High order parametric polynomial approximation of conic sections

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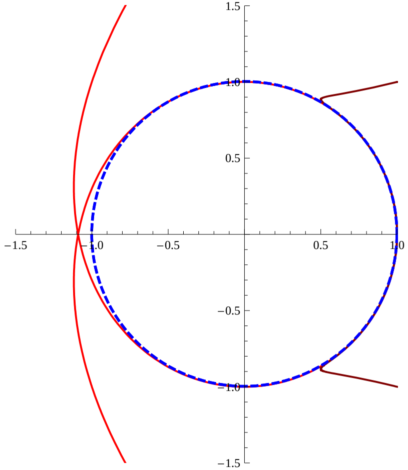
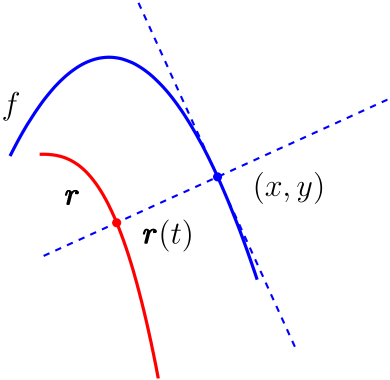


Figure: The unit circle (blue dashed), quintic polynomial approximation given by Lyche and Mørken in 1995 (brown) and the new quintic approximant (red).

Parametric approximation

- Let
$$f(x, y) = 0, \quad (x, y) \in \mathcal{D} \subset \mathbb{R}^2,$$
be a segment of a regular smooth planar curve $\boldsymbol{f}$.
- Suppose
$$\boldsymbol{r} : I \subset \mathbb{R} \to \mathbb{R}^2 : t \mapsto \begin{pmatrix} x_r(t) \\ y_r(t) \end{pmatrix}$$
is a parametric approximation of the curve segment $\boldsymbol{f}$ and
$$f(x_r(t), y_r(t)) = \varepsilon(t), \quad t \in [a, b].$$

- Consider the regular parameterization of $\boldsymbol{r}$ with respect to the normal of $\boldsymbol{f}$, i.e., every point (x, y) on a curve f defines a unique parameter $t := t(x, y)$ on a curve $\boldsymbol{r}$:



- This provides an upper bound on Hausdorff and parametric distance.

Theorem

If the curve $\mathbf{r}$ *can be regularly reparameterized by the normal to* f, *and* ε *is small enough, the distance between curves is bounded by*

$$\max_{(x,y)\in\mathcal{D}} \frac{|\varepsilon(t(x,y))|}{\sqrt{f_x^2(x,y)+f_y^2(x,y)}} + \mathcal{O}(\varepsilon\,(t\,(x,y))^2).$$

Conic sections

- Only ellipse and hyperbola are interesting to consider (no polynomial parameterization).
- A particular conic section is given as

$$f(x, y) := x^2 \pm y^2 - 1 = 0.$$

- The main problem: find two nonconstant polynomials x_n and y_n of degree at most n, such that

$$x_n^2(t) \pm y_n^2(t) = 1 + \varepsilon(t),$$

where ε is a polynomial of degree at most $2\,n$.

- Assume that at least one point is interpolated, i.e., $\varepsilon(0) = 0$. In order to get ε as small as possible, it is natural to choose $\varepsilon(t) := t^{2n}$.
- If also a tangent direction is prescribed at the interpolation point, we have

$$(x_n(0), y_n(0)) = (1, 0), \quad (x_n'(0), y_n'(0)) = (0, 1).$$

- Thus

$$x_n(t) = 1 + \sum_{\ell=1}^{n} a_\ell\, t^\ell,$$
$$y_n(t) = t + \sum_{\ell=2}^{n} b_\ell\, t^\ell,$$

and

- $$x_n^2(t) \pm y_n^2(t) = 1 + \left(a_n^2 \pm b_n^2\right)\ t^{2n}.$$
- A reparameterization $t \mapsto \sqrt[2n]{|a_n^2 \pm b_n^2|}\, t$ gives

$$x_n(t) := 1 + \sum_{\ell=1}^{n} \alpha_\ell\, t^\ell, \quad y_n(t) := \sum_{\ell=1}^{n} \beta_\ell\, t^\ell, \quad \beta_1 > 0,$$

where

$$x_n^2(t) \pm y_n^2(t) = 1 + \mathrm{sign}(a_n^2 \pm b_n^2)\, t^{2n}.$$

Many acceptable solutions exist.

