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RBFs & scaling

A scaling perspective on accuracy and convergence in RBF approximations

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DWCAA09, September 9, 2009

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Outline

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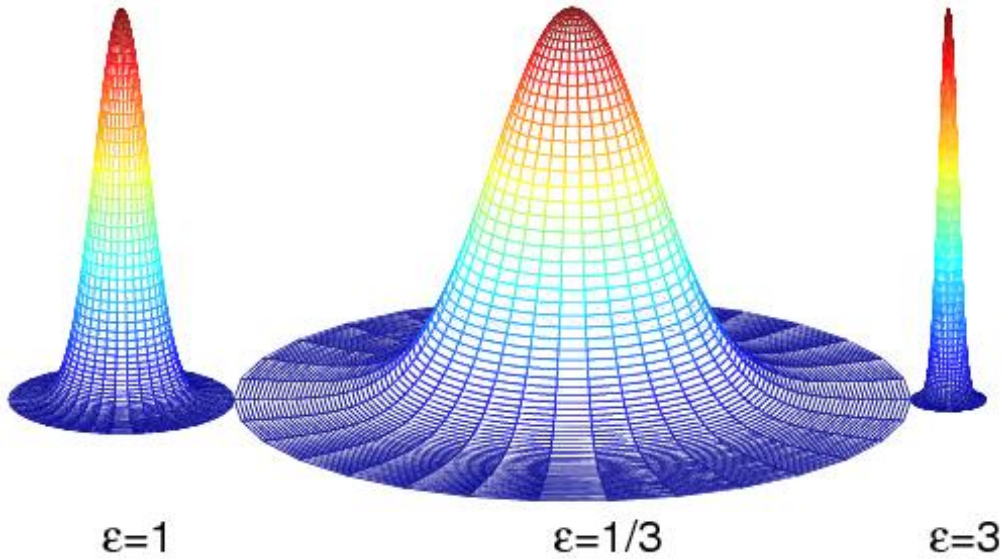
A quick introduction to scaling of RBFs

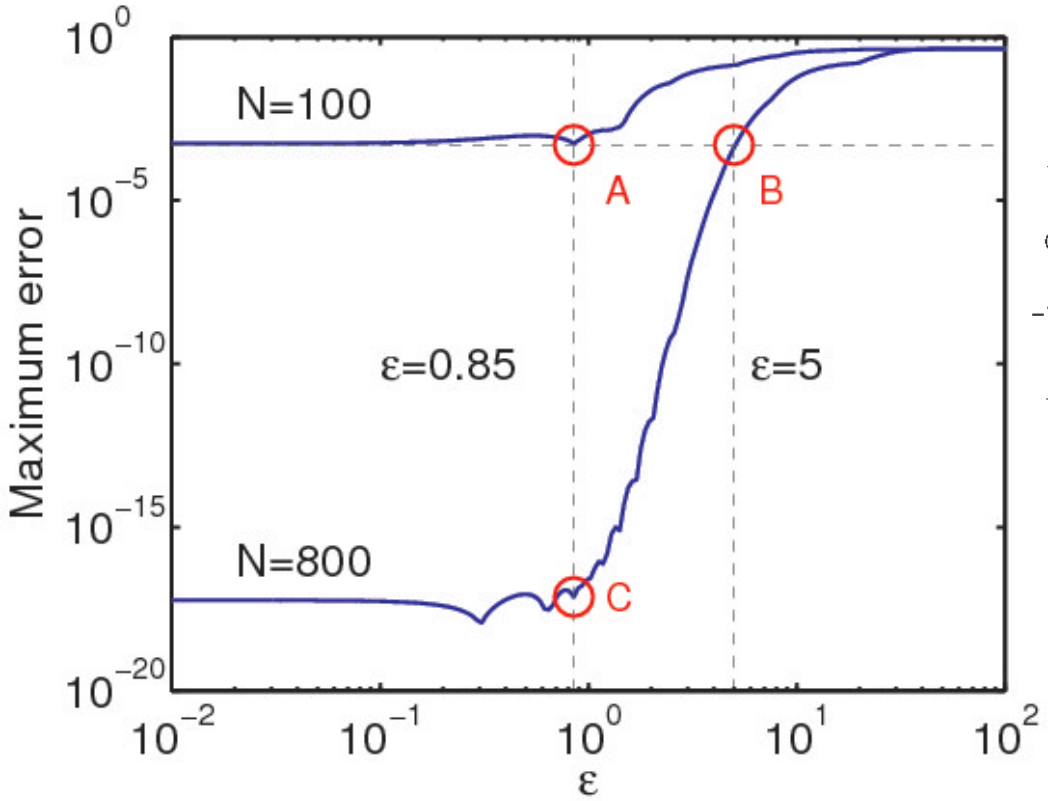
Gaussian RBF (GA): $\phi(\varepsilon r) = \exp(-\varepsilon^2 r^2)$

