EXISTENCE, UNIQUENESS AND REGULARITY FOR SOME CLASSES OF PARABOLIC PROBLEMS

preliminary lesson for the PhD course "Introduction to mean field games"

MAIN LITERATURE:

*) Lieberman "Second order parabolic differential equations"

*) Ladyženskaya, Solonnikov, Ural'ceva "Linear and quasilinear equations of parabolic type.

SETTING OF THE PROBLEM

Consider the Cauchy-Dirichlet problem for a semilinear parabolic equation:

$$(CDP)\quad \begin{cases} Pu := u_t - \Delta u + H(x, Du) = 0 & \text{in } \omega \times (0,T) \\ u = \varphi_1 & \text{on } \omega \times \{0\} \\ u = \varphi_2 & \text{on } \partial\omega \times (0,T) \end{cases}$$

where ⊛ ω is a domain (= open connected) of $\mathbb{R}^n$, bounded

⊛ $u : \omega \times (0,T) \to \mathbb{R}$ is a scalar function

⊛ $u_t \equiv \frac{\partial u}{\partial t}$; Δu = Laplacian of $u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$; Du = gradient of $u = \left(\frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_n}\right)$.

(Rmk: the Laplaciano and the gradient are only in x and not in t)

Notations: $\Omega = \omega \times (0,T)$; if $(x,t) \in \Omega$, we write $X = (x,t)$, and similarly for $Y = (s,t)$.

For simplicity we consider $\omega = B(0,R) = \{x \in \mathbb{R}^n : |x| < R\}$

$B\Omega := \omega \times \{0\}$ = bottom of Ω

$S\Omega := \partial\omega \times (0,T)$ = side of Ω

$C\Omega := \partial\omega \times \{0\}$ = corner of Ω.

$B\Omega \cup S\Omega \cup C\Omega = \mathcal{P}\Omega$ "Parabolic boundary".

Using these notations, the problem (CDP) can be rewritten as:

$$(CPD')\quad \begin{cases} Pu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \mathcal{P}\Omega \end{cases}$$

<u>Aim</u>: we consider H "nice" with subquadratic growth in $p = Du$ and we want to prove that (under suitable compatibility and regularity conditions), problem (CDP) has exactly one classical solution; in other words: $\exists\, u : \bar{\Omega} \to \mathbb{R}$, with $u \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ such that u fulfills (CDP) pointwise, where:

$$C^{2,1}(\Omega) := \left\{ v : \Omega \to \mathbb{R} : v, \partial_t v, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j} \in C^0(\Omega) \ \forall i,j \right\}.$$

Example: $H(x,p) = A(x,t)\,|p|^\theta + V(x,t)$ $\quad \theta \in [0,2]$

with A and V Lipschitz continuous in $\bar{\Omega}$.

<u>Remark</u>: $Pu = 0$ is semilinear and in divergence form; hence we can also introduce a definition of WEAK SOLUTIONS.

u is a subsolution to $Pu \leq 0$ if $H(x, Du)$ is well defined and is in $\mathbb{L}^1_{loc}(\Omega)$ and if

$$\int_\Omega \left[Du\, D\eta - H(x,Du)\,\eta - u\,\eta_t \right] dX \leq 0$$

$\forall\, \eta \in C^1_{\mathcal{P}} := \{ \eta \in C^1(\bar{\Omega}),\ \eta = 0 \text{ on } \mathcal{P}\Omega \}$, $\eta \geq 0$.

By usual arguments we can choose $\eta \in W^{1,2}_{\mathcal{P}}(\Omega) := \left\{ \begin{array}{l} \text{limit of } C^{1}_{\mathcal{P}} \text{ functions} \\ \text{in } W^{1,2}(\Omega) \end{array} \right\}$

We shall follow the theory of quasilinear equations in nondivergence form, given in [Lieberman], namely for equations of the type

$$u_t - \sum a_{ij}(X, u, Du)\frac{\partial^2 u}{\partial x_i \partial x_j} + a(X, u, Du) = 0.$$

UNIQUENESS

Proposition 1 [Lieberman, Corollary 9.2]. Assume: $u, v \in C^{2,1}(\Omega)\cap C^0(\bar\Omega)$, $Pu \le Pv$ in Ω and $u \le v$ on $\mathcal{P}\Omega$. Then $u \le v$ in $\bar\Omega$.

