The Minimal Action Principle and applications

Luca Baracco[1], Olga Bernardi[2]

[1]*Dipartimento di Matematica "Tullio Levi-Civita"*
Email: baracco@math.unipd.it

[2]*Dipartimento di Matematica "Tullio Levi-Civita"*
Email: obern@math.unipd.it

Timetable: 24 hours; First lecture on April 10, 2025, 10:30 (dates already fixed see on https://dottorato.math.unipd.it/calendar), Torre Archimede, Room 2BC30

Credits: 4

Course requirements: Basic notions of Mathematical Physics and Calculus of Variations.

Examination and grading: oral exam.

SSD: MATH-03/A, MATH-04/A

Aim: A sample of various settings, also beyond classical Lagrangian and Hamiltonian framework, where the "Least Action Principle" play a fundamental role. The course include basic notions and proofs of Aubry-Mather Theory and Mather-Mané Theory. Moreover, we discuss connections both with mathematical billiards and Symplectic Topology.

Course contents: For a Lagrangian $L : TM \to \mathbb{R}$ on a tangent bundle, the Euler-Lagrange equations:

$$\frac{d}{dt}\partial_{\dot{x}} L(x, \dot{x}) - \partial_x L(x, \dot{x}) = 0$$

encode the variational character of Lagrangian Mechanics. In fact, it is well-known that a curve $x : [t_0, t_1] \to X$ solves E-L if and only if the first variation of the corresponding Action

$$\int_{t_0}^{t_1} L(x, \dot{x}) dt,$$

with fixed extremals, vanishes. Concerning minimality, in general it holds for short times. In fact, due to the possible presence of conjugated points, critical curves cease to be minimizing for larger times. Only under certain convexity assumptions there are "Action minimizing orbits". For such a distinguished and mechanical relevant class of Lagrangian –the so-called Tonelli Lagrangians– the Legendre transform is a global diffeomorphism and E-L equations equal to Hamilton's equations

$$\begin{cases} \dot{x} = \partial_y H(x, y) \\ \dot{y} = -\partial_x H(x, y) \end{cases}$$

for the corresponding Hamiltonian $H : T^*X \to \mathbb{R}$. For autonomous systems, H gives the conserved energy value along a solution.

Beyond Lagrangian and Hamiltonian setting, the search of dynamically relevant minimal objects is one of the central topics of modern theory of Dynamical Systems. One of the first results in this direction go back to the Eighties with the so-called Aubry-Mather theory for

monotone twist map. An important application of this theory is the study of mathematical billiards, from Birkhoff to –more recent– types of billiards like symplectic and outer billiards. The generalization of such a theory from one to more degrees of freedom have been developed two decades later with Mather-Mané theory, where minimizing measures, instead of trajectories, play a crucial role. This significant theory has connections from Hamilton-Jacobi equation to Symplectic Topology.

The aim of this PhD course is to present –in a self-contained way– the "Minimal Action Principle" in different settings. This principle can be considered a sort of –largely accepted– "thriftiness" of Nature in its motions. In more detail, the plan of the course includes:

1. BASIC NOTIONS OF AUBRY-MATHER THEORY: Monotone twist maps, minimal orbits, the minimal action (or Mather's β function) for monotone twist maps.
2. THE MINIMAL ACTION FOR CONVEX LAGRANGIANS: Basic notions of Mather-Mané theory, Mané critical value, the Aubry set.
3. THE MINIMAL ACTION FOR CONVEX BILLIARDS: Birkhoff, symplectic and outer billiards and their corresponding generating functions. Length and area spectrum invariants.
4. THE MINIMAL ACTION IN SYMPLECTIC TOPOLOGY: A bit of symplectic geometry, the graph selector, boundary rigidity phenomena, the Aubry set and non-removable intersections.

Bibliography: The course follows the main lines of the book (present also in Padova math library):
– Siburg K.F. The principle of least action in geometry and dynamics. Lecture Notes in Mathematics, vol. 1844, xiii+ 128 pp. Berlin, Germany: Springer, (2004).

Moreover, additional related publications include:

– Bangert V. Mather Sets for Twist Maps and Geodesics on Tori Dynamics Reported, Volume 1 Dynamics Reported, (1988).
– Paternain G.P. Polterovich L. Siburg K.F. Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory, Volume 3, Number 2, (2003).
– Sorrentino A. Action-minimizing methods in Hamiltonian dynamics: an introduction to Aubry-Mather theory. Monograph in the Series: Mathematical Lecture Notes Vol. 50, Princeton University Press, (2015).
– Tabachnikov S. Geometry and Billiards (Student Mathematical Library) (Student Mathematical Library, 30), (2005).