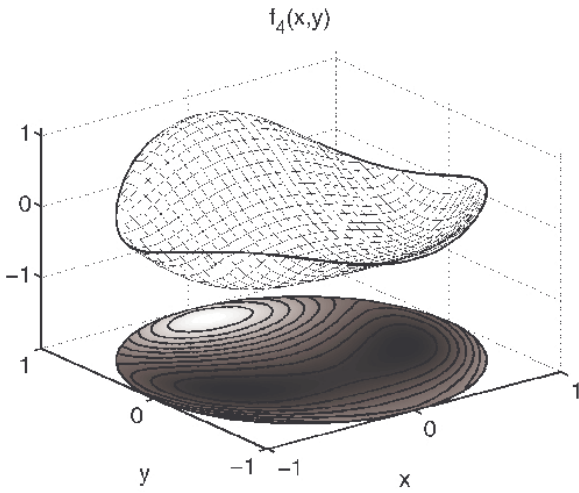


Scaling and accuracy: Motivation

Gaussian RBFs: $s(\underline{x}) = \sum_{j=1}^{N} \lambda_j \phi(\varepsilon \|\underline{x} - \underline{x}_j\|)$

$u(x, y) = \sin(x^2 + 2y^2) - \sin(2x^2 + (y - 1/2)^2)$





A–B Same error. Cost $8^3 = 512$ times higher at B.

B–C Same cost. Error 10^{14} times smaller at C.

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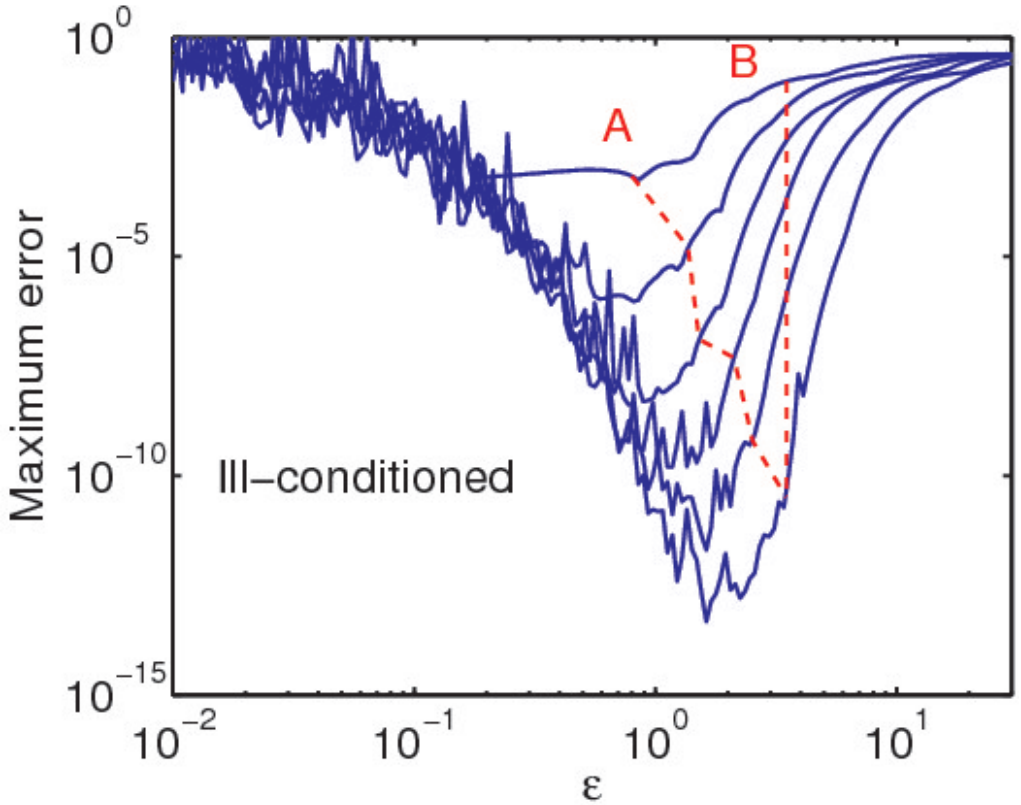
Choosing the shape parameter: Before

