Convex polyhedra and their diameter

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Timetable: 24 hrs, First lecture on November 6, 2024, 12:30 (dates already fixed, see Calendar of Activities on https://dottorato.math.unipd.it/calendar/, Torre Archimede, Room 2BC30.

Credits: 4

Course requirements: Standard Mathematical knowledge will be sufficient

Examination and grading: Seminar

SSD: MAT/09

Aim: A convex polyhedron is defined as the set of solutions to a given system of linear inequalities, and as such, it generalizes to n dimensions the commonly known notion of 3-dimensional polyhedron, which is used, for instance, to describe the structure of molecules and chemical compounds. Besides being natural and interesting mathematical objects, general n-dimensional polyhedra are fundamental in the field of optimization.

In this course we will introduce some basic properties of polyhedra and discuss their importance and usefulness. After reviewing some fundamental structural results, we will focus on the notion of diameter of a polyhedron. In order to understand what the diameter of a polyhedron is, one can visualize the 3-dimensional case, where the notions of vertex and edge should be clear: The diameter of a polyhedron is the maximum distance between any two vertices, where, for a fixed pair of vertices, the distance is measured by counting the minimum number of steps needed to move from one vertex to the other by traversing edges. (This can be properly extended to the n-dimensional case.) We will survey classical and famous results on the diameter of polyhedra, as well as some recent achievements. We will spend some time on the Hirsch conjecture, which was posed in 1957, proved for some special but very important cases in the subsequent decades, and finally disproved by Francisco Santos in 2010 in a paper that won the Fulkerson prize, one of the most important awards in the area of Discrete Mathematics. However, a weaker version, called the polynomial Hirsch conjecture, is still open, and its correctness would be extremely important to assess the efficiency of some algorithms for the solution of linear optimization problems.

The topic of this course can be interesting for students in various fields ofMathematics, as it connects (at least) combinatorial and discrete geometry, graph theory, and optimization. Graphs and their diameter are also studied in algebra and probability (e.g., random walks are sometimes the tool to prove the existence of a short path between two vertices).

Course contents:

- Polyhedra: basic facts, inequality description, vertex/ray description, fundamental examples
- The role of polyhedra in several disciplines
- The skeleton of a polyhedron and the notion of diameter

- Lower and upper bounds on the diameter of a polyhedron
- Hirsch conjecture and its link with the efficiency of linear programming algorithms
- Correctness of the conjecture for 0/1 polytopes and for some other relevant cases
- Santos' counterexample to the Hirsch conjecture (sketch)
- The (weaker) polynomial Hirsch conjecture and some related results
- Time permitting) The recent notion of circuit diameter

Bibliography:
For the introductory part on polyhedra, students can refer to Chapter 3 of the following book, also accessible online:

- M. Conforti, G. Cornuéjols, G. Zambelli, Integer Programming, Springer, 2014.

The notion of diameter and some classical related results are discussed, e.g., in Chapter 3 of the following book:

- G. M. Ziegler, Lecture on Polytopes, Springer, 2007.

More specific references (including those needed to be prepared for the exam, which is in the form of student seminar) will be provided during the course.