Efficient numerical integration schemes for the discretization of hypersingular BIEs related to wave propagation problems

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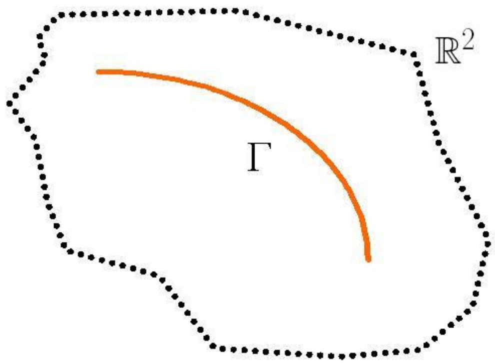
Summary

Time-domain Galerkin BEM for 2D wave propagation:

- Exterior Dirichlet Problem
- Exterior Neumann Problem
- Discretization and Performance of Quadrature Schemes
- Numerical Results

2D Dirichlet exterior problem

- Dirichlet Problem



$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_x u(x,t) = 0, \qquad x \in \mathbb{R}^2 \setminus \Gamma, \quad t \in (0,T),$$

$$u(x,0) = u_t(x,0) = 0, \qquad x \in \mathbb{R}^2 \setminus \Gamma,$$

$$u(x,t) = g(x,t), \qquad (x,t) \in \Sigma_T := \Gamma \times [0,T]$$

- Representation Formula $\quad x \in \mathbb{R}^2 \setminus \Gamma \quad t \in [0,T]$

$$u(x,t) = \int_0^t \int_\Gamma G(x-\xi, t-\tau)\phi(\xi,\tau)\, d\gamma_\xi d\tau =: (V\phi)(x,t) \ ,$$

with $G(x,t) = \frac{1}{2\,\pi} \frac{H[t-|x|]}{(t^2-|x|^2)^{1/2}}$ and $\phi = \left[\frac{\partial u(x,t)}{\partial \mathbf{n}_x}\right]$ across Γ

- Boundary Integral Equation

$$\boxed{(V\phi)(x,t) = g(x,t)} \quad x \in \Gamma \quad t \in [0,T]$$

Energetic weak formulation

- Laplace transform methods [C. Lubich (1994), T. Ha Duong (2003)]
- Time stepping methods [R. Kress (1997), M. Costabel (2004)]
- Space-time integral equations
  - Collocation [A. Frangi (2000)]
  - L^2-weak formulation [E. Bécache (1993)]

  $\longrightarrow$ Numerical instabilities

- Energy Identity

$$\mathcal{E}(T;u) := \frac{1}{2}\int_{\mathbb{R}^n}\left(\frac{\partial u(x,T)}{\partial t}^2 + |\nabla u(x,T)|^2\right)dx = \int_{\Sigma_T}\frac{\partial u}{\partial t}(x,t)\left[\frac{\partial u}{\partial \mathbf{n}}\right](x,t)\,d\gamma_x dt$$

- Bilinear Form $\quad a_{\mathcal{E}}(\phi,\psi) := \int_{\Sigma_T}(V\phi)_t\psi\,d\gamma_x dt$

- Energetic Weak Formulation associated to BIE $\quad (V\phi)(x,t) = g(x,t)$

$$\int_{\Sigma_T}(V\phi)_t\psi\,d\gamma_x dt = \int_{\Sigma_T} g_t\psi\,d\gamma_x dt$$

Theoretical Analysis

$$a_{\mathcal{E}}(\phi,\psi) := < (V\phi)_t, \psi >_{L^2(\Sigma_T)} = < g_t, \psi >_{L^2(\Sigma_T)}$$

- 1D Problem

$a_{\mathcal{E}}$ continuous and coercive $\longrightarrow$ stability and convergence

[A. Aimi and M. Diligenti
"*A new space-time energetic formulation for wave propagation analysis in layered media by BEMs*."
Int. J. for Num. Meth. In Eng., 2008]

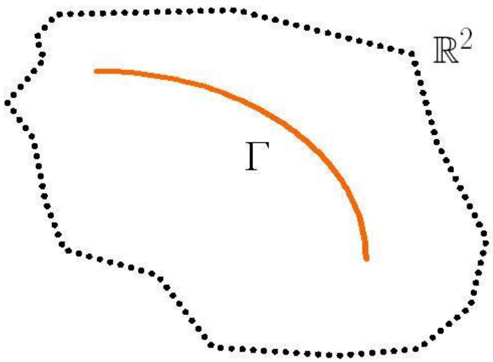
- 2D Problem

$a_{\mathcal{E}}$ continuous and coercive w.r.t. L^2 - norm under a constraint on space oscillation

[A. Aimi, M. Diligenti, C.Guardasoni, I. Mazzieri, S. Panizzi
"An energy approach to space-time Galerkin BEM for wave propagation problems."
Internat. J. Numer. Methods Engrg., 2009.]

2D Neumann exterior problem

- Neumann Problem



$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_x u(x,t) = 0, \qquad x \in \mathbb{R}^2 \setminus \Gamma, \quad t \in (0,T),$$

$$u(x,0) = u_t(x,0) = 0, \qquad x \in \mathbb{R}^2 \setminus \Gamma,$$

$$\frac{\partial u(x,t)}{\partial \mathbf{n}_x} = g(x,t), \qquad (x,t) \in \Sigma_T := \Gamma \times [0,T]$$

- Boundary Integral Equation $(D\phi)(x,t) = g(x,t) \qquad x \in \Gamma \quad t \in [0,T]$

$$(D\phi)(x,t) = \int_0^t \oint_\Gamma \frac{\partial^2}{\partial \mathbf{n}_x \partial \mathbf{n}_\xi} G(x-\xi, t-\tau)\phi(\xi,\tau)\, d\gamma_\xi d\tau \ ,$$

with $G(x,t) = \dfrac{1}{2\,\pi} \dfrac{H[t-|x|]}{(t^2-|x|^2)^{1/2}}$ and $\phi = [u]$ across Γ

- Energetic Weak Formulation: $\mathcal{E}(T;u) = \displaystyle\int_{\Sigma_T} \frac{\partial\,[u]}{\partial t}(x,t)\, \frac{\partial u}{\partial \mathbf{n}}(x,t)\; d\gamma_x dt$

$$\int_{\Sigma_T} (D\phi)\psi_t \, d\gamma_x dt = \int_{\Sigma_T} g\psi_t \, d\gamma_x dt$$

Numerical approximation

- Uniform decomposition of (0,T) $\Delta t = T/N_T, \qquad t_k = k\,\Delta t, \;\; k = 1, \cdots, N_T$

- Decomposition $\Delta_x = \{e_1, \ldots, e_{N_\Gamma}\}$ on Γ

- $$\phi(x,t) \simeq \sum_{i=1}^{N_\Gamma} \sum_{k=0}^{N_T-1} \alpha_{ik} \widetilde{\phi}_i(x) \bar{\phi}_k(t), \qquad \alpha_{ik} \simeq \phi(x_i, t_{k+1})$$

- $\widetilde{\phi}_i(x)$ basis polinomial functions; $\widetilde{\phi}_i|_{e_i} =: \phi_i$ local function on e_i

