Foundations of Derived Analytic Geometry

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SS: MATH-02/B

Credits: 3

Timetable: 16 hrs. Torre Archimede, Room 2BC30, calendar in progress:

Sept 6, 2027	xx:xx-xx:xx	Sept xx, 2027	xx:xx-xx:xx
Sept xx, 2027	xx:xx-xx:xx	Sept xx, 2027	xx:xx-xx:xx
Sept xx, 2027	xx:xx-xx:xx	Sept xx, 2027	xx:xx-xx:xx
Sept xx, 2027	xx:xx-xx:xx	Sept 27, 2027	xx:xx-xx:xx

see also on Calendar of Activities on https://dottorato.math.unipd.it/calendar.

Examination and grading: Seminar on topics complementary to the course, or another form of assessment to be agreed upon with the teacher

Introduction and background requirements: A solid background in category theory is required. Basic knowledge of commutative algebra, algebraic geometry, and functional analysis is helpful but not essential

Aim: Derived analytic and C^∞-geometry are emerging fields of mathematics which have exciting implications for the study of moduli problems appearing in physics, complex geometry, and arithmetic geometry.
The goal of this course is to introduce a model of derived analytic geometry as algebraic geometry, in the sense of [16], relative to the category of complete bornological vector spaces over either an Archimedean or non-Archimedean field. We will define categories of derived analytic stacks, and explain how many important results - existence of cotangent complexes, six functor formalisms, descent for K-theory, representability theorems - translate to the analytic world. We will explain some applications, including the construction of analytic and C^∞-moduli spaces - moduli of Galois representations, of instantons, and of solutions to PDEs. The main reference will be the book draft [3].

Course contents:

Lecture 1: Motivations for derived analytic geometry, and homological algebra for functional analysis. In the first lecture we motivate the development of derived analytic geometry as relative algebraic geometry. For example, we will explain how one can define the category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves on an analytic space in a way which resolves Gabber's counter-example to the naïve definition. The category $\mathbf{QCoh}(X)$ will necessarily be derived, and we will justify why by showing that natural maps in analytic geometry are not flat. We will also give examples of deep theorems which have been proven using this algebro-geometric model of derived analytic geometry, including HKR [9] and Koszul duality [4]. Having mo-

tivated the desire for an algebraic model of (derived) analytic geometry, we make a start on the technical details. The algebras of interest in analytic geometry - e.g., entire functions on complex analytic spaces - have natural topological, or rather bornological, structures. We explain how to marry functional analysis with homological algebra by introducing quasi-abelian categories.

Lecture 2: Homotopy theory in exact categories, and homological characterisations of the analytic topologies. Quasi-abelian categories are canonically Quillen exact categories. Following [7], here we will explain conditions on an exact category such that it is a convenient setting for homological/ homotopical algebra - precisely, such that they are homotopical algebra contexts in the sense of [16]. This will involve defining derived tensor products and derived mapping spaces. Next we will make a connection with geometry. Let $\{U_i \to U\}_{i\in I}$ be a collection of maps of Stein spaces over C or rigid affinoids over a non-Archimedean field K. We will recall the main results of [2], [1], which show how to characterise when such a collection is an open cover, purely in terms of the corresponding collection of maps of algebras of analytic functions $\{\mathcal{O}(U) \to \mathcal{O}(Ui)\}_{i\in I}$. This will necessitate the definition of homotopy epimorphisms of algebras.

Lecture 3: Higher algebra in derived algebraic contexts. In this lecture we will introduce the category-theoretic framework for generalised algebraic geometry furnished by Raksit's derived algebraic contexts [13]. This will include the definition of the category $\mathbf{DA}\log$ **(C)** of derived algebras. We will also explain the relevant example for derived analytic geometry, namely, the ∞-category **Ch** $(\mathrm{CBorn}_{\mathbb{Z}})$ of complexes of complete bornological abelian groups. Next we justify why derived algebraic contexts are convenient settings for higher algebra. In particular, we will prove the existence of the Tor-spectral sequence, and deduce important consequences for homotopy epimorphisms and étale maps.

Lecture 4: Derived algebraic contexts, Lawvere theories, and topologies. This is a technical lecture in which we show how to define rational localisations, étale maps, and smooth maps relative to a Lawvere theory. We will explain the relevance to analytic geometry using the Lawvere theories determined by Steins and affinoids - this will be further developed in later lectures. Finally, we will introduce the Grothendieck topologies of interest on $\mathbf{Aff}^{cn}$ **(C)** - namely, the G- and étale topologies associated to a Lawvere theory.

