From counting points to motivic integration: the geometry behind computing integrals

François Loeser

Ecole normale supérieure, Paris

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The set of k-points of the corresponding (affine) algebraic variety X_F is the set of points in k^N which are common zeroes of the polynomials f_i, that is,

$$X_F(k) = \{x = (x_1, \ldots, x_n) \in k^N \colon (\forall i)(f_i(x_1, \ldots, x_N) = 0)\}.$$

For any ring K containing k, we can also consider the set of K-points

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In particular, if $r = 0$, we get the affine space $\mathbb{A}^N$ with $\mathbb{A}^N(K) = K^N$ for every K containing k.

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of formal germs of arcs on X_F may be extremely powerful.

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We write $X_{F'} \subset X_F$ and we say $X_{F'}$ is a subvariety of X_F.

Do there exist similarly a family $\tilde{F}$ such that

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Yes, add new variables U_i for each g_i and set

$$\tilde{F} = (f_i, g_i U_i - 1).$$

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If we have two families $F = (f_i(T_1, \ldots, T_N))$ and $F' = (f_i'(S_1, \ldots, S_{N'}))$, with no variable in common, we may set

$$X_F \times X_{F'} := X_{F \cup F'}.$$

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For instance, $T \mapsto (T^2, T^3, T^{-2})$ induces an isomorphism between $\mathbb{A}^1 \setminus \{0\}$ and the variety defined by

$$X_1^3 - X_2^2 = 0 \quad \text{and} \quad X_1 X_3 - 1 = 0.$$

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- Multiplicativity:

$$\lambda(X \times X') = \lambda(X) \cdot \lambda(X').$$

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Convention: open balls of dimension 0 are points.

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Proposition

$X \mapsto \mathrm{Eu}\,(X)$ *is an additive* $\mathbb{Z}$*-valued invariant.*

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Is there any relation between $N_{p^e}(X_p)$ and $\mathrm{Eu}(X)$?

The following is a consequence of results by A. Grothendieck going back to the 60's:

Theorem (Crude Form)

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More precisely:

Theorem (Precise Form)

Given a X, for almost all p, there exists finite families of complex numbers α_i, $i \in I$, and β_j, $j \in J$, depending only on X and p, such that

$$N_{p^e}(X_p) = \sum_I \alpha_i^e - \sum_J \beta_j^e$$

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So, Euler characteristics may be computed by counting in finite fields!

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$$\sum_{i \geq -\alpha} a_i t^i,$$

with $\alpha \geq 0$.

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Similarly, the field of p-adic numbers $\mathbb{Q}_p$ is the set of series $\sum_{i\geq -\alpha} a_i p^i$, with a_i in $\{0,\ldots,p-1\}$ and $\alpha \geq 0$.

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However, they are very much the same: by the Ax-Kochen-Eršov principle, we shall discuss later in the talk, they are asymptotically, that is, for $p \gg 0$, the same.

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Proper means $h^{-1}(\text{compact}) = \text{compact}$.

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Note that $E_\emptyset^\circ = Y \setminus F$ and Y is the disjoint union of all the E_I°'s.

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Theorem (Denef and Loeser)

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Remark

The result also holds in the complex analytic setting.

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Challenging problem: Find a direct proof ...

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Inspired by mirror symmetry, physicists were led to conjecture the following statement:

"Birational Calabi-Yau have the same Betti numbers."

In 1995, this was proved by V. Batyrev by using p-adic integrals in a way similar to Denef and Loeser and the part of the Weil conjectures proved by Deligne (which allows for projective varieties to deduce not only Euler characteristics, but also Betti numbers, from counting in finite fields).

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Motivic integration was born . . .

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$$X \mapsto [X]$$

from the category of algebraic varieties over k to some universal ring M_k.

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there exists a unique ring morphism $\alpha : M_k \to R$ such that

$$\lambda(X) = \alpha([X])$$

for every X.

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Note that, in general $\mathcal{L}(X)$ is infinite dimensional!

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Such attempts are doomed to fail immediately since, as soon as k is infinite, $k((t))$ is not locally compact.