- if $\phi = \left[\dfrac{\partial u(x,t)}{\partial \mathbf{n}_x}\right] \qquad \bar{\phi}_k(t) = H[t - t_k] - H[t - t_{k+1}],$

  if $\phi = [u]$

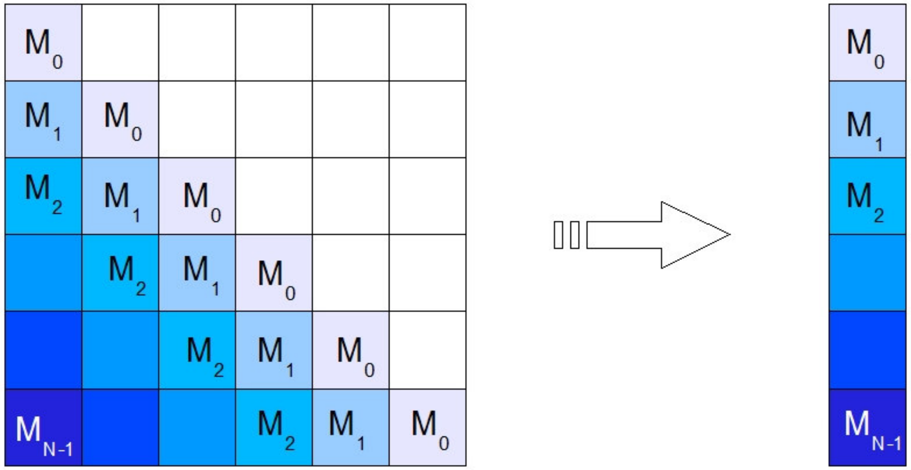
$$\bar{\phi}_k(t) = \frac{t - t_k}{\Delta t} H[t - t_k] - 2\frac{t - t_{k+1}}{\Delta t} H[t - t_{k+1}] + \frac{t - t_{k+2}}{\Delta t} H[t - t_{k+2}],$$

Numerical approximation

- The resulting linear system matrix element is

$$\int_\Gamma \int_0^T \widetilde{\phi}_j(x)\bar{\phi}_h(t) \int_\Gamma \int_0^t \mathcal{K}(x,\xi,t,\tau)\widetilde{\phi}_i(\xi)\bar{\phi}_k(\tau)\,d\gamma_\xi\, d\tau\, d\gamma_x\, dt$$

- Two analytical integrations in time
- Two numerical integrations in space
- The resulting linear system matrix has a Toeplitz structure



- Block forward substitution being M_0 symmetric and positive definite.

Numerical approximation

- Two analytical integrations in time

Dirichlet Problem:

$$\sum_{\alpha,\beta=0}^{1}(-1)^{\alpha+\beta}\int_\Gamma \widetilde{\phi}_j(x)\int_\Gamma \mathcal{B}(r,t_{h+\alpha},t_{k+\beta})\widetilde{\phi}_i(\xi)d\gamma_\xi d\gamma_x, \quad \text{where} \quad r=\|x-\xi\|_2$$

$$\mathcal{B}(r,t_h,t_k)=\frac{1}{2\pi}H[(t_h-t_k)-r]\left\{\log\left[(t_h-t_k)+\sqrt{(t_h-t_k)^2-r^2}\right]-\log[r]\right\}$$

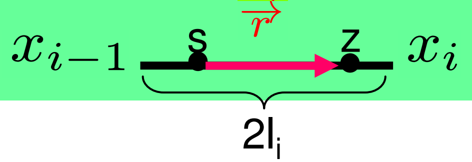
Neumann Problem:

$$\sum_{\alpha,\beta,\delta=0}^{1}(-1)^{\alpha+\beta+\delta}\int_\Gamma \widetilde{\phi}_j(x)\int_\Gamma \mathcal{C}(r,t_{h+\alpha},t_{k+\beta+\delta})\widetilde{\phi}_i(\xi)d\gamma_\xi d\gamma_x, \quad \text{where}$$

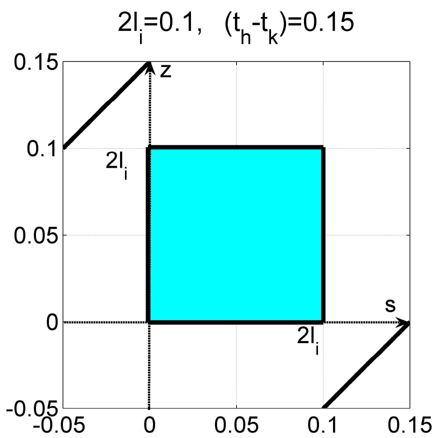
$$\begin{aligned}\mathcal{C}(r,t_h,t_k)=&\frac{1}{4\pi\Delta t}H[(t_h-t_k)-r]\,(\mathbf{n}_\xi\cdot\mathbf{n}_x)\\ &\cdot\left\{\log\left[(t_h-t_k)+\sqrt{(t_h-t_k)^2-r^2}\right]-\log[r]-\frac{(t_h-t_k)\sqrt{(t_h-t_k)^2-r^2}}{r^2}\right\}\\ &+\frac{1}{2\pi\Delta t}H[(t_h-t_k)-r]\,\frac{(\mathbf{r}\cdot\mathbf{n}_x)(\mathbf{r}\cdot\mathbf{n}_\xi)}{r^2}\,\frac{(t_h-t_k)\sqrt{(t_h-t_k)^2-r^2}}{r^2}\end{aligned}$$

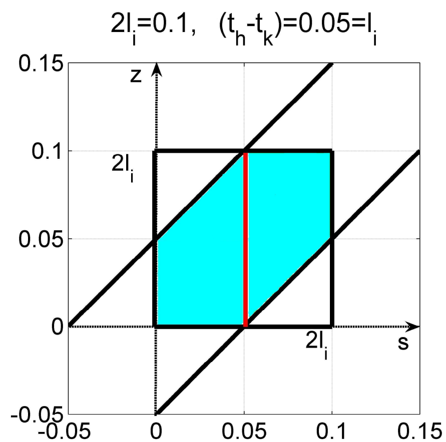
- Space integral singularities as in elliptic problems

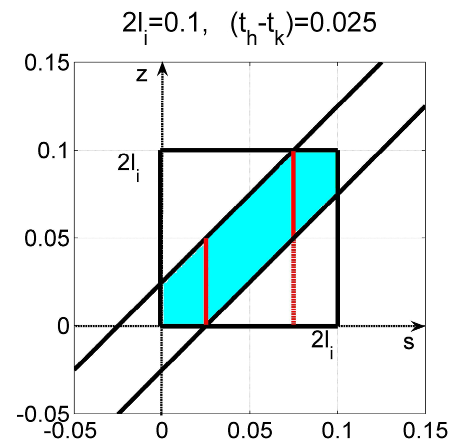
Element by element technique: coincident elements



