On the Optimal Harvesting of Marine Resources

Alberto Bressan

Department of Mathematics

Penn State University

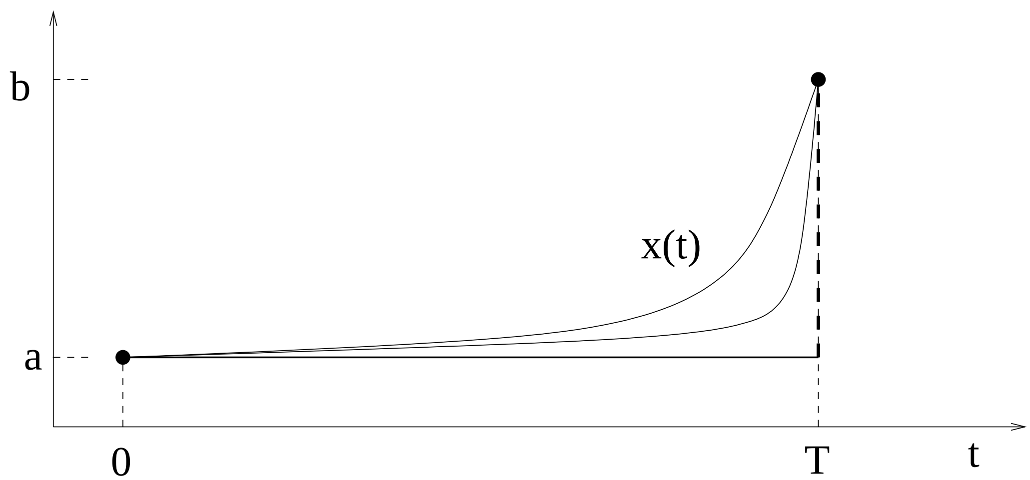
Variational problems with slow growth

$$\text{minimize:} \quad \int_0^T L\big(t,\, x(t),\, \dot{x}(t)\big)\, dt$$

$$x(0) = a \qquad\qquad x(T) = b$$

Tonelli's assumption: $$\frac{L(t,x,p)}{|p|} \to \infty \qquad \text{as} \quad |p| \to \infty$$

otherwise optimal solution can have impulsive nature



The Scalar Optimization Problem

$$\frac{d}{dt}\phi = a\phi\left(b - \phi\right) - \phi\, u$$

$\phi(t)$ = total amount of fish (or marine resource)
$u(t)$ = intensity of fishing (or harvesting)
a = reproduction rate
b = maximum population supported by the habitat

Steady state, normalized with $b = 1$:

$$\phi(1 - \phi) = \phi u$$

Optimization problem:

$$\max_{u \geq 0} \, (\phi - c)\, u$$

c = unit cost of fishing (harvesting) effort $\qquad 0 \leq c \leq 1$

Optimal solution:

$$\phi = 1 - u \qquad\qquad \max_{u \geq 0} \, (1 - u - c)u$$

$$u = \frac{1 - c}{2} \qquad\qquad \phi = \frac{1 + c}{2}$$

A non-cooperative game

$$\phi(1-\phi) - \sum_{i=1}^{n} \phi\, u_i = 0$$

u_i = intensity of fishing by the i-th company

c_i = cost for the i-th company

Nash equilibrium solution : $(u_1^*, \ldots, u_n^*)$

Optimization problem for the i-th player:

$$\max_{\omega \geq 0} \; (\phi - c_i)\omega$$

$$\phi = 1 - \sum_{j \neq i} u_j^* - \omega$$

$$u_i^* = \arg\max_{\omega \geq 0} \; \Big(1 - \sum_{j \neq i} u_j^* - \omega - c_i\Big)\omega$$

$$u_i^* = \frac{1 - \sum_{j \neq i} u_j^* - c_i}{2} \qquad\qquad i = 1, \ldots, n$$