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In fact a very general theory of motivic integration within the framework of "Constructible motivic functions" has been recently developed by Cluckers and Loeser. It allows to consider integrals with parameters and avoids using a completion M_k.

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The basic idea is to use the truncation morphisms:

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Note that $\mathcal{L}_n(X)$ is finite dimensional and $[\mathbb{A}^1]$ is invertible in M_k.

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Similarly, one can avoid the use of p-adic integration in the proof of the Denef-Loeser Theorem.

What is the underlying idea?

If $h : Y \to X$ is a birational morphism, one can express the motivic volume of $\mathcal{L}(X)$ as a motivic integral on $\mathcal{L}(Y)$ involving the order of vanishing of the jacobian.

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This is achieved by using an analogue of the "change of variables formula" in this setting.

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At the level of arc spaces h induces a morphism between $\mathcal{L}(Y)$ and $\mathcal{L}(X)$ which restricts to a bijection between $\mathcal{L}(Y) \setminus \mathcal{L}(F)$ and $\mathcal{L}(X) \setminus \mathcal{L}(h(F))$.

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This is the reason why measure theoretic arguments seem to be so well adapted to birational geometry.

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But there exists one at the level of arc spaces!

Indeed, let x be a point of X and denote by $G(x)$ the isotropy subgroup at x, Indeed, let x be a point of X and denote by $G(x)$ the isotropy subgroup at x, consisting of those elements of G fixing x.

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with $\mathrm{Conj}\, G(x)$ the set of conjugacy classes in $G(x)$ and B a subset of infinite codimension in $\mathcal{L}(X)_x$.

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Work of Batyrev, Kontsevich, Denef-Loeser, Yasuda, ...

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$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \to B(0, \eta) \setminus \{0\}.$$

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Here $B(a, r)$ denotes the closed ball of center a and radius r.

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$$f(\varphi(t)) = t^n + \text{(higher order terms)}.$$

Theorem (Denef-Loeser)

For $n \geq 1$,

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In fact, the spaces $\mathcal{X}_n$ do contain much more information about the Milnor fiber and the monodromy:

Denef and Loeser proved that the series

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Work of Denef-Loeser, Guibert, Bittner, Guibert-Loeser-Merle, etc.

A first order ring formula is a formula written with symbols 0, $+$, $-$, 1, $\times$, $=$, A first order ring formula is a formula written with symbols 0, $+$, $-$, 1, $\times$, $=$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), A first order ring formula is a formula written with symbols 0, $+$, $-$, 1, $\times$, $=$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), quantifiers $\exists$, $\forall$, and variables.

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A (first order ring) sentence is a formula with no free variable.

Theorem (Ax-Kochen-Eršov)

Let φ be a first order sentence. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.

Let d be a positive integer. A field k is called $C_2(d)$ if every homogenous polynomial of degree d in $n > d^2$ variables with coefficients in k has a non trivial solution in k^n.

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Question (E. Artin): Does $\mathbb{Q}_p$ have the $C_2(d)$ property?

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Now consider a formula φ with n free variables.

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More generally one may extend the valued ring language which admits symbols to express that the valuation is larger than something, or the initial coefficient is equal to someting.

Theorem (Denef-Loeser)

Let φ be a formula in the valued ring language. Then, for almost all p, the sets $X_\varphi(\mathbb{Q}_p)$ and $X_\varphi(\mathbb{F}_p((t)))$ have the same volume. Furthermore this volume is equal to the number of points in $\mathbb{F}_p$ of a motive M_φ canonically attached to φ.

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There is a similar statement for integrals.

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By the above mentioned work of Cluckers and Loeser, all p-adic integrals depending on parameters that are definable in a precise sense may be obtained by specialization of canonical motivic integrals for almost all p, and similarly for $\mathbb{Q}_p$ replaced by $\mathbb{F}_p((t))$.

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To conclude, let us note that integrals of the above type are ubiquitous in harmonic analysis over non archimedean fields, p-adic representation Theory and the Langlands Program.