Table: The number of appropriate solutions in all three cases for $n = 1, 2, \ldots, 10$.

n	1	2	3	4	5	6	7	8	9	10
elliptic	1	1	3	6	15	27	63	120	246	495
hyp. $a_n^2 < b_n^2$	1	0	1	2	5	8	19	32	68	120
hyp. $a_n^2 > b_n^2$	0	1	0	2	0	9	0	32	0	125

Solutions

- Solving the equation

$$x_n^2(t) \pm y_n^2(t) = 1 \pm t^{2n}$$

  is equivalent to solving

$$x_n^2(t) \pm y_n^2(t) = 1$$

  in the factorial ring $\mathbb{R}[t]/t^{2n}$.
- There are additional restrictions, classic algebraic tools can not be applied.

Idea

- Rewrite

$$(x_n(t) + \boldsymbol{i}\, y_n(t))\,(x_n(t) - \boldsymbol{i}\, y_n(t)) = \prod_{k=0}^{2n-1}\left(t - e^{\boldsymbol{i}\,\frac{2k+1}{2n}\pi}\right),$$

  where the right-hand side is the factorization of $1 + t^{2n}$ over $\mathbb{C}$.

- Thus

$$x_n(t) + \boldsymbol{i}\, y_n(t) = \gamma \prod_{k=0}^{n-1}\left(t - e^{\boldsymbol{i}\,\sigma_k\frac{2k+1}{2n}\pi}\right), \quad \gamma \in \mathbb{C}, \quad |\gamma| = 1,$$

  where $\sigma_k = \pm 1$.

Best solution

- The best solution should have the minimum error term ε.
- It turns out that this happens if

$$\beta_1 = \frac{1}{\sin \frac{\pi}{2n}}.$$

- Surprisingly, any solution for the elliptic case, for which x_n is even and y_n is odd, can be transformed to the hyperbolic solution by the map

$$x_n(t) \mapsto x_n(\boldsymbol{i}\ t),$$
$$y_n(t) \mapsto -\boldsymbol{i}\ y_n(\boldsymbol{i}\ t).$$

- In particular, this is true for the best solution too, thus it is enough to consider the elliptic case only.

Theorem

Coefficients of the best solution for the elliptic case are

$$\alpha_k = \begin{cases} \displaystyle\sum_{j=0}^{k(n-k)} P(j,k,n-k)\cos\left(\frac{k^2\pi}{2n}+\frac{\pi}{n}j\right); & k \text{ is even}, \\ 0; & k \text{ is odd}, \end{cases}$$

$$\beta_k = \begin{cases} 0; & k \text{ is even}, \\ \displaystyle\sum_{j=0}^{k(n-k)} P(j,k,n-k)\sin\left(\frac{k^2\pi}{2n}+\frac{\pi}{n}j\right); & k \text{ is odd}, \end{cases}$$

where $P(j,k,r)$ *denotes the number of integer partitions of* $j \in \mathbb{N}$ *with* $\leq k$ *parts, all between* 1 *and* r*, where* $k, r \in \mathbb{N}$*, and* $P(0,k,r) := 1$*.*

Examples

Table: Polynomial approximation of the unit circle and maximal normal (radial) error.

n	x_n and y_n	error
3	$x_3(t)=1-2t^2$ $y_3(t)=2t-t^3$	2
4	$x_4(t)=1+(-2-\sqrt{2})t^2+t^4$ $y_4(t)=(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}})(t-t^3)$	0.414213
5	$x_5(t)=1+(-3-\sqrt{5})t^2+(1+\sqrt{5})t^4$ $y_5(t)=(1+\sqrt{5})t+(-3-\sqrt{5})t^3+t^5$	0.089987
6	$x_6(t)=1-2(2+\sqrt{3})t^2+2(2+\sqrt{3})t^4-t^6$ $y_6(t)=(\sqrt{2}+\sqrt{6})t-\sqrt{2}(3+2\sqrt{3})t^3+(\sqrt{2}+\sqrt{6})t^5$	0.013886
$\vdots$	$\vdots$	$\vdots$
15	$\dots$	$1.07280 \cdot 10^{-15}$

It can be shown that the error is $\mathcal{O}(n^{-2n})$.