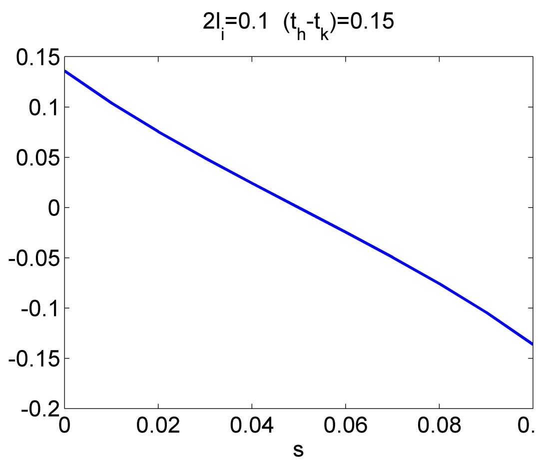
$$H[(t_h - t_k) - |s - z|]$$

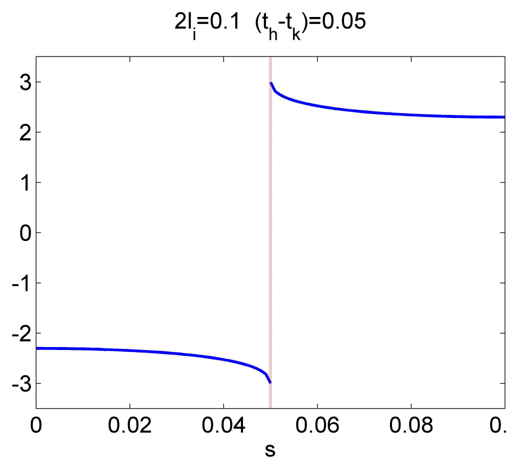


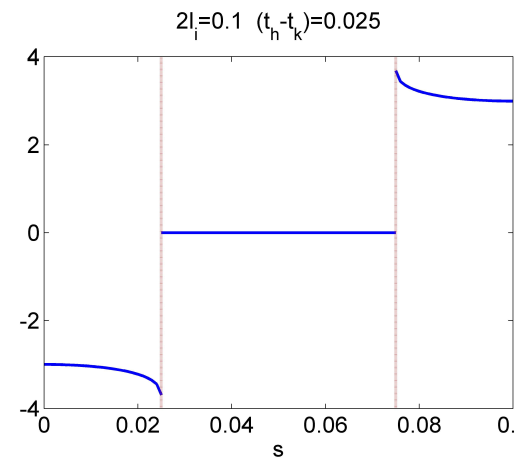




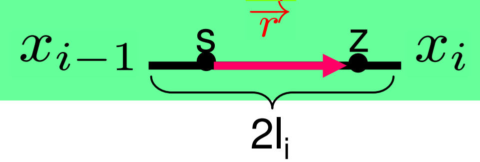
$$\frac{d}{ds}\int_{e_i} H[(t_h - t_k) - |s - z|] \log\left[(t_h - t_k) + \sqrt{(t_h - t_k)^2 - r^2}\right] dz$$



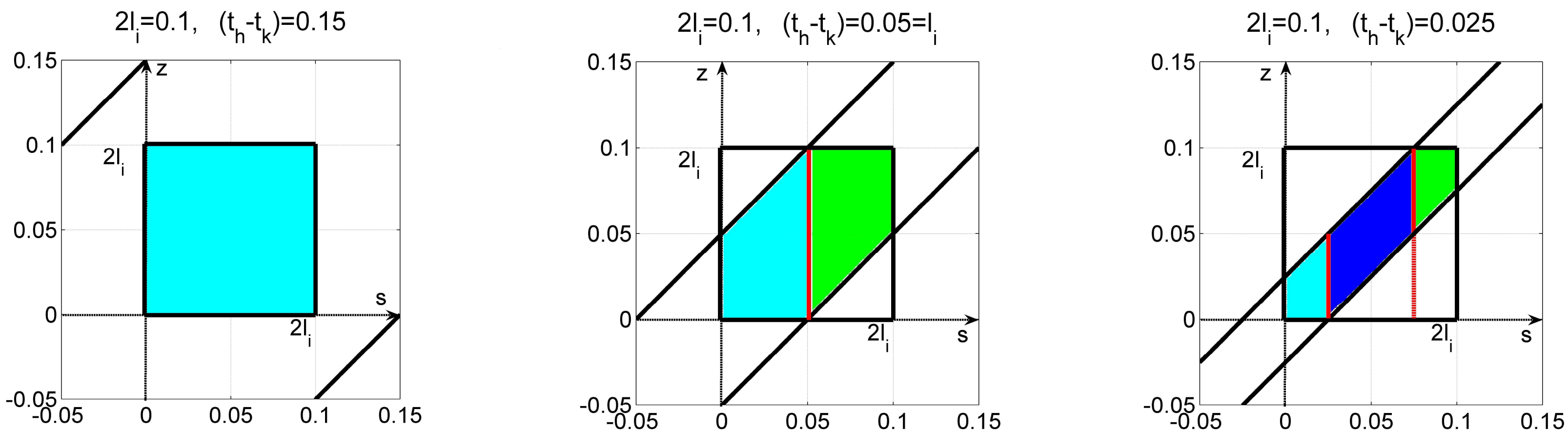




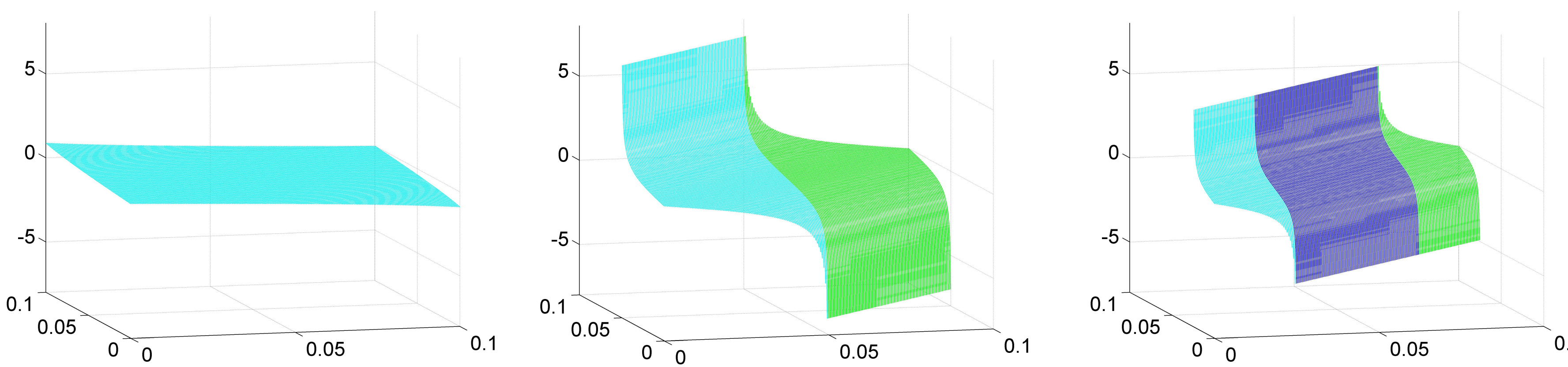
Element by element technique: coincident elements