Proof: case 1: $u, v \in C^{2,1}(\bar\Omega \setminus \mathcal{P}\Omega) \cap C^0(\bar\Omega)$ (i.e.: u and v are regular on $\partial\Omega \setminus \mathcal{P}\Omega$).

We set $w := (u-v)e^{\lambda t}$ for some $\lambda < 0$. Since $u, v \in C^0(\bar\Omega)$ the function w reaches its maximum at some point $(x_0, t_0) = X_0 \in \bar\Omega$.

If $X_0 \in \mathcal{P}\Omega$, then $\max w \le 0 \Rightarrow$ OK.

If $X_0 \in \bar\Omega \setminus \mathcal{P}\Omega$ then we have:

$$Du(X_0) = Dv(X_0)$$
$$\Delta u(X_0) \le \Delta v(X_0)$$
$$u_t(X_0) - v_t(X_0) \ge -\lambda\big(u(X_0) - v(X_0)\big)$$

Therefore, we have:

$$0 \ge \Big[(u-v)_t - \Delta(u-v) + H(x, Du) - H(x, Dv)\Big]_{|X_0}$$
$$\ge -\lambda\big(u(X_0) - v(X_0)\big) \Rightarrow \text{OK}$$

case 2: general case.

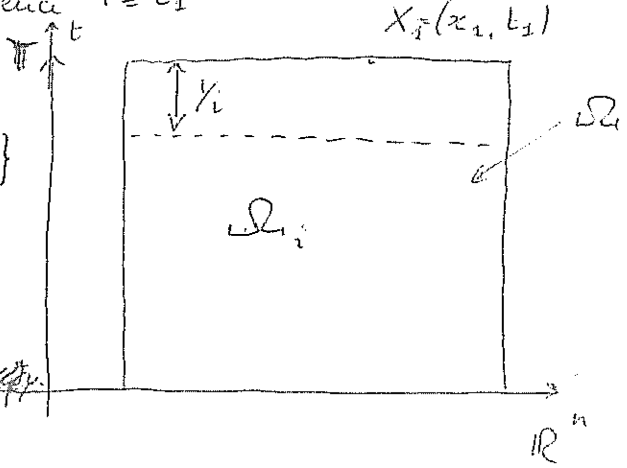
Consider $M := \max_{\bar\Omega}(u-v) = u(X_1) - v(X_1)$ for some $X_1 \in \bar\Omega$.

If $M \le 0 \Rightarrow$ OK. Assume by contradiction $M > 0$. By case 1 $X_1 \notin \Omega \cup \mathcal{P}\Omega$; in other words, if $(u-v)$ has a positive max it must occur in $\bar\Omega \setminus \mathcal{P}\Omega$. Hence $T = t_1$

Set $t_i := T - 1/i$

$\Omega_i := \{X \in \Omega : X = (x,t),\ t \le t_i\}$

$M_i := \max_{\bar\Omega_i}(u-v)$



For i sufficiently large, Ω_i is non empty and $M_i > 0$.

Set $u(X_i) - v(X_i) = M_i$ for some $X_i \in \bar\Omega_i$. Impossible by case 1

Corollary 1 [L., corollary 9.3]. Assume $u, v \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ with

$$\begin{cases} Pu = Pv = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega \end{cases}.$$

Then $u = v$ in $\overline{\Omega}$.

Lemma 1 [L., lemma 9.4]. Suppose $u, v \in C^{2,1}(\overline{\Omega} \setminus \partial\Omega) \cap C^0(\overline{\Omega})$ with $Pu < Pv$ in Ω and $u < v$ on $\partial\Omega$. Then: $u < v$ on $\overline{\Omega}$.

Proof: Assume $u(X_0) = v(X_0)$ for some $X_0 \in \Omega \setminus \partial\Omega$ and

$u(x,t) < v(x,t)$ $\forall (x,t) \in \overline{\Omega}$ with $t < t_0$.

(i.e.: t_0 is the first time when $u = v$ in some point).