Computations with RBF-Direct in quad precision.



N
100
200
300
500
800
1600

A Hover at the edge of ill-conditioning. $\varepsilon h = C \Rightarrow$ Stationary interpolation, in the end no convergence.

B Choose large constant epsilon (sub-optimal). Spectral convergence, but will hit ill-conditioning.

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The RBF-QR method: Idea

- The Gaussian RBFs are expanded in terms of

$$\begin{cases} T^c_{j,m}(\underline{x}) = e^{-\varepsilon^2 r^2} r^{2m} T_{j-2m}(r) \cos((2m+p)\theta), \\ T^s_{j,m}(\underline{x}) = e^{-\varepsilon^2 r^2} r^{2m} T_{j-2m}(r) \sin((2m+p)\theta), \end{cases}$$

leading to $\Phi(\underline{x}) = C \cdot D \cdot T(\underline{x})$, where c_{ij} is $\mathcal{O}(1)$ and $D = \mathrm{diag}(\mathcal{O}(\varepsilon^0, \varepsilon^2, \varepsilon^2, \varepsilon^4, \varepsilon^4, \varepsilon^4, \varepsilon^6, \ldots))$.

- Then C is QR-factorized so that

$$\Phi(\underline{x}) = Q \cdot \begin{bmatrix} R_1 & R_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \cdot T(\underline{x})$$

- Form a new basis (in the same space)

$$\Psi(\underline{x}) = D_1^{-1} R_1^{-1} Q^H \Phi(\underline{x}) = \begin{bmatrix} I & \tilde{R} \end{bmatrix} \cdot T(\underline{x}).$$

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The RBF-QR method: Capabilities

- Works for Gaussian RBFs in 2D.
- Works for ε all the way down to zero.
- How large N we can solve for depends on the required accuracy.

Accuracy	Double precision	Quad precision
1e-16	–	2700
1e-6	611	6000
1e-2	1400	$\approx$ 8000

- Not sensitive to the shape of the domain.
- Computational cost is $\mathcal{O}(N^3)$ (as RBF-direct)

ε	0	0.1	1
RBF-Direct (quad)	–	1	1
RBF-QR (quad)	3	5	7.6

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The RBF-QR method: Extensions

The flat limit QR with column pivoting is needed for non-unisolvent cases.

Other RBFs For Bessel RBFs, yes.
Unclear for MQ, IQ, IMQ.

Other geometries The sphere: Fornberg & Piret 2007.

More dimensions Some hope for the 3D-problem.

Preprint available for the 2-D case



B. Fornberg, E. Larsson, and N. Flyer,
Stable Computations with Gaussian RBFs in 2-D
`http://www.it.uu.se/research/publications/reports/`

