Topics of Control Theory from a Differential Geometric point of view

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Timetable: 16 hrs. First lecture on March 20, 2025, 16:30 (dates already fixed, see on Calendar of Activities at https://dottorato.math.unipd.it/calendar), Torre Archimede, Room 2BC30.

Credits: 4

Course requirements: Even if the course concerns topics in Control Theory and Differential Geometry, the lectures will be designed to be accessible by a general audience, with no special training in either of those two areas.

Examination and grading: seminar about a research paper in the topic.

SSD: MAT/05-07-03

Aim:
We would like to offer an introduction to a few fundamental results of Control Theory and Differential Geometry, with the purpose of illustrating how certain classical Differential Geometric tools can be used to find solutions to problems in Control Theory. We also have the intent of discussing some very recent results by Cardin, Giannotti, Spiro and Zoppello on the Pontryagin Maximum Principle and the local controllability problem of systems with non-linear controls.

Course contents:

- *Preliminaries (I).* A quick overview of some basic notions and facts of Differential Geometry, as e.g. manifolds, submanifolds, vector fields, push-forwards under diffeomorphisms, tensor fields, p-forms and their integrations on submanifolds, flows, Lie derivatives and Lie brackets, etc.
- *Preliminaries (II).* A brief introduction to a few topics of Control Theory: first order control systems in normal forms, controllability and accessibility, the three variants (Mayer, Lagrange and Bolza) of cost minimising problems under first order differential constraints, Kalman Theorem, needle variations, Pontryagin Maximum Principle (PMP).
- Comparison between the classical and the differential geometric proofs of the PMP for Mayer problems.
- Frobenius Theorem and its applications to the integrability problems of systems of partial differential equations.
- Non-integrable distributions, Rashevskij's Theorem and its generalisations by Chow and Sussmann.

- Chow's criterion for the controllability of systems which are linear in controls; Controllability of systems on Lie groups; Discussions of several examples of control systems and comparisons of criterions for local controllabliity. The Chaplygin sleigh and its fluidodynamic variant: results and open problems.
- Non-linear control systems and representation of solutions as oriented curves in the extended state-control space. Graphic completions. New criterions for local accessibility and small time local controllability. Examples.

Bibliography:

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