Therefore X_0 is a maximum point for $w := u - v$. Hence, we have

$$Du(X_0) = Dv(X_0) \qquad \Delta w(X_0) \leq 0 \qquad w_t(X_0) \geq 0$$

In particular, we have,

$$0 > \left[(u-v)_t - \Delta(u-v) + H(X, Du) - H(X, Dv)\right]_{|X_0} \geq 0 .$$ Contradiction ▨

For weak solution, we have:

Proposition 2 [L., theorem 9.7]. Suppose $H \in C^1$. If $u, v \in C^1(\overline{\Omega})$ are

$$\begin{cases} Pu \leq Pv & \text{in } \Omega \quad \text{in weak sense} \\ u \leq v & \text{on } \partial\Omega \end{cases}$$

then $u \leq v$ in $\overline{\Omega}$.

MAXIMUM ESTIMATES

Let us now estimate the supremum of u in Ω in terms of the supremum of u on the parabolic boundary $\mathcal{P}\Omega$.

Theorem 1 [L., Theorem 9.5]. Let $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ with $Pu \leq 0$ in Ω.

Assume that $k, b_1 \in \mathbb{R}$ are such that:

$$z\, H(X, 0) \geq -k z^2 - b_1 \quad \forall X \in \Omega,\ z \in \mathbb{R}.$$

Then:

$$\sup_{\Omega} u \leq e^{(k+1)T} \left(\sup_{\mathcal{P}\Omega} u^+ + b_1^{1/2} \right). \qquad (\text{here } u^+ := \max\{u, 0\})$$

Remark: In this case, assumption only gives the constants k and b_1. In more general equation where $H = H(X, u, Du)$, it is:

$$z\, H(X, z, 0) \geq -k z^2 - b_1 \quad \forall X \in \Omega,\ \forall z \in \mathbb{R}$$

and it imposes a growth condition in z.

Proof: We want to estimate $\sup u$. Hence, if $u \leq 0 \Rightarrow$ Ok. Otherwise, for $\lambda > 0$ (to be chosen), let $\overline{X} \in \overline{\Omega}$ be a point when the function $v := e^{\lambda t} u$ attains a positive maximum.

- If $\overline{X} \in \mathcal{P}\Omega \Rightarrow$ Ok. Actually

$$u(X)\, e^{\lambda t} \leq u(\overline{X})\, e^{\lambda \overline{t}} \leq \sup_{\mathcal{P}\Omega} u^+ e^{\lambda \overline{t}} \qquad \forall X \in \Omega.$$

- Assume $\overline{X} \in \Omega$. We have

$$0 \geq \Big[u_t - \Delta u + H(X, Du) \Big]_{|\overline{X}} \geq u_t(\overline{X}) - k\, u(\overline{X}) - \frac{b_1}{u(\overline{X})}$$

where the latter inequality is due to $\Delta u(\overline{X}) \leq 0$ and to with $z = u$. Moreover, by $u_t = v_t e^{-\lambda t} - \lambda u$, we obtain:

$$0 \geq v_t(\overline{X}) e^{-\lambda \overline{t}} - \lambda u(\overline{X}) - k u(\overline{X}) - \frac{b_1}{u(\overline{X})} \geq -(k + \lambda) u(\overline{X}) - \frac{b_1}{u(\overline{X})}$$

where the last inequality is due to $v_t(\overline{X}) \geq 0$.

Now we choose $\lambda = -k-1$ and we get

$$0 \geq u(\bar{X}) - \frac{b_1}{u(\bar{X})}$$

i.e.: $b_1^{1/2} \geq u(\bar{X}) \quad \Rightarrow$ OK.

⊗ Assume $\bar{X} \in \partial\Omega_1 \setminus \mathcal{P}\Omega_1$. In this case we proceed by approximation in $\Omega_{1i} := \{X \in \Omega_1 : t \leq T - 1/i\}$, and we use the previous case in each Ω_i

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<u>Remark</u>: The previous theorem provides a maximum estimates which is dependent on T. In the following one, we establish a maximum estimate independent of T.