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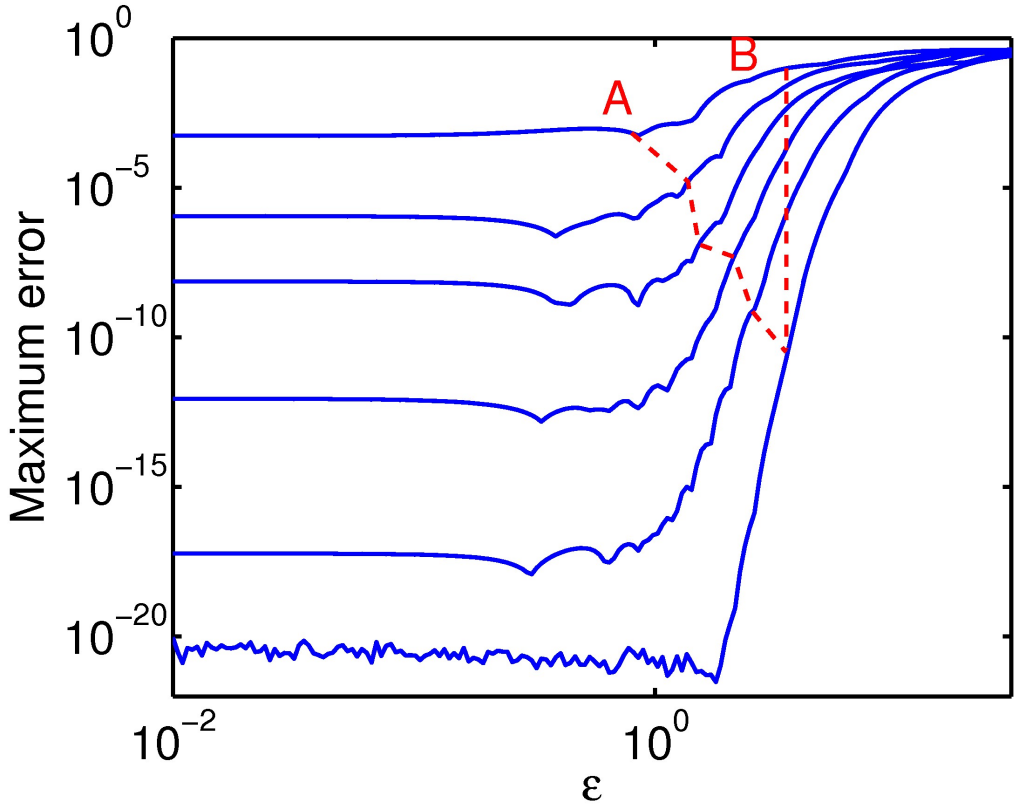
Choosing the shape parameter: After

Computations with RBF-QR in quad precision.



N
100
200
300
500
800
1600

A and B are the choices we could make with RBF-Direct.

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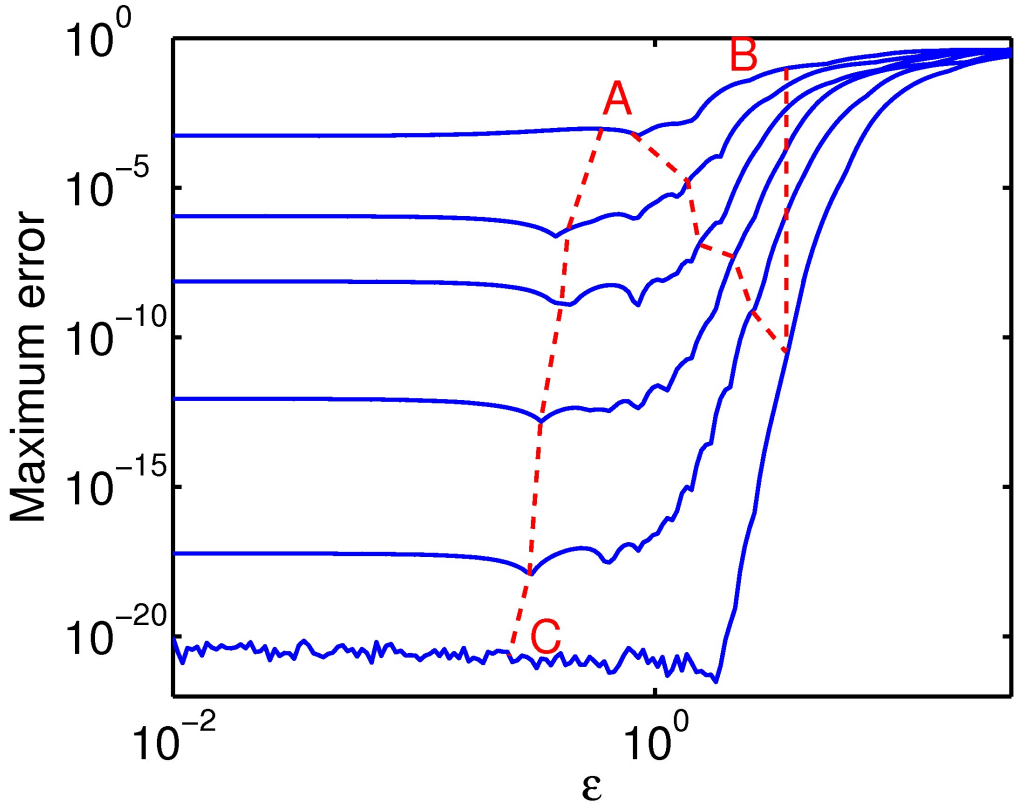
Choosing the shape parameter: After