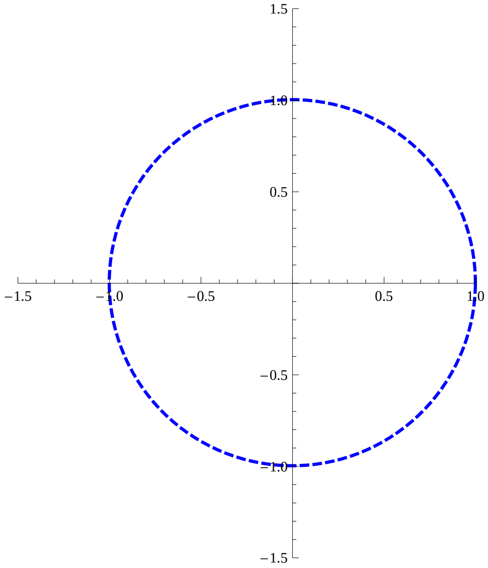


Figure: The unit circle.

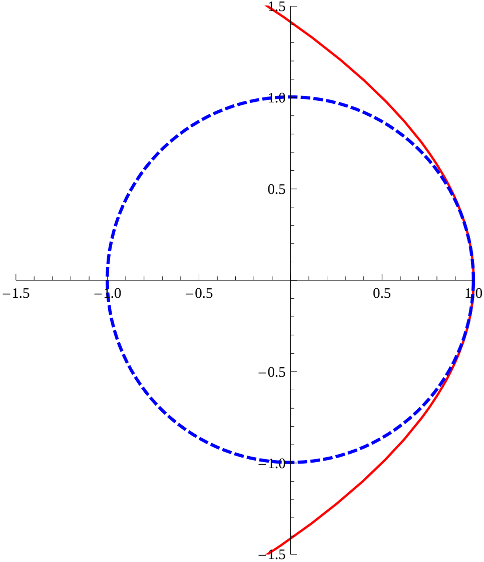


Figure: The unit circle and its polynomial approximant for $n = 2$.

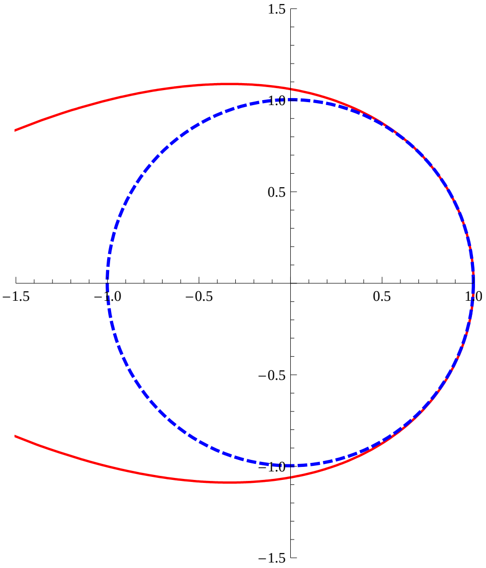


Figure: The unit circle and its polynomial approximant for $n = 3$.

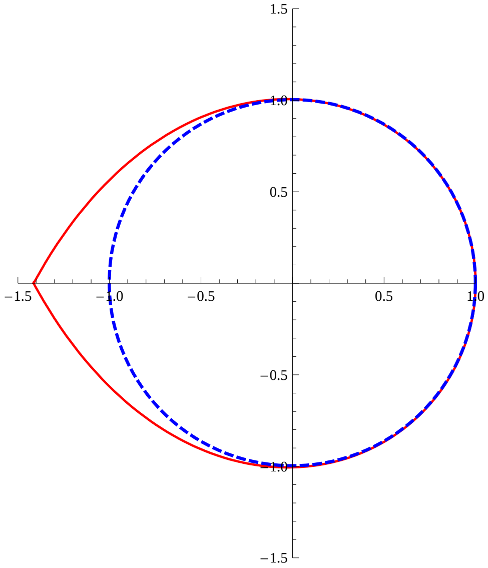


Figure: The unit circle and its polynomial approximant for $n = 4$.

