Fully Nonlinear Partial Differential Equations: an introduction

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SS: MATH-03/A

Credits: 3

Timetable: 16 hrs. Torre Archimede, Room 2BC30:

Nov 25, 2026	10:30-12:30	Dic 10, 2026	10:30-12:30
Nov 26, 2026	10:30-12:30	Dic 14, 2026	11:30-13:30 (Room 1AD100)
Nov 30, 2026	10:30-12:30	Dic 16, 2026	10:30-12:30
Dic 3, 2026	10:30-12:30	Dic 21, 2026	10:30-12:30

see also on Calendar of Activities on https://dottorato.math.unipd.it/calendar.

Examination and grading: seminar on a paper

Introduction and background requirements: I will start giving the main examples of fully nonlinear PDEs and their motivations. In the first part of the course I'll introduce the definitions and basic properties of viscosity solutions to degenerate elliptic equations. I'll state the main results on existence and uniqueness of solutions to Dirichlet and Cauchy boundary value problems, focusing in particular on the comparison principles, that are the cornerstone of the theory. The proofs will be given in some special cases, interesting for applications. The second part of the course will concern two important qualitative properties of solutions: the Strong Maximum Principle and the Liouville property. They are standard results for classical solutions of linear elliptic PDEs, and many extensions are known, especially for divergence form equations. My goal is explaining how the viscosity methods allow to turn around the difficulties of non-smooth solutions, fully nonlinear equations, and their possible degeneracies.

A solid **background** in advanced calculus is enough; some previous knowledge on PDEs helps to appreciate better the contents of the course.

Aim: Introducing the students to a theory of weak solutions for nonlinear problems with a very wide range of applications

Course contents:

- Second order fully nonlinear PDEs: notions of ellipticity. Main examples of equations: Hamilton-Jacobi, Monge-Ampere, geometric and Hessian equations, subelliptic PDEs,....
- Definition of viscosity solution, with motivations.
- Comparison principles: a review and some examples.

- Some results on existence and uniqueness of solutions to Dirichlet and Cauchy boundary value problems.
- Connections of some Hamilton-Jacobi equations with Calculus of Variations, deterministic and stochastic optimal control, and differential games.
- The Strong Maximum Principle (SMP): from classical to viscosity solutions; propagation of maxima for degenerate equations
- Examples of subelliptic PDEs: H¨ormander vector fields and the Heisenberg group. Subunit vector fields and a general SMP
- Liouville properties: from solutions of uniformly elliptic equations to subsolutions of degenerate equations.

References:

- D. Gilbarg and N.S. Trudinger: Elliptic PDEs of 2nd order, Springer 1983.
- M. Crandall, H. Ishii, P.L. Lions: Users guide to viscosity solutions, Bull. AMS 1992.
- L.A. Caffarelli and X. Cabre: Fully nonlinear elliptic equations, AMS Colloquium Publication, vol. 43, American Mathematical Society, 1995
- M. Bardi and I. Capuzzo-Dolcetta: Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkh¨auser 1997.
- S. Koike: A beginner's guide to the theory of viscosity solutions, MSJ Memoirs 2004.
- M. Bardi, A. Goffi: New strong maximum and comparison principles for fully nonlinear degenerate elliptic PDEs, Calc. Var. PDE (2019)