Computations with RBF-QR in quad precision.



N
100
200
300
500
800
1600

A and B are the choices we could make with RBF-Direct.

C Conditioning is no longer ε-dependent. Choice is free, but not given. Here $\varepsilon^* = Ch^{0.6}$.

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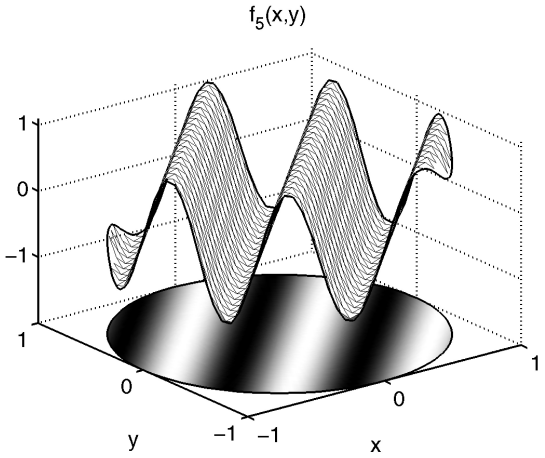
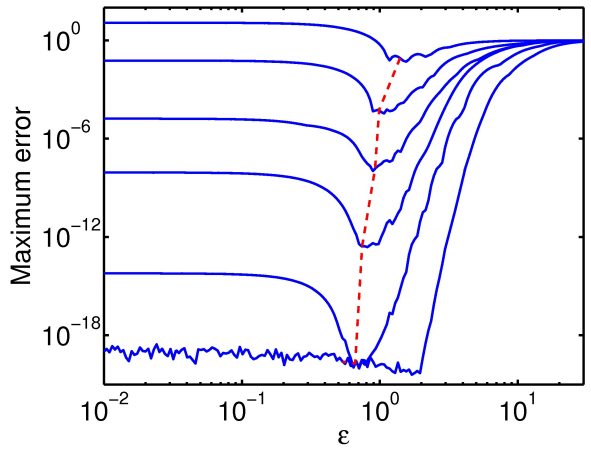
Other examples: Decreasing optimal ε