Symmetric case: $c_i = c \qquad u_i^* = u^* \qquad i = 1, \ldots, n$

$$u^* = \frac{1 - (n-1)u^* - c}{2}$$

$$u^* = \frac{1-c}{1+n} \qquad\qquad \phi = \frac{1+cn}{1+n} \quad \to c \quad \text{as} \quad n \to \infty$$

A Spatially Dependent Model

$\phi(t,x)$ = density of fish population
$u_k(t,x)$ = intensity of fishing by k-th company

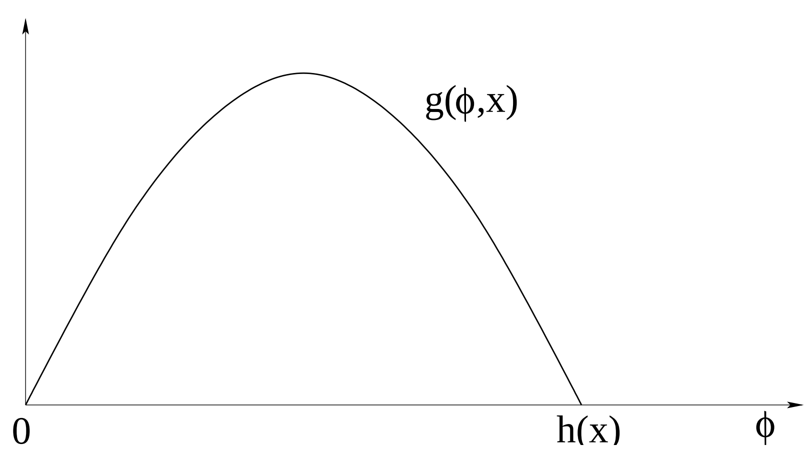
$$\phi_t = \sum_{i,j=1}^{m} \big(a^{ij}(x)\phi_{x_i}\big)_{x_j} - \sum_{i=1}^{m} \big(b_i(x)\phi\big)_{x_i} + g(x,\phi) - \phi \sum_{k=1}^{n} u_k$$

$$x \in \Omega \subset I\!R^m$$

$$\mathbf{n} \cdot \nabla\phi = 0 \qquad\qquad x \in \partial\Omega$$

$g(x,\phi) = \alpha(x)\big(h(x)-\phi\big)\phi$ = local growth rate of the fish population

$h(x)$ = maximum population sustained by the habitat at x



The steady state solution

$$\sum_{i,j=1}^{m} \left(a^{ij}(x)\phi_{x_i}\right)_{x_j} - \sum_{i=1}^{m} \left(b_i(x)\phi\right)_{x_i} + g(x,\phi) - \phi \cdot \sum_{k=1}^{n} u_k = 0 \qquad x \in \Omega \tag{1}$$

$$\mathbf{n} \cdot \nabla\phi = 0 \qquad x \in \partial\Omega \tag{2}$$

Assume: $g(x,\phi) = f(x,\phi)\,\phi$, f strictly decreasing

- Given $u_1, \ldots, u_n$, a nontrivial positive solution of (1)-(2) exists iff there exists a positive solution to the linearized problem

  $$\sum_{i,j=1}^{m} \left(a^{ij}(x)\phi_{x_i}\right)_{x_j} - \sum_{i=1}^{m} \left(b_i(x)\phi\right)_{x_i} + \left(f(x,0) - \sum_{k=1}^{n} u_k\right)\phi = \lambda\phi \qquad x \in \Omega$$

  for some $\lambda > 0$.