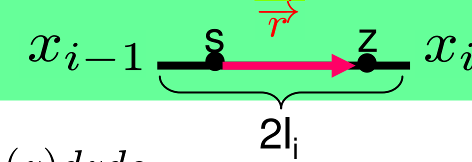
$$H[(t_h - t_k) - |s - z|]$$



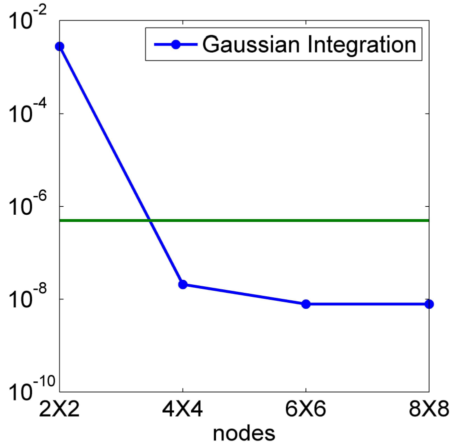
$$\frac{\partial}{\partial z}\sqrt{(t_h - t_k)^2 - |s - z|^2}$$

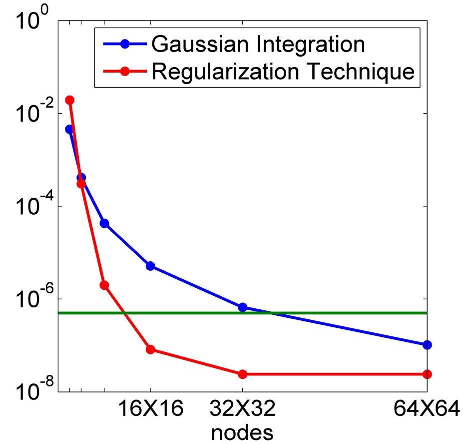


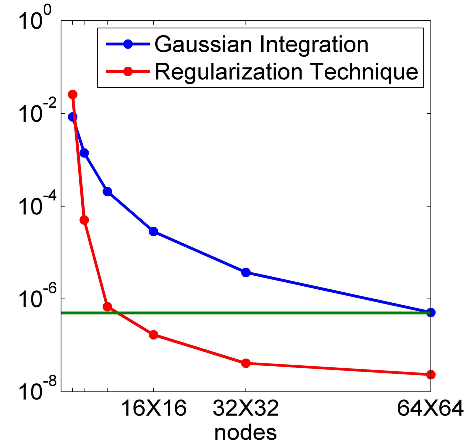
Element by element technique: coincident elements



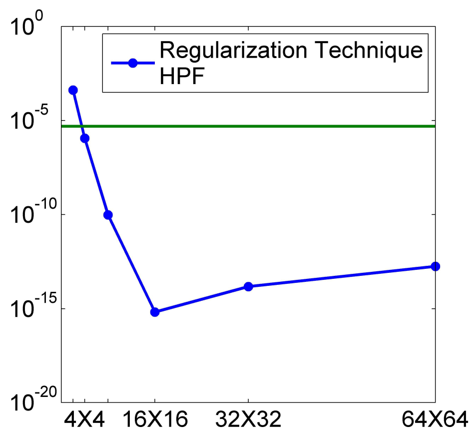
$$\int_{e_i} \phi_i(s) \int_{e_i} \log\left[(t_h-t_k)+\sqrt{(t_h-t_k)^2-|s-z|^2}\right]\phi_i(z)\,dz\,ds$$

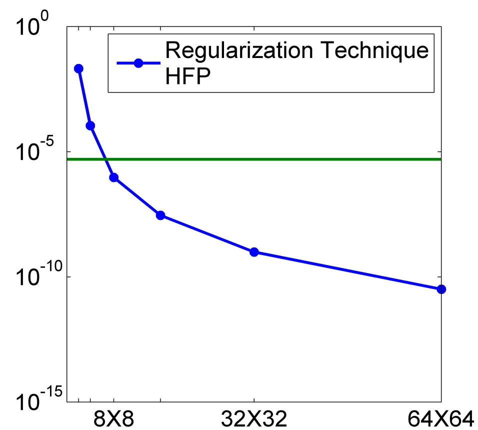


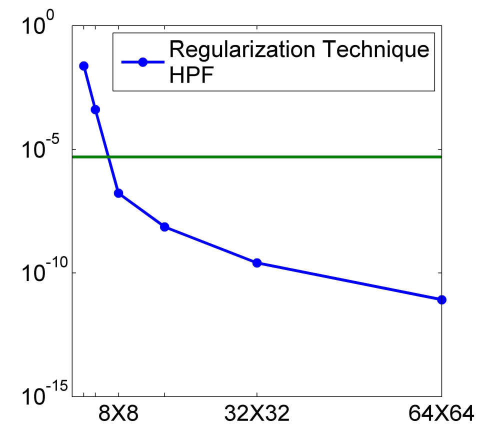




$$\fint_{e_i} \phi_i(s) \fint_{e_i} \frac{(t_h-t_k)\sqrt{(t_h-t_k)^2-|s-z|^2}}{|s-z|^2}\,\phi_i(z)\,dz\,ds$$







- Quadrature formulas for space integrals [A. Aimi, M. Diligenti, G. Monegato, (1997)]

Numerical approximation: Quadrature formulas

- Regularization technique [Monegato G., Scuderi L., 1999]

$$\int_0^1 f(s)ds = \int_0^1 f(\varphi(\tilde{s}))\varphi'(\tilde{s})d\tilde{s}$$

with

$$\varphi(\widetilde{s}) = \frac{(p+q-1)!}{(p-1)!(q-1)!}\int_0^{\widetilde{s}} u^{p-1}(1-u)^{q-1}du, \qquad p,q \geq 1$$

$$\boxed{\varphi^{(i)}(0) = 0, \quad \varphi^{(j)}(1) = 0, \quad i = 1,...,p-1, \quad j = 1,...,q-1}$$

- Hadamard finite part rules [Monegato G., 1994]

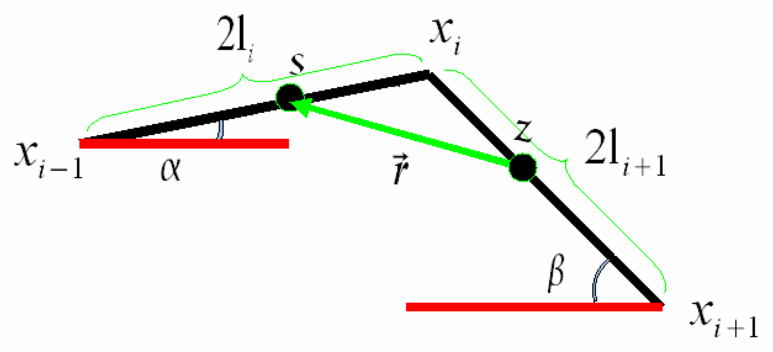
$$\neq\!\!\!\!\!\!\int_0^1 \frac{f(s)}{s}\,\mathrm{d}s \approx w_0^{GR} f(0) + \sum_{k=1}^{n} w_k^{GR} f(s_k^{GR})$$