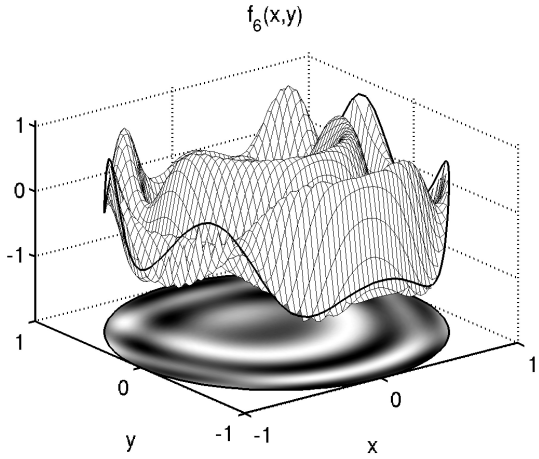
$\sin(2\pi(x-y))$

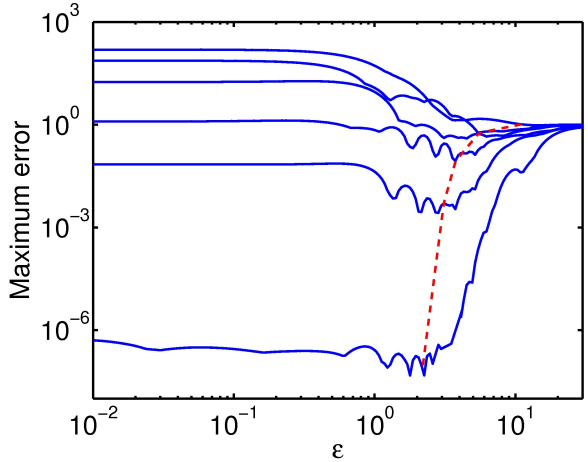




$\varepsilon^* = Ch^{0.65}$

$\sin(2\pi(x^2+2y^2)) - \sin(2\pi(2x^2+(y-1/2)^2))$





$\varepsilon^* = Ch^{1.1}$

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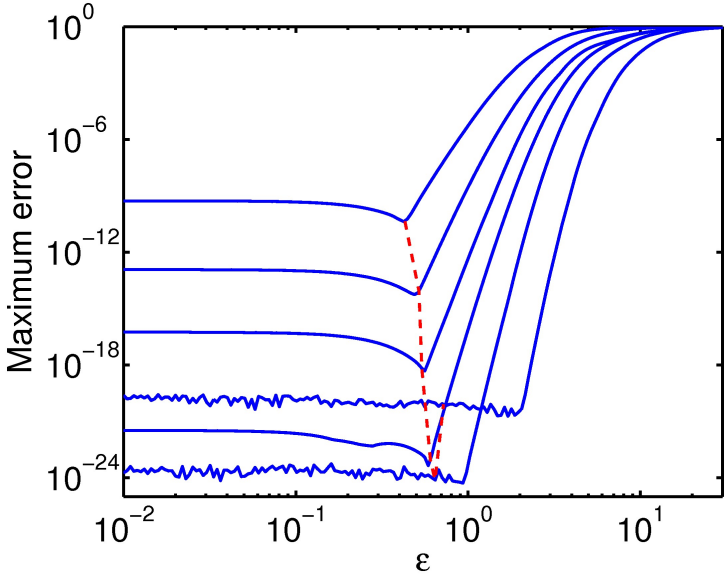
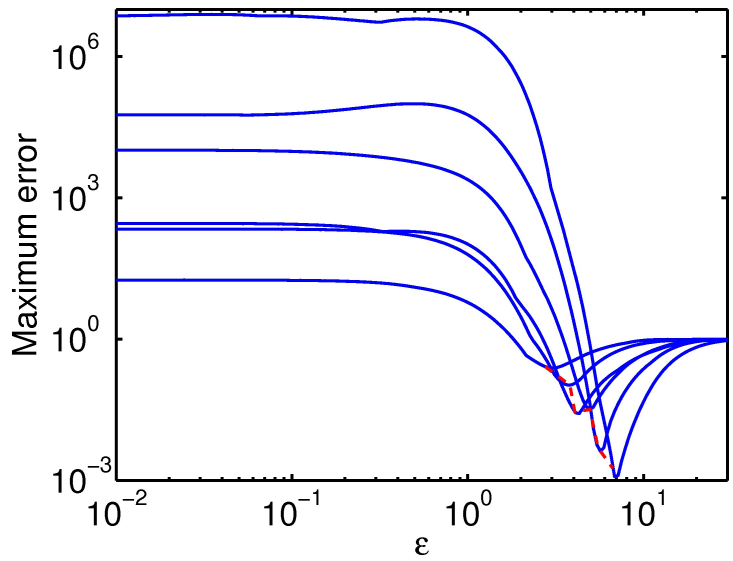
Other examples: Increasing optimal ε

$$\frac{25}{25+(x-0.2^2)+2y^2}$$



$$\varepsilon^* = C/h^{0.35}$$

$$\frac{\operatorname{atan}(2(x+3y-1))}{\operatorname{atan}(2\sqrt{10}+2)}$$



$$\varepsilon^* = C/h^{0.6}$$

- For the atan function, we are chased by the Runge phenomenon *(cf. Fornberg & Zuev 2007)*.
RBF-QR does not help in this case.
- Node clustering and/or variable ε can help.