<u>Theorem 2</u> [L., Theorem 9.6]. Assume:

$$H(X, p) \geq -\mu_1 |p| - \mu_2 \qquad \text{for some } \mu_1, \mu_2 \geq 0.$$

If $Pu \leq 0$ in Ω_1, then:

$$\sup_{\Omega_1} u \leq \sup_{\mathcal{P}\Omega} u^+ + C(\mu_1, R)\mu_2$$

(Recall that $R = \frac{1}{2}\operatorname{diam} \omega$)

<u>Proof</u>: Suppose without any loss of generality that $0 \leq x_1 \leq \rho$. Set

$$v = \mu_2\left(e^{\alpha\rho} - e^{\alpha x_1}\right) + \sup_{\mathcal{P}\Omega_1} u^+ \qquad \text{for } \alpha = \mu_1 + 1.$$

Suppose first that $\mu_2 > 0$. In $\Omega_1^+ := \{X : u > 0\}$ we have

$$Pv = \mu_2 \alpha^2 e^{\alpha x_1} + H(X, Dv)$$

$$\geq \mu_2 \alpha^2 e^{\alpha x_1} - \mu_1 |Dv| - \mu_2 .$$

Since $Dv = (-\mu_2 \alpha e^{\alpha x_1}, 0, \dots, 0)$, we have

$$|Dv| = \mu_2 \alpha e^{\alpha x_1}$$

and furthermore

$$Pv \geq \mu_2 \alpha^2 e^{\alpha x_1} - \mu_1 \mu_2 \alpha e^{\alpha x_1} - \mu_2 = \mu_2 \alpha^2 e^{\alpha x_1}\left[1 - \frac{\mu_1}{\alpha} - \frac{1}{\alpha^2 e^{\alpha x_1}}\right]$$

$$\geq \mu_2 \alpha^2 e^{\alpha x_1}\left[1 - \frac{\mu_1}{\alpha} - \frac{1}{\alpha}\right] > 0$$

where the last inequality is due to our choice of α.

By Proposition 1 (which still holds in general domains Ω_1) we get $u \leq v$ in Ω_1.

For $\mu_2 = 0$, we just take the limit as $\eta_2 \to 0^+$ $\blacksquare$

EXISTENCE AND REGULARITY

Notations: We need to introduce some classes of "parabolic" Hölder continuous functions:

For $X=(x,t)$, $Y=(y,s)$, we set $|X-Y| = \max\{|x-y|, |t-s|^{1/2}\}$

A function $f: \Omega \to \mathbb{R}$ is Hölder continuous at $X_0=(x_0,t_0)$ with exponent $\alpha \in (0,1]$ if

$$[f]_{\alpha, X_0} := \sup_{X \in \Omega \setminus \{X_0\}} \frac{|f(X)-f(X_0)|}{|X-X_0|} < +\infty$$

$$[f]_{\alpha,\Omega} := \sup_{X_0 \in \Omega} [f]_{\alpha, X_0}$$

For parabolic equations, roughly speaking the rule "2 x-derivative equal to 1 t-derivative" holds. This equivalence makes the definition of higher order Hölder seminorms more complicated than the elliptic counterpart.

For $\beta \in (0,2]$, we set

$$\langle f \rangle_{\beta, X_0} := \sup \left\{ \frac{f(x_0,t)-f(X_0)}{|t-t_0|^{\beta/2}} \, , \quad \begin{matrix}(x_0,t)\in\Omega \\ t \neq t_0\end{matrix} \right\}$$

$$\langle f \rangle_{\beta,\Omega} := \sup_{X_0 \in \Omega} \langle f \rangle_{\beta, X_0}$$

For $a>0$ we write $a = k+\alpha$, with $k \geq 0$, $k \in \mathbb{Z}$, $\alpha \in (0,1]$ and we set:

$$\langle f \rangle_{a,\Omega} := \sum_{|\beta|+2j=k-1} \langle D_x^\beta D_t^j f \rangle_{\alpha+1}$$

$$[f]_{a,\Omega} := \sum_{|\beta|+2j=k} [D_x^\beta D_t^j f]_\alpha$$

$$|f|_{a,\Omega} := \sum_{|\beta|+2j \leq k} \sup |D_x^\beta D_t^j f| + [f]_{a,\Omega} + \langle f \rangle_{a,\Omega}.$$

$|\cdot|_{a,\Omega} \equiv |\cdot|_a$ defines a norm on $H_a := \{f : |f|_a < +\infty\}$. Moreover,

H_a endowed with this norm is a Banach space.

We also need some weighted Hölder spaces, where the weight is given in terms of the distance from the parabolic boundary $\mathcal{P}\Omega$.

