Products of random matrices: theory and applications

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Timetable: 24 hrs, First lecture on November 12, 2024, 10:30 (dates already fixed, see Calendar of Activities on https://dottorato.math.unipd.it/calendar/, Torre Archimede, Room 2BC30.

Credits: 4

Course requirements:
A first course in probability with measure theory, some knowledge of stochastic processes with discrete time (Markov chains, martingales) is welcome, but the important concepts will be (re)introduced in some detail.

Examination and grading: Oral exam

SSD: MAT/06

Aim: Learning fundamental of products of random matrices.

Course contents:
Products of random matrices appear as a fundamental model and/or tool in a number of mathematical contexts, ranging from very abstract to very applied ones. The theory is rich already in the case in which one deals with products of independent and identically distributed (IID) random $d \times d$ matrices $(M_n)_{n=1,2,\dots}$, with d an integer larger than 1: one of the main issue in this context is understanding the leading asymptotic behavior, for $n \to \infty$, of $\log ||M_1 M_2 \dots M_n||$, with $||\cdot||$ a matrix norm, assuming that $\log ||M_1||$ is in L^1. This is the natural generalization of the case in which $d = 1$, where $M_1 M_2 \dots M_n$ is just a product of IID real random variable and the issue we raise is easily solved by applying the Law of Large Numbers, and the answer is very explicit. The situation is more involved when $d = 2$ or more: in fact, the issue we just raised is the identification of the top Lyapunov exponent of the product of random matrices we consider and, while the theory is very complete, the answer is somewhat involved and definitely not as explicit as for $d = 1$. It involves in particular the construction of an auxiliary process that is interesting in its own right.

In the first part of the course develops the the theory that leads to a formula, the Furstenberg formula, for the top Lyapunov exponent. While some results will be given (and proven) for general d, the full theory will be developed only for $d = 2$. We will also study the second Lyapunov exponent (i.e., for $d = 2$ we will study all the Lyapunov exponents, since there are d Lyapunov exponents): the key result that we will prove is Oseledets Theorem.

The second part of the course focuses on applications. Possible topics include:

1. Anderson localization in one dimension: the Schrödinger operator with random potentials
2. Products of random matrices and random walks on groups

3. The transfer matrix method in statistical mechanics and application to some disordered models.

In reality the two parts of the course will be to a certain extent entangled: the random matrices that are relevant for the applications will be used as examples in the theoretical part of the course.

Bibliography:

There will be lecture notes, mostly based (for the first part of the course) on

- P. Bougerol and J. Lacroix, Products of random matrices with applications to Schrödinger operators, Progress in Probability and Statistics, 8, Birkhäuser, 1985.
- M. Viana, Lectures on Lyapunov exponents, Cambridge Studies in Advanced Mathematics, 145 Cambridge University Press, 2014.