- If there exists a uniformly positive solution of (1)-(2), this is unique

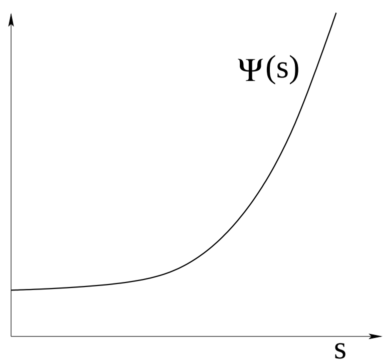
Net profit for the k-th fishing company

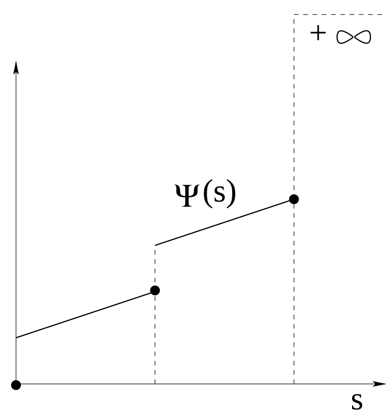
$$J_k = \int_\Omega \big(\phi(x) - c_k(x)\big)\, u_k(x)\, dx$$

$c_k(x) = +\infty$ if k-th company is not allowed to fish at x (inside a marine reserve)

Possible additional cost:

$$[\text{cost to the } k\text{-th company}] = \Psi\left(\int u_k(x)\, dx\right)$$





Optimization Problems

$$\sum_{i,j=1}^{m} \big(a^{ij}(x)\phi_{x_i}\big)_{x_j} - \sum_{i=1}^{m} \big(b_i(x)\phi\big)_{x_i} + g(x,\phi) - \phi \sum_{k=1}^{n} u_k = 0 \qquad x \in \Omega \subset I\!R^m$$

$$\mathbf{n} \cdot \nabla\phi = 0 \qquad\qquad x \in \partial\Omega$$

1 - Optimal control problem: $(n = 1)$

$$\text{maximize:} \quad \int_\Omega \big(\phi(x) - c(x)\big)\, u(x)\, dx$$

$$\text{subject to} \qquad \Delta\phi + (1-\phi)\phi - \phi\, u = 0 \qquad\qquad x \in \Omega$$

If the control u does not uniquely determine $\phi > 0$, one can seek an optimal pair (u, ϕ)

2 - Non-cooperative game:

Study Nash equilibrium solutions $(u_1^*, \ldots, u_n^*)$

Strategy $u_k^*(x)$ for the k-th player solves the optimization problem

$$\max_{\omega \geq 0} \int_\Omega \big(\phi(x) - c(x)\big)\, \omega(x)\, dx$$

subject to:

$$\Delta\phi + (1-\phi)\phi - \phi \sum_{i \neq k} u_i^* - \phi\, \omega = 0 \qquad\qquad x \in \Omega$$

3 - Optimization of the non-cooperative equilibrium solution

If a set $S \subset \Omega$ is set aside as marine park, then

$$c_i(x) = +\infty \qquad\qquad x \in S\,, \quad i = 1, \ldots, n$$

Problem: select S so that the corresponding Nash equilibrium solution $(u_1^*, \ldots, u_n^*)$ maximizes the total catch

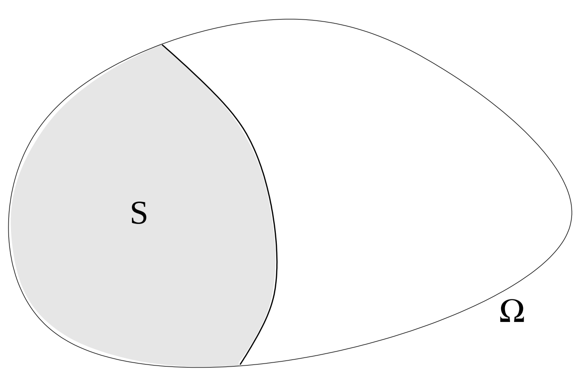
$$\max_{S \subset \Omega} \int_{\Omega \setminus S} \phi(x) \cdot \sum_{i=1}^{n} u_i^*(x)\, dx$$

Remark: in all these problems, cost functionals have **slow growth** w.r.t. the pointwise values of u.