· For $X, Y \in \Omega$, we set

$$d(X) = \mathrm{dist}\left(X, \mathcal{P}\Omega \cap \{t < t_0\}\right), \quad d(X,Y) = \min\{d(X), d(Y)\}.$$

For $b \in \mathbb{R}$, we define:

$$|f|_0^{(b)} := \begin{cases} \sup_{\Omega}\{d^b(X)\,|f(X)|\} & \text{if } b \geq 0 \\ \mathrm{diam}\,\Omega \,\sup_{\Omega} |f(X)| & \text{if } b < 0 \end{cases}$$

For $a > 0$ and $a + b \geq 0$, we set ($a = k + \alpha$ as before)

$$[f]_a^{(b)} := \sup\left\{\sum_{|\beta|+2j=k} d(X,Y)^{a+b} \frac{|D_x^\beta D_t^j f(X) - D_x^\beta D_t^j f(Y)|}{|X-Y|^\alpha}, \; X \neq Y, \; X, Y \in \Omega\right\}$$

$$\langle f \rangle_a^{(b)} := \sup\left\{\sum_{|\beta|+2j=k-1} d(X,Y)^{a+b} \frac{|D_x^\beta D_t^j f(X) - D_x^\beta D_t^j f(Y)|}{|X-Y|^{1+\alpha}}\right\}$$

$$|f|_a^{(b)} := \sum_{|\beta|+2j\leq k} |D_x^\beta D_t^j f|_0^{(|\beta|+2j+b)} + [f]_a^{(b)} + \langle f \rangle_a^{(b)}.$$

For $a > 0$ and $(a+b) < 0$, we define $[f]_a^{(b)}$ and $\langle f \rangle_a^{(b)}$ by replacing $d(X,Y)$ with $\mathrm{diam}\,\Omega$.

The set $H_a^{(b)}$ endowed with the norm $|\cdot|_a^{(b)}$ is a Banach space.

——— end notations ———

Theorem 3 [L., Theorem 12.16]

Assume:

- $\varphi \in H_\delta(\mathcal{P}\Omega)$ for some $\delta \in (1,2)$
- $H \in H_\alpha(K)$ for every bounded subset K of $\Omega \times \mathbb{R}^n$
- $|H| = \mathcal{O}(|p|^2)$ as $|p| \to +\infty$.

Then the Cauchy-Dirichlet problem

$$\begin{cases} Pu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \mathcal{P}\Omega \end{cases}$$

has a solution $u \in C^0(\bar{\Omega}) \cap C^{2,1}(\Omega)$ which belongs to the Hölder space $H^{(-\delta)}_{2+\alpha}$.

Furthermore, if the datum φ satisfies:

- $\varphi \in H_{2+\alpha}$
- $P\varphi = 0$ in $C\Omega$ (compatibility condition)

then the solution u belongs to $H_{2+\alpha}$.

Proof: The proof is based on theorem 6 and 7 below.
A very short sketch of the proof is given at page 15.

RECALL

Theorem 4 (Schauder fixed point theorem) [L., theorem 8.1].
Let S be a compact and convex subset of a Banach space $\mathcal{B}$. Let $J: S \to S$ be a continuous map. Then, J has a fixed point.

Theorem 5 (existence and regularity for heat equation) [L., theorem 5.14 and Theorem 5.15].

Consider
$$(*)\begin{cases} u_t - \Delta u = f(X) & \text{in } \Omega \\ u = \varphi & \text{on } \mathcal{P}\Omega. \end{cases}$$

1) If, for some $\alpha \in (0,1)$ and $\delta \in (1,2)$, $f \in H_\alpha^{(2-\delta)}$ and $\varphi \in H_\delta(\mathcal{P}\Omega)$, then there exists a unique $u \in C^{2,1}(\Omega) \cap C^0(\bar\Omega)$ solution to $(*)$ which belongs to $H_{2+\alpha}^{(-\delta)}$. Moreover, we have
$$|u|_{2+\alpha}^{(-\delta)} \leq C(n)\left(|f|_\alpha^{(2-\delta)} + |\varphi|_\delta\right).$$

2) If, for some $\alpha \in (0,1)$, $\varphi \in H_{2+\alpha}(\mathcal{P}\Omega)$ and $f \in H_\alpha$ with
$$\varphi_t - \Delta\varphi = f \quad \text{on } C\Omega,$$
then problem $(*)$ has a unique solution $u \in C^{2,1}(\Omega) \cap C^0(\bar\Omega)$. Moreover $u \in H_{2+\alpha}$ and fulfills:
$$|u|_{2+\alpha} \leq C(n, \alpha, \Omega)\left(|f|_\alpha + |\varphi|_{2+\alpha}\right)$$

——— end recall ———

Let us now come back to the existence of a solution to our problem

$$(\mathrm{CPD}')\begin{cases} Pu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \mathcal{P}\Omega \end{cases}$$

We first establish an existence result for small time.