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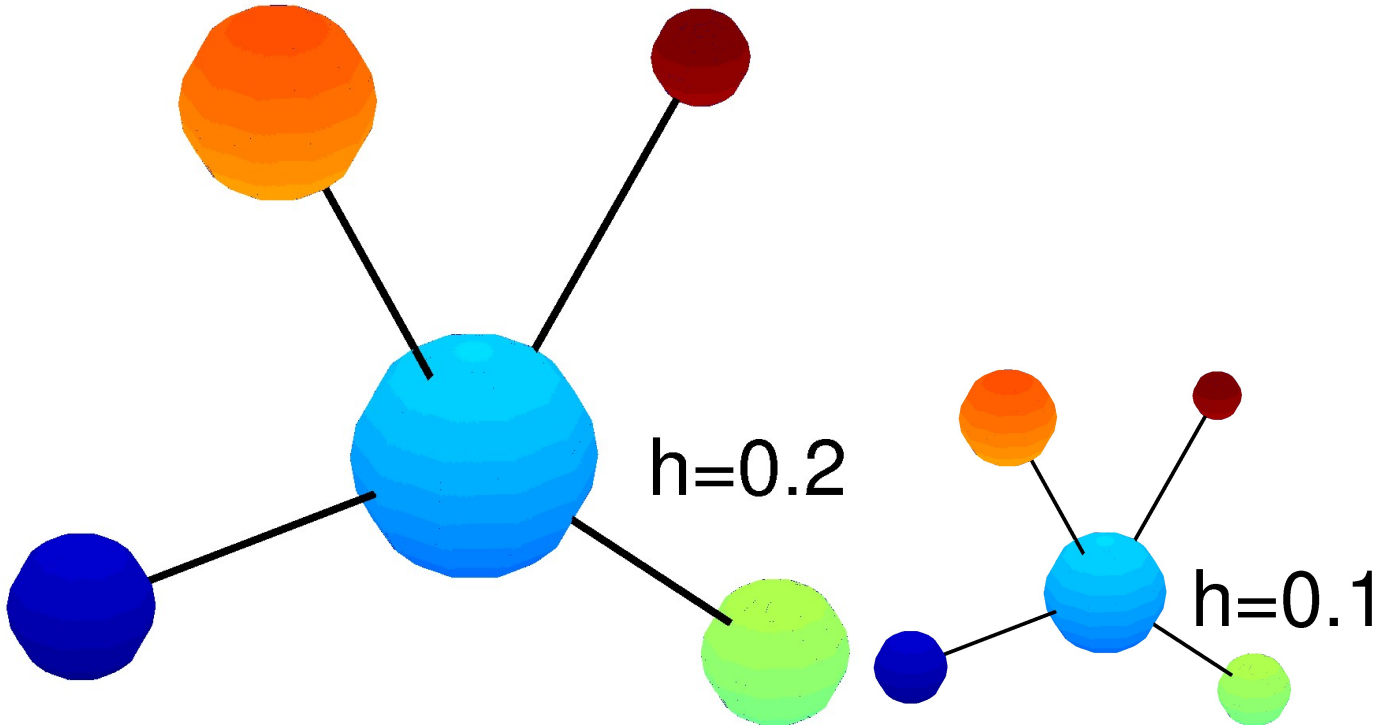
Scaling in RBF-generated finite difference stencils for scattered nodes

Globally: Use constant global ε for fast convergence.



Locally: Corresponds to subproblem for stencil of unit size with decreasing ε.
Using a constant subproblem leads to constant εh globally and stationary interpolation.