The optimal strategies μ_i may well be measure valued !

$$\Delta\phi + (1-\phi)\phi = \phi \cdot \sum_i \mu_i$$

A positive amount of harvesting is expected along the boundary of a marine reserve



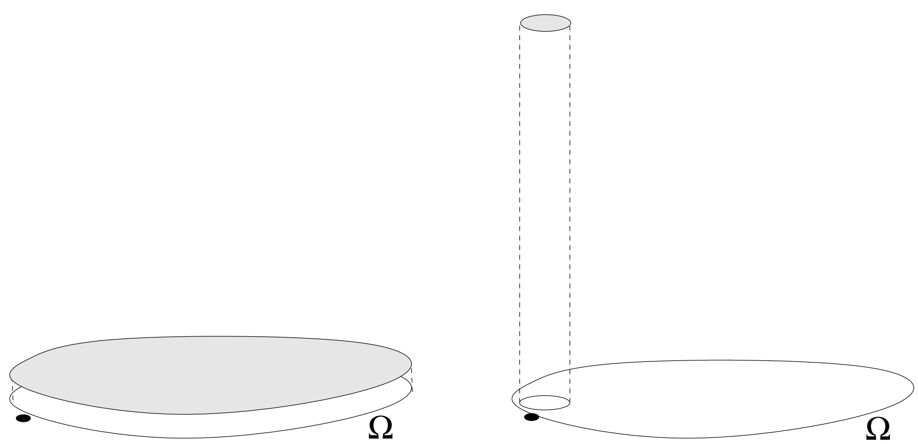
Results on the Optimal Control Problem

$$\Delta\phi + \phi(1-\phi) - \phi\, u = 0 \qquad\qquad x \in \Omega$$

$$\phi = 0 \qquad\qquad x \in \partial\Omega$$

$$\text{maximize: } \; J(u) = \int_\Omega \phi(x)u(x)\,dx - \int_\Omega \big(B_1 u(x) + B_2 u^2(x)\big)\,dx$$

(cost has quadratic growth)



Marine Parks ?

Michael G. Neubert, Marine reserves and optimal harvesting
Ecology Letters **6** (2003)

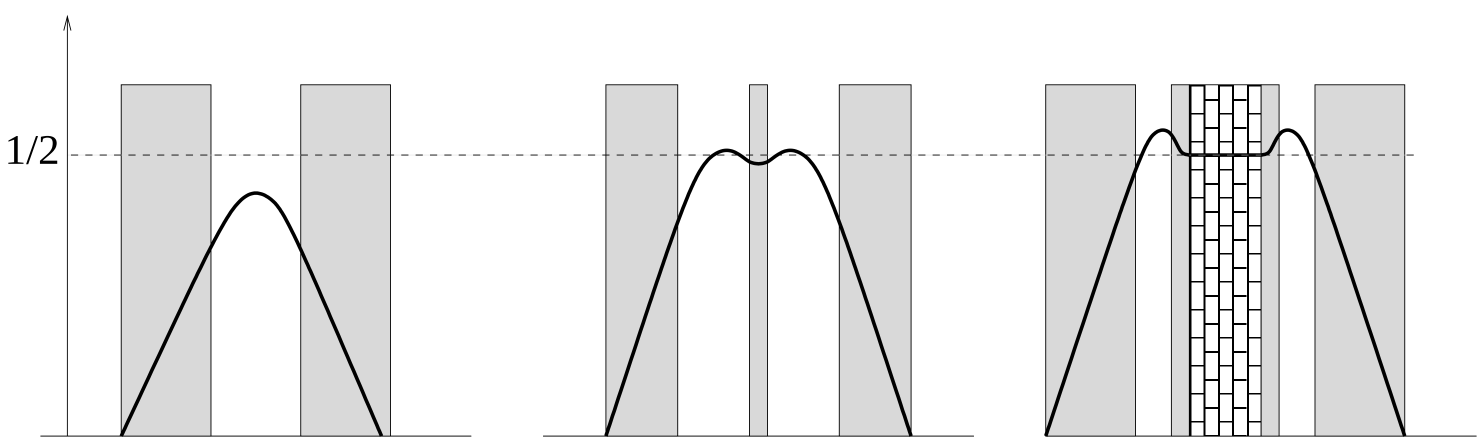
$$\phi'' + \phi(1-\phi) = \phi u \qquad\qquad x \in [0,b]$$

