Degenerations of abelian varieties

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Timetable: 16 hrs. First lecture in week 12 (21–25/3/2022). The precise schedule will be announced later. The plan is to have 4 hours of lectures in each of the weeks 12, 13 and 14, and 2 hours in each of the weeks 17 and 18. Most of the lectures will be given in-class, but a couple of the lectures might be given via zoom.

Course requirements:

1. Algebraic geometry/scheme theory roughly at the level of Chapters 2 and 3 of Hartshorne’s book.
2. Basic knowledge of abelian varieties.

Examination and grading: Seminar on a course related topic.

SSD: MAT/03

Course contents: The course will give an introduction to Néron models of abelian varieties. This theory forms an important tool in the study of abelian varieties, with numerous applications in arithmetic questions as well as in moduli theory.

Let $R$ be a discrete valuation ring with field of fractions $K$ and residue field $k$. If $A$ is an abelian variety defined over $K$, there exists a canonical way to extend $A$ to a group scheme $\mathcal{A}$ over $R$ – the Néron model of $A$. The special fiber $\mathcal{A}_k = \mathcal{A} \times_R k$ is again a smooth commutative group scheme, but its geometric structure can be much more complicated than that of $A$. For instance, it might be non-proper and even disconnected. Informally, one can say that the shape of $\mathcal{A}_k$ reflects the arithmetic properties of $A$ over the field $K$ (or, in geometric terms, the way in which $A$ degenerates at the closed point of $\text{Spec}R$).

Central themes that will be discussed in the course are as follows.

- Structure of the special fiber $\mathcal{A}_k$. In particular, the Chevalley decomposition of the connected component of identity $\mathcal{A}^0_k$ of $\mathcal{A}_k$, and the group of connected components $\Phi(A) = \mathcal{A}_k/\mathcal{A}^0_k$.
- Grothendieck’s semi-abelian reduction theorem, as well as applications/consequences of this result.
- Néron models of Jacobians of curves, and the link to degenerations/models of curves.
- Examples of applications to arithmetic questions for curves and abelian varieties.