<u>Theorem 6</u> (Existence and regularity - small time) [L., theorem 8, 2]

Assume:

① $H \in H_\alpha(K)$, for every $K \subset\subset \Omega \times \mathbb{R}^n$, for some $\alpha \in (0,1)$

② $\varphi \in H_\delta(\mathcal{P}\Omega)$ for some $\delta \in (1,2)$.

Then $\exists\, \varepsilon > 0$ such that the Cauchy-Dirichlet problem

$$\begin{cases} Pu = 0 & \text{in } \Omega_\varepsilon := \{X \in \Omega : X = (x,t),\ 0 < t < \varepsilon\} \\ u = \varphi & \text{on } \mathcal{P}\Omega_\varepsilon \end{cases}$$

has a solution $u \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$. Moreover $u \in H_{2+\alpha}^{(-\delta)}$.

Furthermore if the datum φ also verifies

③ $\varphi \in H_{2+\alpha}(\mathcal{P}\Omega)$ with $P\varphi = 0$ on $C\Omega$

Then $u \in H_{2+\alpha}$.

<u>Proof</u>: Let $\theta \in (1, \delta)$; set $M_0 := 1 + |\varphi|_\theta$. For $\varepsilon > 0$ small enough (to be chosen later), we set

$$S := \{ v \in H_\theta(\Omega_\varepsilon) : |v|_\theta \leq M_0 \},$$

and we define the map $J : S \longrightarrow S$, $v \longmapsto Jv = u$

(⚠ The fact that the image of S is in S will be checked soon!)

where u is the solution to the linear problem

$$(**)\begin{cases} u_t - \Delta u = H(x, Dv) & \text{in } \Omega_\varepsilon \\ u = \varphi & \text{on } \mathcal{P}\Omega_\varepsilon. \end{cases}$$

Observe that, since $v \in H_\theta$ and $H \in H_\alpha$, the function $H(\cdot, Dv(\cdot))$ belongs to $H_{\alpha(\theta-1)}$. Applying Theorem 5-(1), we obtain the existence (and uniqueness) of a solution u to problem (**) which belongs to $H^{(-\delta)}_{2+\alpha(\theta-1)}$ with

$$|u|^{(-\delta)}_{2+\alpha(\theta-1)} \leq C(M_0)$$

By the definition of our weighted Hölder space $H^{(b)}_a$ (and recall also that $\delta \in (1,2)$), we have

$$|u|_1 \leq |u|_\delta \leq C\,|u|^{(-\delta)}_{2+\alpha(\theta-1)} \leq C(M_0)$$

It follows:

$$|u-\varphi| \leq C\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$|u-\varphi|_\delta \leq C(M_0) + |\varphi|_\delta \quad \text{in } \Omega_\varepsilon .$$

By interpolation among Hölder spaces, we get

$$|u-\varphi|_\theta < C\,\varepsilon^{\frac{\delta-\theta}{\delta}}$$

Choosing ε sufficiently small to have $C\,\varepsilon^{\frac{\delta-\theta}{\delta}} < 1$ (so ε is now fixed), we get $|u|_\theta \leq |u-\varphi|_\theta + |\varphi|_\theta \leq 1 + |\varphi|_\theta = M_0$;

in other words, $u = Jv$ belongs to the set S.

Since S is a convex, compact subset of H_1, it follows that the map J has a fixed point u which belongs to $H^{(-\delta)}_{2+\alpha(\theta-1)}$ and solves (CPD').

Applying again Theorem 5-(1) to (**) with v replaced by u, ensures that $u \in H^{(-\delta)}_{2+\alpha}$.