$$\phi(0) = \phi(b) = 0$$

$$\text{maximize: } J(u) = \int_0^b \phi(x)u(x)\,dx$$

$$\text{with pointwise constraint:} \qquad 0 \le u(x) \le M$$

- fish instantly killed as it reaches the boundary
- zero harvesting cost
- pointwise upper bound on harvesting effort



- The problem with boundary conditions $\phi'(0) = \phi'(b) = 0$ has the trivial solution

$$u(x) \equiv \frac{1}{2} \qquad\qquad \phi(x) \equiv \frac{1}{2}$$

Asymptotic limit - large number of fishermen

Optimal strategy for each individual player:

$$\max_{\mu} \int_{\Omega} \big(\phi(x) - c(x)\big)\, d\mu$$

subject to

$$\Delta\phi + \phi(1-\phi) = (n-1)\phi\mu^* - \phi\mu \qquad x \in \Omega$$

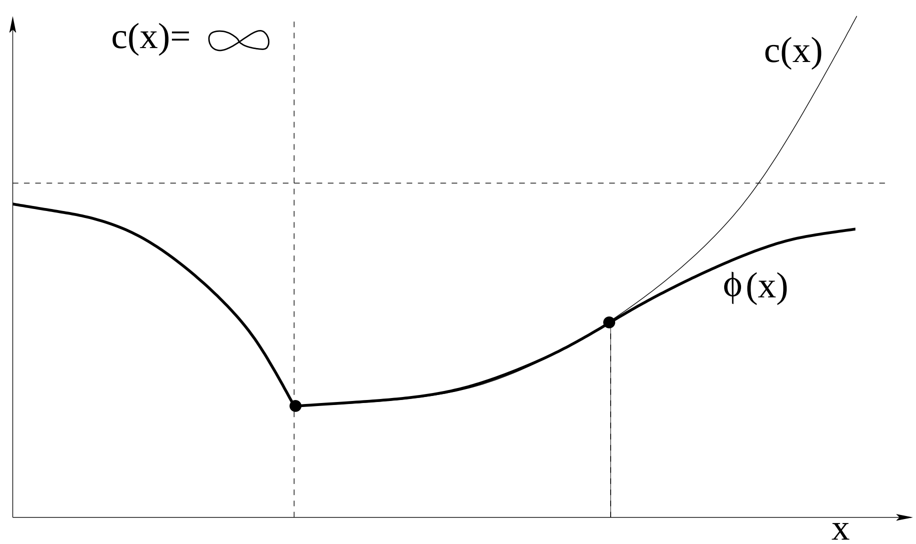
$$\mathbf{n} \cdot \nabla\phi = 0 \qquad x \in \partial\Omega$$

Nash equilibrium: $\mu = \mu^*$

Expect: as $n \to \infty$, fish density $\phi(\cdot)$ approaches the solution to a free boundary problem

$$\phi(x) = c(x) \qquad x \in Supp(\mu)$$

$$\phi(x) < c(x), \qquad \Delta\phi + \phi(1-\phi) = 0 \qquad x \notin Supp(\mu)$$



Optimal design of a marine park

$$c(x) = \begin{cases} c & \text{if} \quad x \notin S \\ +\infty & \text{if} \quad x \in S \end{cases}$$

Choose S such that the corresponding solution ϕ^S satisfies

$$\max_{S \subset \Omega} \int_\Omega \phi^S (1 - \phi^S) \, dx$$