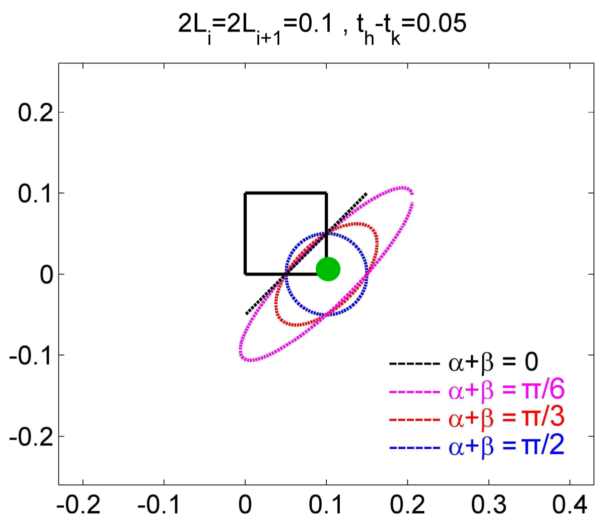
$$s_k^{GR} = \frac{1+x_k}{2} \qquad w_k^{GR} = \frac{\lambda_k}{2s_k^{GR}} \quad k = 1,...,n \qquad w_0^{GR} = -\sum_{k=1}^{n} w_k^{GR},$$

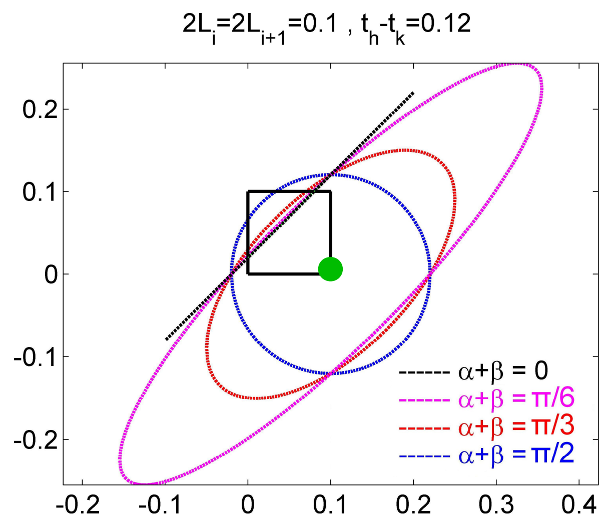
x_k zeros of the Legendre polynomial of degree n

λ_k Christoffel numbers associated with the n-point Gauss-Legendre formula

Element by element technique: contiguous elements







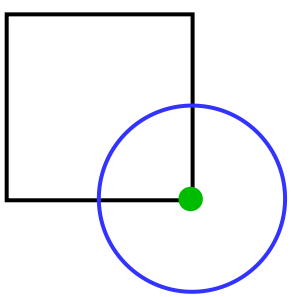
Element by element technique: contiguous elements

$$I = \int_0^{2l_i} \frac{\phi_i(s)}{4\pi\Delta} \int_0^{2l_{i+1}} (\mathbf{n}_z \cdot \mathbf{n}_s) \frac{\overbrace{\boxed{\phi_{i+1}(z)\sqrt{\Delta^2 - r^2}}}^{F(s,z)}}{r^2} H[\Delta - r]\, dz\, ds =$$

$$= \int_0^{2l_i} \frac{\phi_i(s)}{4\pi\Delta} \int_0^{2l_{i+1}} (\mathbf{n}_z \cdot \mathbf{n}_s) \frac{H[\Delta - r]}{r^2} \left\{ \boxed{\left[F(s,z) - \sum_{k=0}^{1} F_z^{(k)}(2l_i, 0) \frac{z^k}{k!} \right]} + \right.$$

$$\left. + F(2l_i, 0) + F_z^{'}(2l_i, 0) z \right\} dz\, ds = I_1 + I_2 + I_3$$

$$I_1 = \int_0^{2l_i} \frac{\phi_i(s)}{4\pi\Delta} \int_0^{2l_{i+1}} (\mathbf{n}_z \cdot \mathbf{n}_s) \frac{H[\Delta - r]}{r^2} \boxed{\left[F(s,z) - \sum_{k=0}^{2} F_z^{(k)}(2l_i, 0) \frac{z^k}{k!} \right]} dz\, ds$$



$$\alpha + \beta = \frac{\pi}{2} \qquad 2l_i = 2l_{i+1} = 0.1 \qquad \Delta = t_h - t_k = 0.05$$

Gauss+Reg.Tech.
$6.156878 \cdot 10^{-3}$

Relative Error I_1

n=m	p=q=1	p=q=2	p=q=3
8	$1.051507 \cdot 10^{-3}$	$1.804700 \cdot 10^{-3}$	$2.610697 \cdot 10^{-3}$
16	$3.925552 \cdot 10^{-5}$	$3.467667 \cdot 10^{-5}$	$1.969021 \cdot 10^{-5}$
32	$2.071282 \cdot 10^{-6}$	$2.857438 \cdot 10^{-7}$	$2.749069 \cdot 10^{-8}$

Element by element technique: contiguous elements

$$I_2 = \frac{F(2l_i,0)}{4\pi\Delta}\int_0^{2l_i}\phi_i(s)\int_0^{2l_{i+1}}(\mathbf{n}_z\cdot\mathbf{n}_s)\frac{1}{r^2}dzds = \frac{F(2l_i,0)}{4\pi\Delta}-\!\!\!\!\!\!\int_0^{2l_i}\phi_i(s)\frac{\mathcal{Q}_2(s)}{2l_i-s}ds$$

HFP
3.476736·10^{-1}

n	1	2
Relative Error I_2	$1.068267\cdot10^{-2}$	$1.232387\cdot10^{-15}$

$$I_3 = \frac{F_z'(2li,0)}{4\pi\Delta}\int_0^{2l_i}\phi_i(s)\int_0^{2l_{i+1}}(\mathbf{n}_z\cdot\mathbf{n}_s)\frac{z}{r^2}dzds = \frac{F_z'(2li,0)}{4\pi\Delta}\int_0^{2l_i}\phi_i(s)\mathcal{Q}_3(s)ds$$

Gauss-Legendre
2.261957·10^{-2}

n	2	4	6
Relative Error I_3	$3.128988\cdot10^{-3}$	$2.048653\cdot10^{-5}$	$1.005310\cdot10^{-9}$

1st numerical example (Dirichlet)

Boundary condition on $\Gamma = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (\cos\alpha, \sin\alpha), \alpha \in [0,\pi]\}$

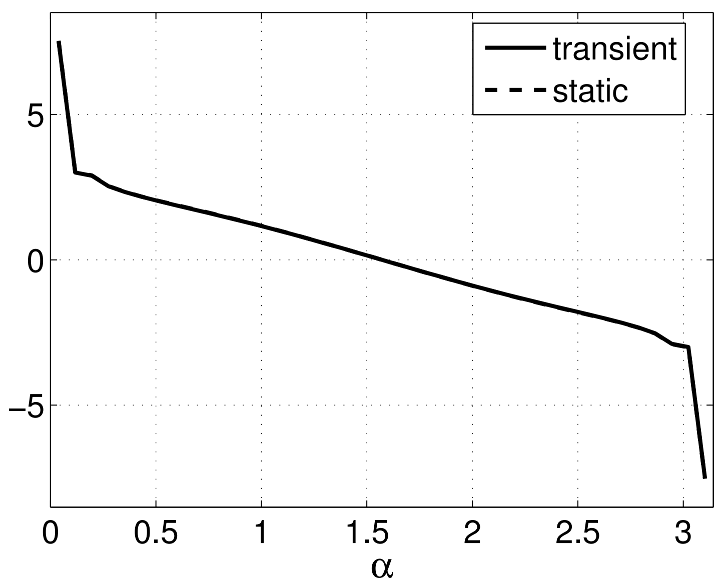
$$\boxed{g(\alpha,t) = H[t]\, f(t)\, \cos\alpha \qquad f(x,t) = \begin{cases} \sin^2(4\pi t)\,, & \text{if } 0 \le t \le 1/8 \\ 1\,, & \text{if } t \ge 1/8 \end{cases}}$$