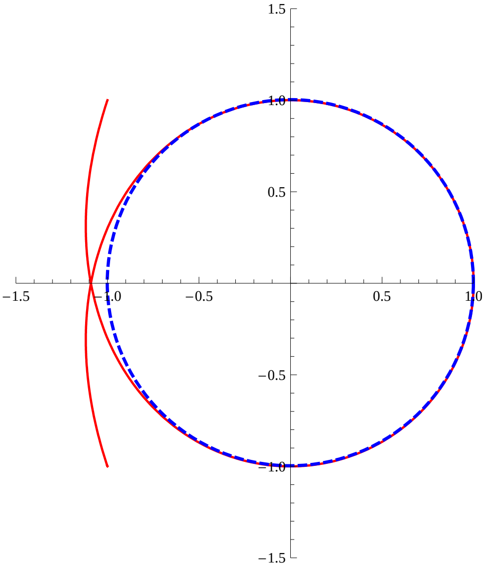


Figure: The unit circle and its polynomial approximant for $n = 5$.

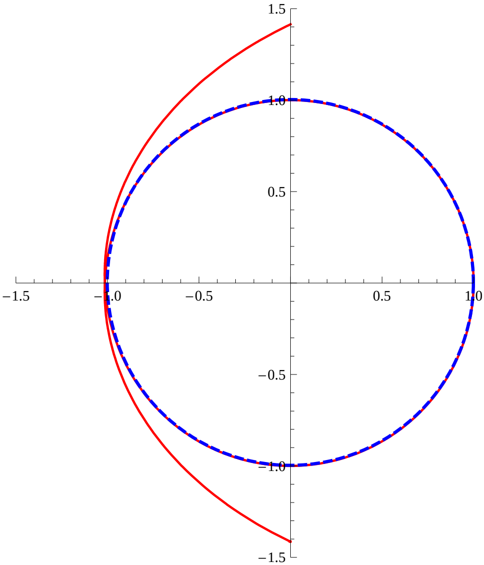


Figure: The unit circle and its polynomial approximant for $n = 6$.

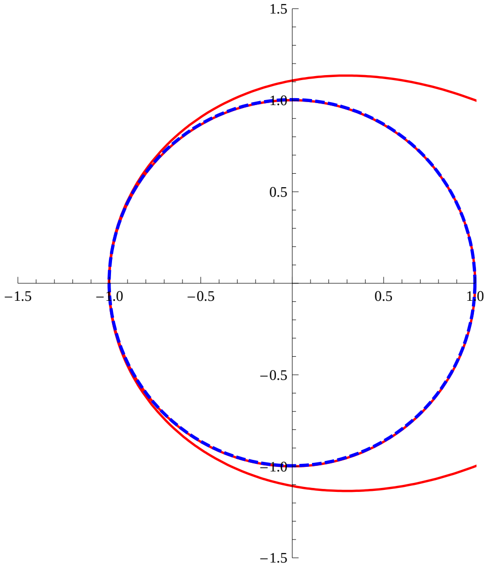
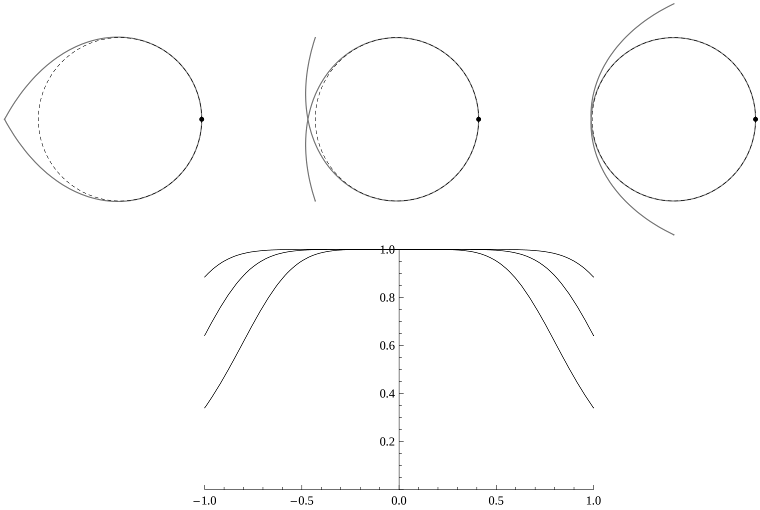


