Curves in Hilbert Modular Surfaces

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Timetable: 16 hrs. First lecture on March 5, 2025, 14:00 (dates already fixed see on https://dottorato.math.unipd.it/calendar), Torre Archimede, Room 2BC30

Credits: 3

Course requirements: Basic algebraic number theory and algebraic geometry (algebraic varieties). Previous exposure to the concept of moduli spaces is helpful but not required.

Examination and grading: seminar about a research paper on the subject

SSD: MAT/03

Aim: Belyi theorem states that every algebraic curve defined over $\overline{\mathbb{Q}}$ is a branched cover of the projective line minus three points. Such a curve is, by Poincaré's unifomization theorem, a (possibly compactified) quotient $\mathbb{H}/\Gamma$ of the upper half-plane by a discrete group $\Gamma \subset SL_2(\mathbb{R})$. In some cases, the group Γ is *arithmetic*, i.e., it comes from a algebra (the curve is then a modular or Shimura curve), but in most cases it is not. An important example of non-arithmetic curves is provided by Teichmüller curves, a class of curves first discovered in relation with the dynamics of rational billiard tables. The first aim of this course is to provide a unified treatment of algebraic curves over $\mathbb{Q}$ as totally geodesic curves in the moduli space of abelian surfaces, with respect to the Kobayashi metric. The second aim is to discuss in detail two classes of such curves, the Hirzebruch-Zagier curves, and the Teichmüller curves in genus two. If time permits, we will discuss the special points on these curves.

Course contents:

1. The moduli space $\mathcal{A}_g$ of abelian varieties of dimension g and the locus of real multiplication. Hilbert modular varieties (an overview).
2. Kobayashi metric on $\mathcal{A}_g$ and totally geodesic curves (Kobayashi curves). They are defined over number fields and lie in the locus of real multiplication.
3. Further motivation: Belyi's theorem and Ellenberg-McReynolds's result ("every curve is a Teichmlller curve").
4. Kobayashi curves as curves in Hilbert modular varieties and modular embeddings. Arithmetic Fuchsian groups.
5. Example I. Totally geodesic curves in $\mathcal{A}_2$ for the Riemannian metric: Hirzebruch-Zagier curves. Their relation with modular curves and moduli interpretation.
6. Teichmüller space and Teichmüller metric; mapping class group and moduli space of curves (an overview).
7. Example II. Totally geodesic curves in $\mathcal{A}_2$ for the Teichmüller metric: Teichmüller curves. Their relation with rational billiards (flat surfaces) and moduli interpretation.

8. If time permits, special (complex multiplication) points in the arithmetic and nonarithmetic cases.

Bibliography:

1. Hirzebruch, F. , Zagier, D. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, Invent. Math.36(1976), 57–113.

2. McMullen, C. Billiards and Teichmüller curves on Hilbert modular surfaces, J. Amer.Math. Soc.16(2003), no.4, 857–885.

3. Möller, M. , Viehweg, E. Kobayashi geodesics in Ag , J. Differential Geom.86(2010), no.2, 355–379.