Lecture 5: Derived analytic geometry I - Affines and descent for modules. Here we discuss descent for modules. We introduce the notion of descendability, due to [10]. We explain why this is a reasonable condition to impose, and why it is satisfied by the topologies introduced in the previous lecture. We will then specialise to the usual topologies of open covers and of étale covers respectively, restricted to algebras of analytic functions on Steins/ affinoids. We will define categories of derived Stein algebras and derived affinoid algebras, and prove that, when restricted to these algebras, the G- and étale topologies can be characterised in the following way: $\{$ Spec$(_i) \to$Spec$(A)\}_{i\in I}$ is a cover precisely if $\{$ Spec$(\pi_0(A_i))$ß Spec $(\pi_0(A))\}_{i\in I}$ is a cover, and for each $i \in \mathcal{I}$ and $n \in \mathbb{N}$, the natural map $\pi_0(A_i)\hat{\otimes}^{\mathbb{L}}_{\pi_0(A)}\pi_n(A) \to \pi_n(A_i)$ is an equivalence. In particular the topology really is a derived enhancement of the classical one.

Lecture 6: Derived analytic geometry II - analytic stacks and comparisons. We are finally ready to define schemes and Deligne-Mumford stacks. We explain why categories of classical analytic spaces embed fully faithfully in these new derived ones. We further show that a Deligne-Mumford stack admits an obstruction theory. In particular, such a stack has a cotangent complex. We recall related results for schemes such as the HKR theorem [9]. We will also sketch the existence of a moduli stack of Deligne-Mumford stacks, and a procedure for

constructing derived analytic blow-ups. Finally, we will compare our stacks with those of Porta and Yue Yu ([12]), and of Clausen and Scholze ([15]). In particular, in the complex analytic setting, we will show that Porta and Yu's geometric stacks embed fully faithfully in ours.

Lecture 7: Other geometries. In this lecture we explain how other geometries can be realised using the bornological approach. This will include C^∞-geometry, formal geometry, and a bornological enhancement of algebraic geometry. We will further explain how the formal geometry case allows us to compare with the non- Archimedean derived stacks of Porta and Yue Yu.

Lecture 8: Horizons - representability, K-theory, and six-functor formalisms. In the final lecture we explain the myriad possibilities presented by this theory of derived analytic and C^∞-geometry. We will survey recent results including K-theory [8], D-modules, and six functor formalisms (both [11]). We will also recall the representability theorem of [14] for derived stacks, and suggest how this can be used to construct moduli stacks in derived analytic and C^∞-geometry, including moduli of Galois representations and solutions to PDEs (following Pardon [6] and Steffens [5]).

References:

[1] Oren Ben-Bassat and Kobi Kremnizer, Non-Archimedean analytic geometry as relative algebraic geometry, Ann. Fac. Sci. Toulouse Math. (6) 26 (2017), no. 1, 49–126, doi: 10.5802/afst.1526 (English, with English and French summaries). MR 3626003

[2] Federico Bambozzi, Oren Ben-Bassat, and Kobi Kremnizer, Stein domains in Banach algebraic geometry, J. Func. Anal. 274 (2018), no. 7, 1865–1927.

[3] Oren Ben-Bassat, Jack Kelly, and Kobi Kremnizer, A perspective on the foundations of derived analytic geometry, arXiv preprint arXiv:2405.07936 (2024).

[4] Jack Kelly, Koszul duality in exact categories, arXiv preprint arXiv:1905.10102 (2019).

[5] Pelle Steffens, Representability of Elliptic Moduli Problems in Derived C^∞-Geometry, arXiv preprint arXiv:2404.07931 (2024).

[6] John Pardon, Derived moduli spaces of pseudo-holomorphic curves (2024).

[7] Jack Kelly, Homotopy in exact categories, Mem. Amer. Math. Soc. 298 (2024), no. 1490, v+160, doi: 10.1090/memo/1490. MR 4756824

[8] Jack and Mukherjee Kelly Devarshi, Localising invariants in derived bornological geometry, arXiv preprint arXiv:2505.15750 (2025).

[9] Jack Kelly, Kobi Kremnizer, and Devarshi Mukherjee, An analytic Hochschild-Kostant-Rosenberg theorem, Adv. Math. 410 (2022), 108694.

[10] Akhil Mathew, The Galois group of a stable homotopy theory, Advances in Mathematics 291 (2016), 403–541.

[11] Arun Soor, A six functor formalism for quasi-coherent sheaves and crystals on rigid analytic varieties, arXiv preprint arxiv:2409.07592 (2024).

[12] Mauro Porta, Derived complex analytic geometry I: GAGA theorems, arXiv preprint arXiv:1506.09042 (2015).

[13] Arpon Raksit, Hochschild homology and the derived de Rham complex revisited, arXiv preprint arXiv:2007.02576 (2020).

[14] Rhiannon Savage, A Representability Theorem for Stacks in Derived Geometry Contexts, arXiv preprint arXiv:2405.08361 (2024).

[15] Peter Scholze and Dustin Clausen, Condensed mathematics and complex geometry, Lecture notes.

[16] Bertrand Toën and Gabriele Vezzosi, Homotopical algebraic geometry I: Topos theory, Advances in mathematics 193 (2005), no. 2, 257–372.