Figure: The unit circle and its polynomial approximant for $n = 7$.

Approximants and curvatures



Cycling

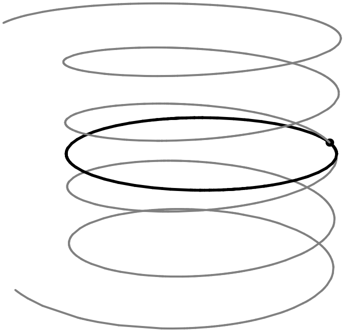


Figure: Unit circle together with the cycles of the approximant for $n = 20$ and $t \in [-1, 1]$.

Höllig-Koch conjecture

Conjecture

A polynomial planar parametric curve of degree n can interpolate $2n$ given points with an approximation order $2n$.

Theorem

Höllig-Koch conjecture holds true for conic sections.

Idea of a proof:

- asymptotic approach,
- a particular nonlinear system has to be studied,
- an existence of a solution guarantees the optimal approximation order,
- solution is obtained by canonical form and optimal solutions for ellipse and hyperbola.

Ellipse

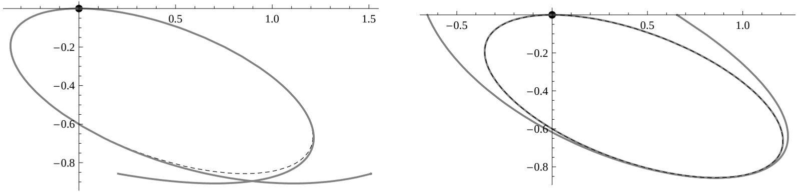


Figure: Approximation of the ellipse $\frac{1}{2}x^2 + xy + \frac{5}{3}y^2 + y = 0$ with the best approximant of degree $n = 5, 7$.

Hyperbola

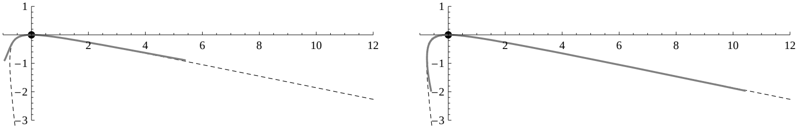
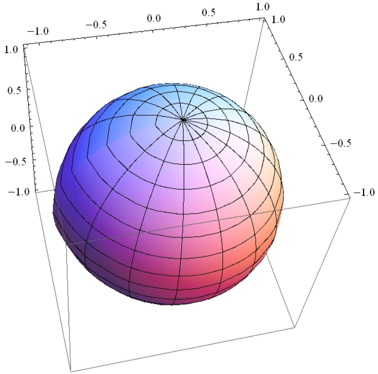


Figure: Approximation of the hyperbola $\frac{1}{5}x^2 + xy + \frac{1}{8}y^2 + y = 0$ with the best approximant of degree $n = 3, 4$.

Sphere approximation



Particular polynomials of degree 5 in u and v yield:

$$x(u,v) = (1 + (-3-\sqrt{5})u^2 + (1+\sqrt{5})u^4)(1 + (-3-\sqrt{5})v^2 + (1+\sqrt{5})v^4)$$

$$y(u,v) = ((1+\sqrt{5})u + (-3-\sqrt{5})u^3 + u^5)(1 + (-3-\sqrt{5})v^2 + (1+\sqrt{5})v^4)$$

$$z(u,v) = (1+\sqrt{5})v + (-3-\sqrt{5})v^3 + v^5$$