Introduction to constructive mathematics

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Timetable: 8 hrs. First lecture on April 11, 2013, 11:00 (dates already fixed, see the calendar), Torre Archimede, Room 2BC/30.

Course requirements: Basic knowledge of the real analysis, metric spaces, normed spaces, functional analysis, and - for the later lectures - topological spaces. Some familiarity with logic would be good but by no means necessary.

Examination and grading: Students will receive a scientific publication to be reviewed.

SSD: MAT/01 Mathematical logic, MAT/05 Mathematical Analysis

Aim: Many, perhaps most, existence proofs in classical (i.e. standard) mathematics start by assuming the non-existence of the object in question, and then derive a contradiction from which we conclude that the object exists after all. This procedure is nonconstructive: that is, it does not provide an algorithm for the construction/computation of the desired object. In constructive mathematics all proofs must be algorithmic; in particular, if you want to prove that a certain object exists, you must produce an algorithm for its construction. In this course we deal with Bishop-style constructive mathematics, in which a change of logic, from classical to intuitionistic, enables us to develop large parts of analysis, algebra, topology, ... in a natural way, that is, without framing everything in terms of, for example, recursive function theory. The lectures begin with an informal outline of the differences between constructive and nonconstructive arguments; this leads to informal intuitionistic logic and set theory. Then we deal with the construction of the real line R and with its completeness and uncountability. From there we move to metric, normed, and Hilbert spaces, including the Stone-Weierstrass theorem, Chebyshev approximation theory, separation theorems, the Hahn-Banach theorem, and results about operators on a Hilbert space. The final lectures will introduce the constructive theory of apartness spaces, an approach to constructive topology rather different to the Sambin one of formal topology. This will include discussion of the primary examples of apartness spaces: uniform spaces. To summarize: the aim of the course is to introduce and explore the distinction between nonconstructive existence and constructive existence, and to show that a fully constructive approach to mathematics leads to significant results with algorithmic proofs (from which programs can be extracted, though this aspect will not be dealt with here).

Course contents:

1. Fundamentals of constructive mathematics: Introduction to constructive logic and foundations; the BHK interpretation of logic; omniscience principles; varieties of constructive mathematics: BISH, INT, and RUSS. The real line R; metric and normed spaces; the fundamental theorem of approximation theory. Hilbert space, projections, and compact operators.

2. Constructive functional analysis: Convexity, boundary crossings, and separation theorems; the Hahn-Banach theorem and its applications; locally convex spaces. The Banach-Alaoglu theorem; the weak operator topology and the characterization of weak-operator continuous linear functionals on B(H).

3. Informal constructive reverse mathematics: What is (constructive) reverse mathematics? Fan theorems and their equivalents relative to BISH; uniform continuity theorems; anti-Specker properties; application to Peano's existence theorem for ordinary differential equations.
4. Apartness and uniformity: Point-set apartness; apartness from topology; topology from apartness; types of continuity. Quasi-uniform spaces. Set-set apartness; apartness and uniformity; strong and uniform continuity; proximal convergence of sequences of functions.

References:

- E. Bishop, *Foundations of Constructive Analysis*, McGraw- Hill, New York 1967.
- E. Bishop and D.S. Bridges, *Constructive Analysis*, Grundlehren der Math. Wissenschaften 279, Springer-Verlag, Heidelberg, 1985.
- D.S. Bridges and L.S. Vîţă, *Techniques of Constructive Analysis*, Universitext, Springer New York, 2006.
- D.S. Bridges and L.S. Vîţă, *Apartness and Uniformity-a Constructive Development*, in: CiE series "Theory and Applications of Computability", Springer Verlag, Heidelberg, 2011.