An invitation to Fractional Sobolev Spaces

Giorgio Stefani[1]

[1] *Dipartimento di Matematica "Tullio Levi-Civita", Universitęgli Studi di Padova*
Email: giorgio.stefani@unipd.it

SS: MATH-03/A

Credits: 3

Timetable: 16 hrs. Torre Archimede, Room 2BC30, calendar in progress:

Mar xx, 2027	xx:xx-xx:xx	Mar xx, 2027	xx:xx-xx:xx
Mar xx, 2027	xx:xx-xx:xx	Mar xx, 2027	xx:xx-xx:xx
Mar xx, 2027	xx:xx-xx:xx	Mar xx, 2027	xx:xx-xx:xx
Mar xx, 2027	xx:xx-xx:xx	Mar xx, 2027	xx:xx-xx:xx

see also on Calendar of Activities on https://dottorato.math.unipd.it/calendar.

Examination and grading: Oral examination on the content of the course or presentation of a seminar based on a research paper related to the topics covered.

Introduction and background requirements: Basic knowledge ofmeasure theory and Lebesgue spaces, classical Sobolev spaces $W^{1,p}$, and elementary notions of partial differential equations.

Aim: The aim of the course is to introduce fractional Sobolev spaces $W^{s,p}$ and to present their main analytical properties, with a particular focus on applications to nonlocal partial differential equations. At the end of the course, students will be able to:

- understand and compare the definitions and characterizations of fractional Sobolev spaces;
- use the main functional-analytic tools associated with fractional Sobolev spaces, including embedding, approximation, compactness, and extension theorems;
- analyze variational problems and nonlocal partial differential equations, including problems involving the fractional Laplacian.

Course contents: The course provides an introduction to fractional Sobolev spaces and their applications to nonlocal partial differential equations. The following topics will be discussed:

(1) motivation, basic definitions, and comparison with integer-order Sobolev spaces;

(2) density of smooth functions and the Meyer–Serrin theorem;

(3) the Rellich–Kondrachov compactness theorem and the Poincaré inequality;

(4) extension theorems, embedding results, and interpolation inequalities;

(5) Bourgain–Brezis–Mironescu and Maz'ya–Shaposhnikova formulas;

(6) the fractional Laplacian, Riesz potentials, and the fractional Poisson equation;

(7) the Caffarelli–Silvestre extension.

References:

[1] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455. MR3586796

[2] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245–1260, DOI 10.1080/03605300600987306 MR2354493

[3] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), no. 5, 521–573, DOI 10.1016/j.bulsci.2011.12.004. MR2944369

[4] G. Leoni, A first course in fractional Sobolev spaces, Graduate Studies in Mathematics, vol. 229, American Mathematical Society, Providence, RI, 2023. MR4567945

[5] V.Maz'ya and T. Shaposhnikova, On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Funct. Anal. 195 (2002), no. 2, 230–238, DOI 10.1006/jfan.2002.3955. MR1940355