Time interval $(0,10)$, $\Delta t = 0.1$, $\qquad \mathbf{x}_i = (\cos\frac{i\pi}{40}, \sin\frac{i\pi}{40}) \quad i = 0, 40$

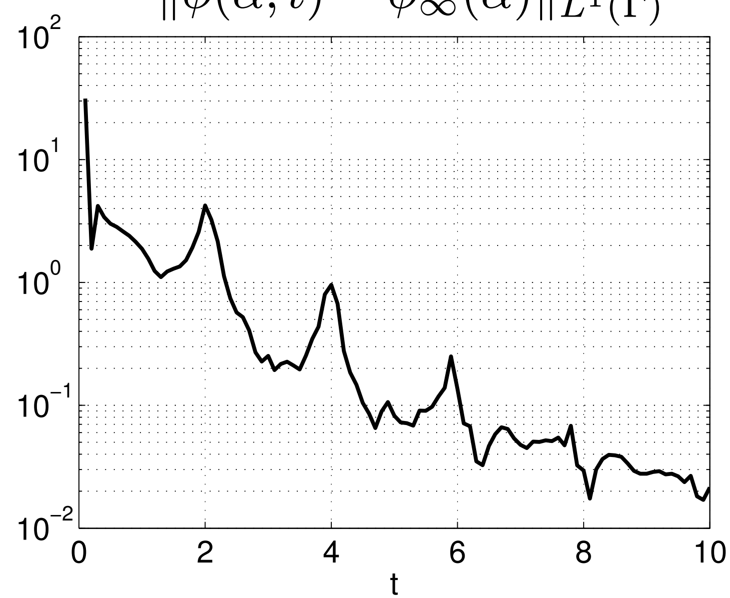
Limit for $t \to \infty \qquad g(\alpha,t) \to g_\alpha = \cos\alpha$

$$\begin{cases} -\triangle u_\infty = 0 & \text{in} \quad \mathbb{R}^2 \setminus \Gamma\,, \quad u(\mathbf{x}) = O(1) \quad \text{for} \quad \|\mathbf{x}\|_2 \to \infty \\ u_\infty = \cos\alpha & \text{on} \quad \Gamma \end{cases} .$$

$\phi(\alpha, 10)\,, \quad \phi_\infty(\alpha)$

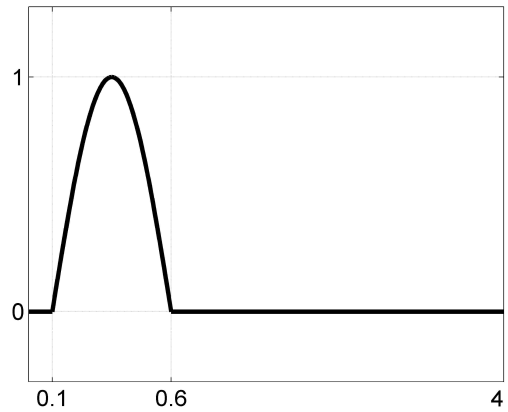


$\|\phi(\alpha,t) - \phi_\infty(\alpha)\|_{L^1(\Gamma)}$



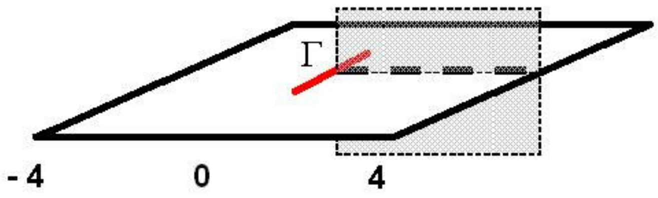
2nd numerical example (Dirichlet)

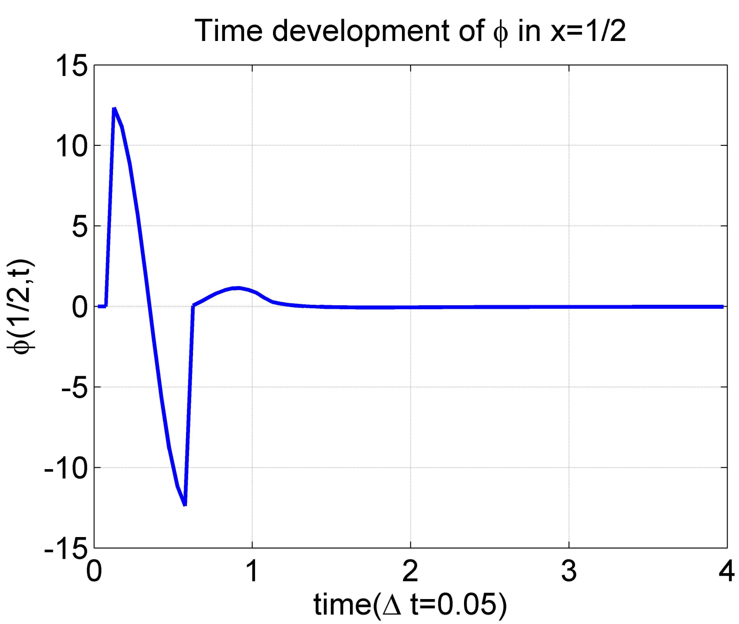
Boundary condition:

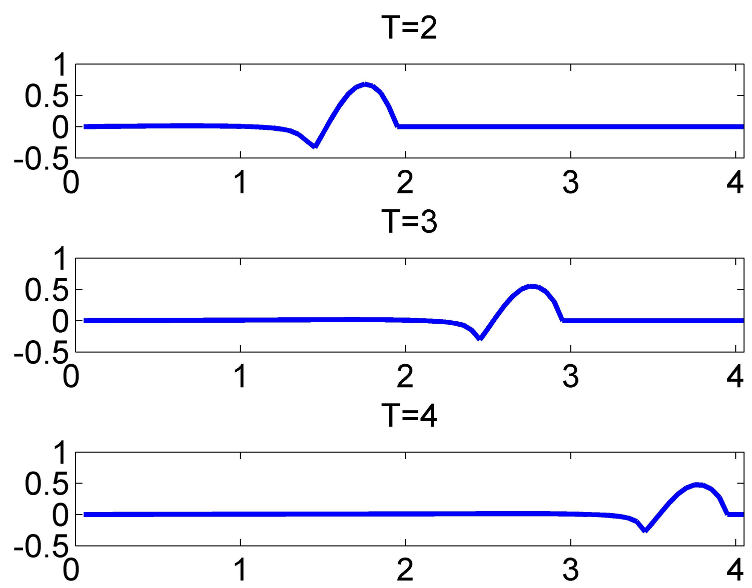


$$\Gamma = \{(x,0),\ x \in [0,1]\}$$ Constant shape function

Time interval $(0,4),\ \Delta t = 0.05,\ \Delta x = 0.05$







3rd numerical example (Neumann)

