases}$$

By a solution of (7) we mean a function ψ of class $C^1(\Pi) \cap C^2(\Pi \setminus \partial A)$ which solves (7) almost everywhere.

Theorem. (A.A.-M.Struwe) For all $k, W > 0$ there exists a solution $\psi(r, z) = \psi(r, -z)$ of (7) such that the corresponding vortex core $A = \{(r, z) \in \Pi : \psi(r, z) > \frac{1}{2}Wr^2 + k^2\}$ is non-empty and bounded.

Roughly, we approximate (7) Dirichlet problems on balls B_R. The corresponding Euler functional J_R behaves as follows.

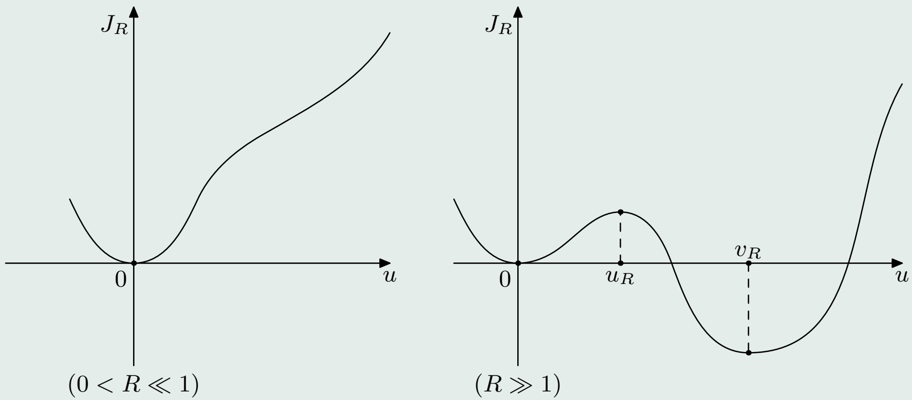


Figure 2: Behavior of J_R

For $R \gg 1$, J_R has a local minimum v_R and a mountain pass critical point u_R.

• the minimum v_R "blows up" as $R \to \infty$,

while

• the MP critical point u_R converges to the solution of (7).

Roughly, the MP critical level $c(R)$ is non-increasing and hence almost everywhere differentiable. Taking a sequence $R_k \to +\infty$ where $c'(R_k)$ exists, one shows that

$$\|u_{R_k}\|^2_{H^1_0(B_{R_k})} \leq 5\,[c(R_k) + 2R|c'(R_k)| + 4] \leq Const.$$

Then u_{R_k} converges to a solution u of (7) with the properties listed before.