Assume now that the datum φ also verifies (3). In this case, following the same arguments as before (but now we use Theorem 5 - (2)), we get

$$|Jv|_{2+\alpha(\theta-1)} \leq C(M_0)$$

and therefor the solution u belongs to $H_{2+\alpha(\theta-1)}$.

Applying again Theorem - (2) to problem (**) with v replaced by u, we infer that u belongs to $H_{2+\alpha}$ ■

<u>Remark</u>: For general domains, one has to require some regularity / shape condition on Ω.

The previous theorem establishes the existence only for small time. In the following theorem, we shall use some a priori estimates (not obvious at all!) to obtain global existence.

<u>Theorem 7</u> (L, Theorem 8.3).

Assume hypotheses ① and ② of theorem 6. If there is a constant M_δ (indipendent of ε) such that any solution to

(***) $\begin{cases} Pu = 0 & \text{in } \Omega_\varepsilon := \{X \in \Omega : X = (x,t),\ 0 < t < \varepsilon\} \\ u = \varphi & \text{on } \mathcal{P}\Omega_\varepsilon \end{cases}$

verifies $|u|_\delta \leq M_\delta$, then there is a solution of $\begin{cases} Pu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \mathcal{P}\Omega. \end{cases}$

<u>Proof</u>:

Let ε_0 be the supremum of values ε such that problem (***) has a solution. Observe that Theorem 6 ensures: $\varepsilon_0 > 0$. We want to prove that $\varepsilon_0 = T$; by contradiction, we assume $\varepsilon_0 < T$.

Let $\{\varepsilon_i\}$ be an increasing sequence with $\varepsilon_i \to \varepsilon_0$ and we denote u_i the solution to $\begin{cases} Pu = 0 & \text{in } \Omega_{\varepsilon_i} \\ u = \varphi & \text{on } \mathcal{P}\Omega_{\varepsilon_i}. \end{cases}$

Our hypothesis "$|u|_\delta \le M_\delta$", and Ascoli-Arzelà theorem ensure that there exists a subsequence (still denoted $\{u_i\}$) which converges to some function u.

By interpolation, we get $u_i \to u$ in $H^{(-\tau\delta)}_{(2+\alpha)\tau}$ $\forall \tau \in (0,1)$. In particular, choosing $\tau > \max\{\frac{1}{\delta}, \frac{2}{2+\alpha}\}$, we obtain:

$$\left.\begin{array}{l} u_i \to u \\ Du_i \to Du \end{array}\right\} \begin{array}{l} \text{uniformly} \\ \text{in } \Omega \end{array} \qquad \left.\begin{array}{l} \partial_t u_i \to \partial_t u \\ D^2 u_i \to D^2 u \end{array}\right\} \begin{array}{l} \text{uniformly in any} \\ \text{cylinder } Q \text{ with } \overline{Q} \subset \Omega. \end{array}$$

Therefore, passing to the limit in the equation, we get that u solves $\begin{cases} Pu = 0 & \text{in } \Omega_{\varepsilon_0} \\ u = \varphi & \text{on } \mathcal{P}\Omega_{\varepsilon_0} \end{cases}$ and it is defined on $t = \varepsilon_0$.

We can use theorem 6 to solve (for η sufficiently small). $\begin{cases} Pu^* = 0 & \text{in } \Omega^* := \{X \in \Omega : \varepsilon_0 < t < \varepsilon_0 + \eta\} \\ u^* = \varphi & \text{on } S\Omega^* \\ u^* = u & \text{on } B\Omega^* \end{cases}$

The function $w(X) := \begin{cases} u(X) & \text{if } t \le \varepsilon_0 \\ u^*(X) & \text{if } t \in (\varepsilon_0, \varepsilon_0 + \eta) \end{cases}$ is a solution to $\begin{cases} Pw = 0 & \text{in } \Omega_{\varepsilon_0+\eta} \\ w = \varphi & \text{on } \mathcal{P}\Omega_{\varepsilon_0+\eta} \end{cases}$. This fact contradicts the definition of ε_0.

<u>Proof of Theorem 3</u> (Short plan).

In order to apply theorem 7, we need to:

1) estimate the maximum of u in Ω (see Theorem 1, 2)

2) estimate the maximum of the gradient near $\mathcal{P}\Omega$

3) estimate the maximum of the gradient in Ω

4) estimate the Hölder constant in Ω