[Bécache E., Ha Duong T. (1993,1994)]

<u>Boundary condition</u>: *plane linear wave*

$$g(x,t) = -\frac{\partial}{\partial \mathbf{n_x}} f(t - \mathbf{k}\cdot\mathbf{x})\big|_{\mathbf{\Gamma}},$$
$$\mathbf{k} = (\cos\theta, \sin\theta), \quad \theta = \pi/3$$
$$f(t) = tH(t)$$

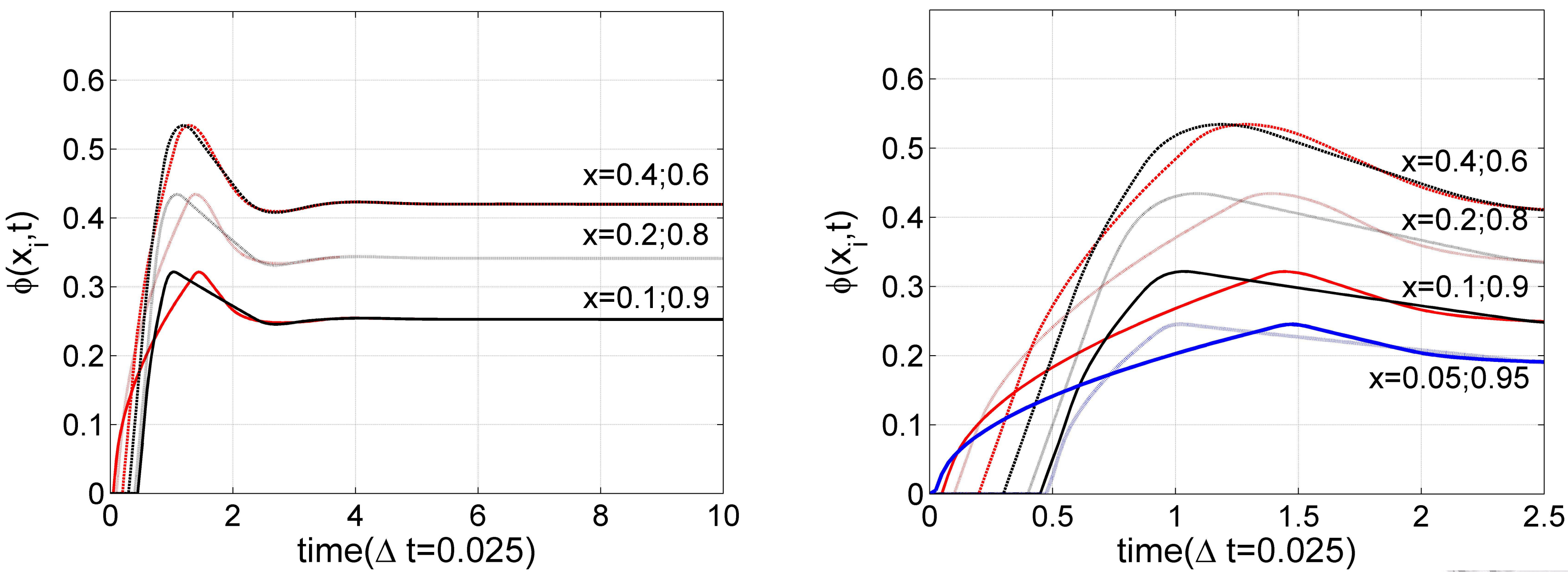
$$\Gamma = \{(x,0),\ x \in [0,1]\}$$

Time interval $(0,10)$,

$\Delta t = 0.025,\ \Delta x = 0.05$

Linear shape function

Time evolution in some point of Γ



3rd numerical example (Neumann)

[Bécache E., Ha Duong T. (1993,1994)]

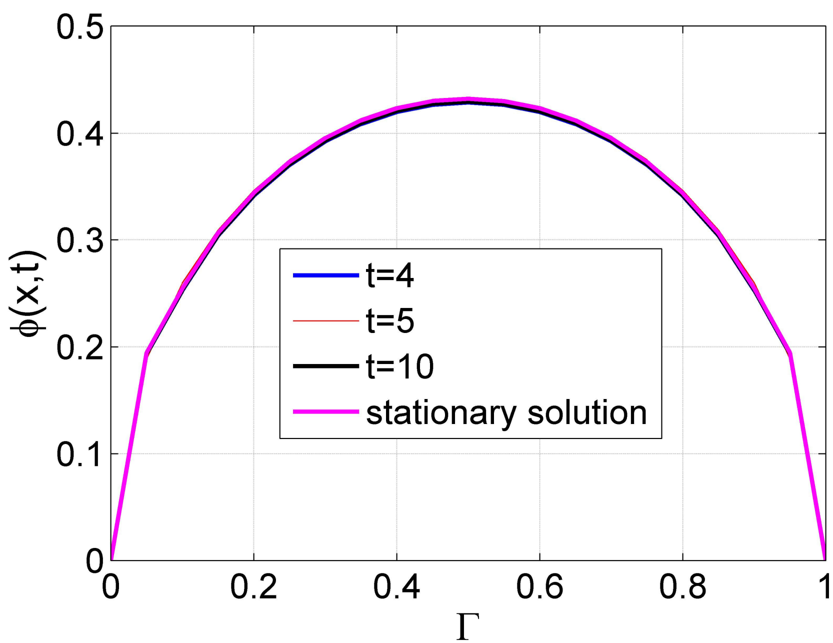
Limit for $t \to \infty$ $\qquad g(x,t) \to g_\theta = \sin\theta$

$$\begin{cases} \Delta u_\infty = 0 & \text{in} \quad \mathbb{R}^2 \setminus \Gamma \\ \dfrac{\partial u_\infty}{\partial \mathbf{n_x}} = g_\theta & \text{on} \quad \Gamma \end{cases}$$

Analytical static solution

$$\phi_\theta^\infty = [u_\infty] = \sin\theta \sqrt{x(1-x)}$$

$$\text{COD} = [u]_\Gamma$$



4th numerical example (Neumann)

[Bécache E., Ha Duong T. (1993,1994)]

Boundary condition: *plane harmonic wave*

$$g(x,t) = -\frac{\partial}{\partial \mathbf{n_x}} f(t - \mathbf{k}\cdot\mathbf{x})\Big|_\Gamma ,$$

$$f(t) = \begin{cases} 0 & \text{if} \quad t \le 0 \\ \sin^2(\frac{\omega t}{2}) & \text{if} \quad 0 \le t \le \frac{\pi}{\omega} \\ \sin(\frac{\omega t}{2}) & \text{if} \quad t \ge \frac{\pi}{\omega} \end{cases}$$

$$\mathbf{k} = (\cos\theta, \sin\theta), \ \ \theta = \pi/3, \ \ \omega = 8\pi$$

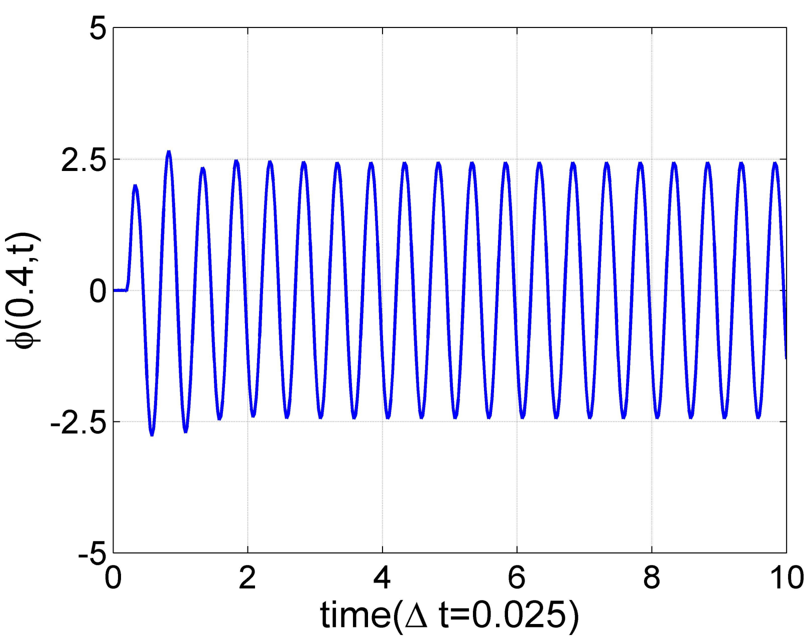
$\Gamma = \{(x,0),\ x \in [0,1]\}$

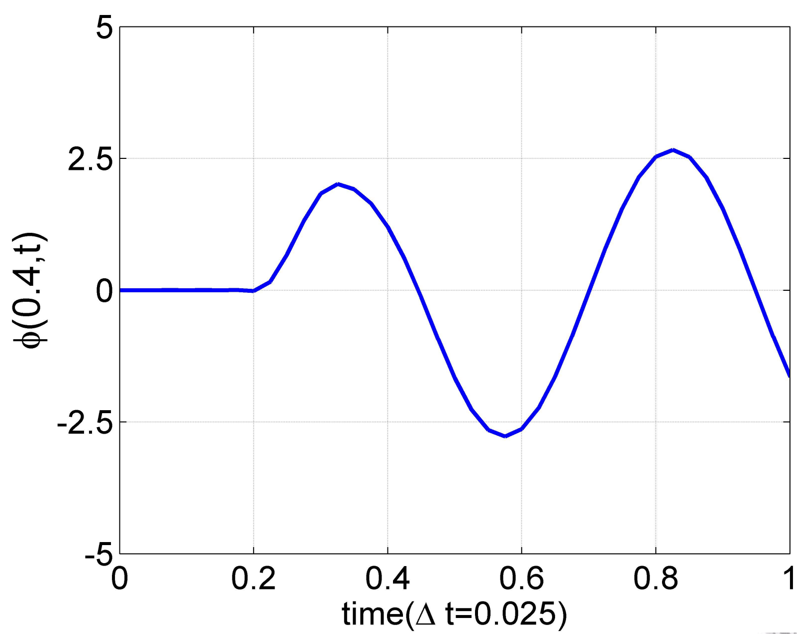
Time interval $(0, 10)$,

$\Delta t = 0.025,\ \Delta x = 0.05$

Linear shape function

Time evolution in $x = 0.4$



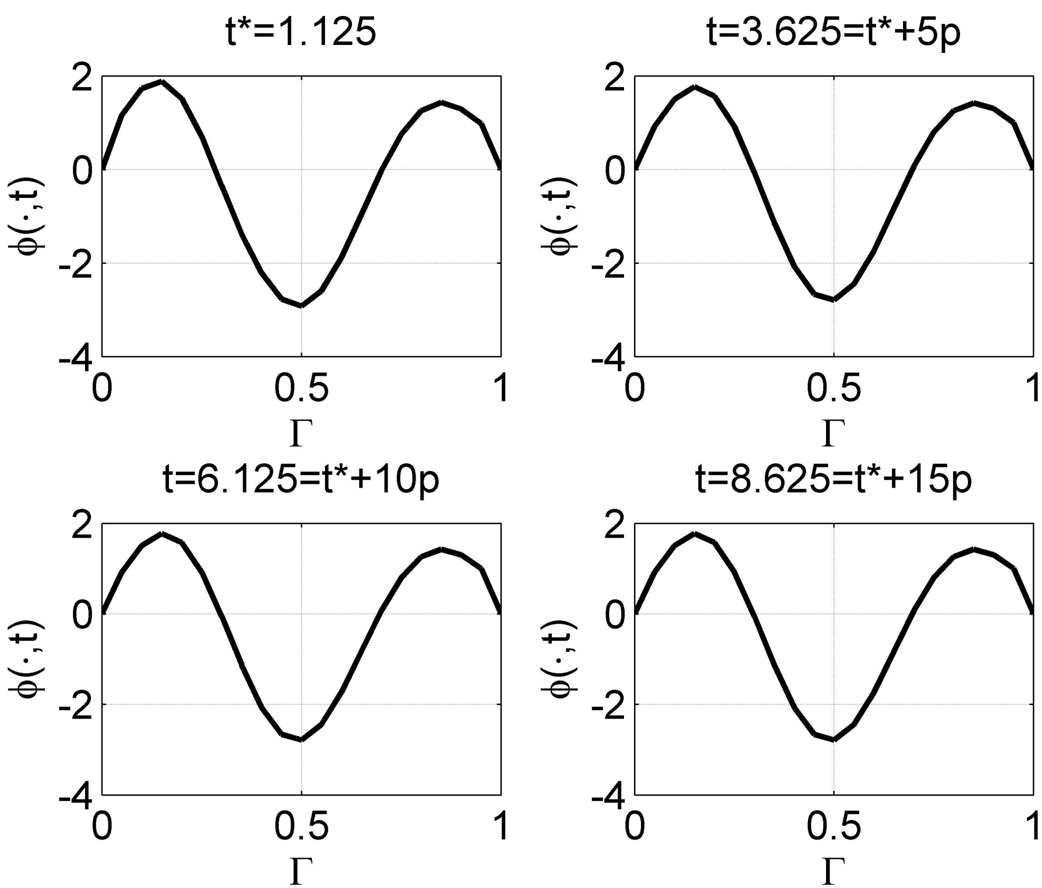


4th numerical example (Neumann)

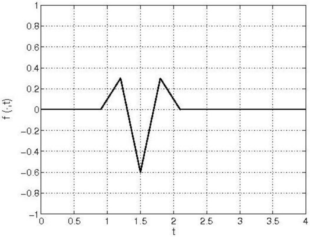
[Bécache E., Ha Duong T. (1993,1994)]

Expected behavior for **large times**:

ϕ harmonic with the same period $p = 0.5$ of Neumann datum



Boundary condition



Time interval $(0, 4)$
$\Delta t = 0.1 \quad \Delta\alpha = \frac{\pi}{40}$
Linear shape functions

Future works

- Complete the analysis of energetic bilinear form for 2D problems
- BEM-FEM coupling in 2D elastodynamic
- Energetic weak formulation to 3D problems in the context of applied seismology