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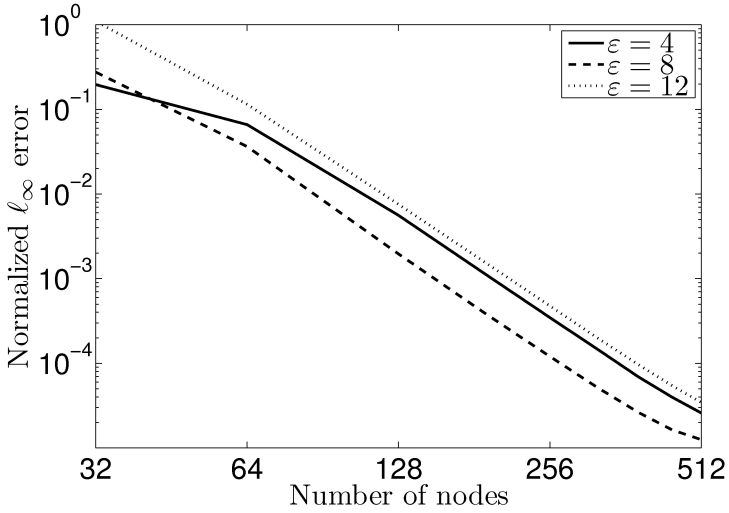
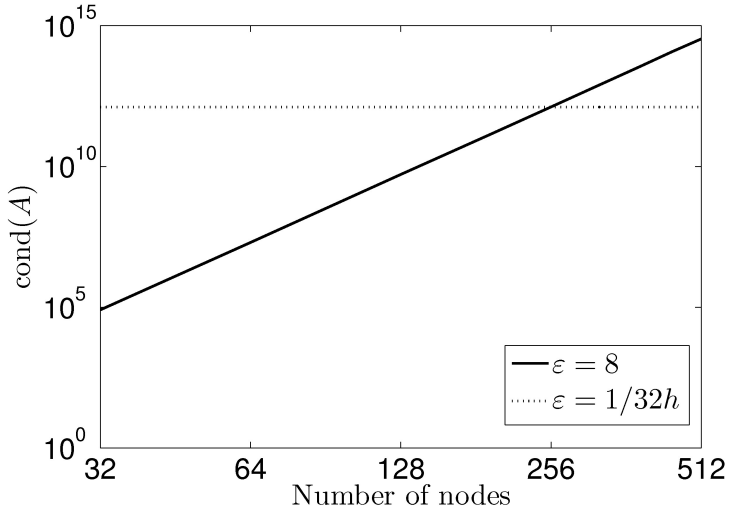
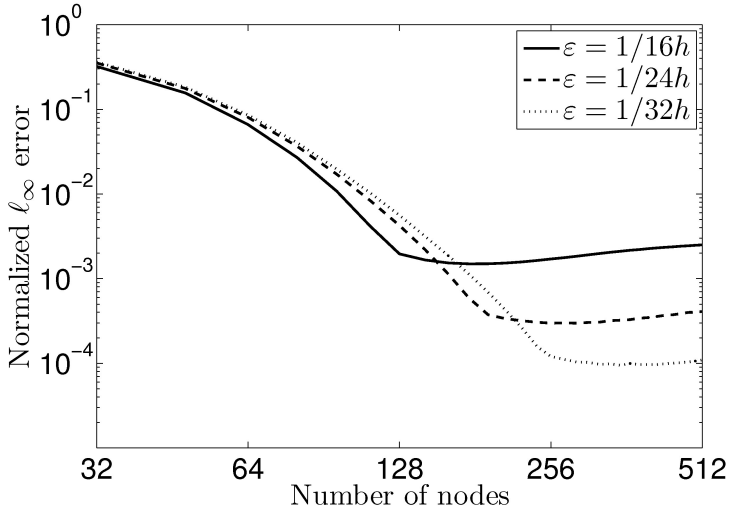
Convergence results for RBF-FD

Constant ε





Scaled ε



1-D periodic advection, pictures by Erik Lehto

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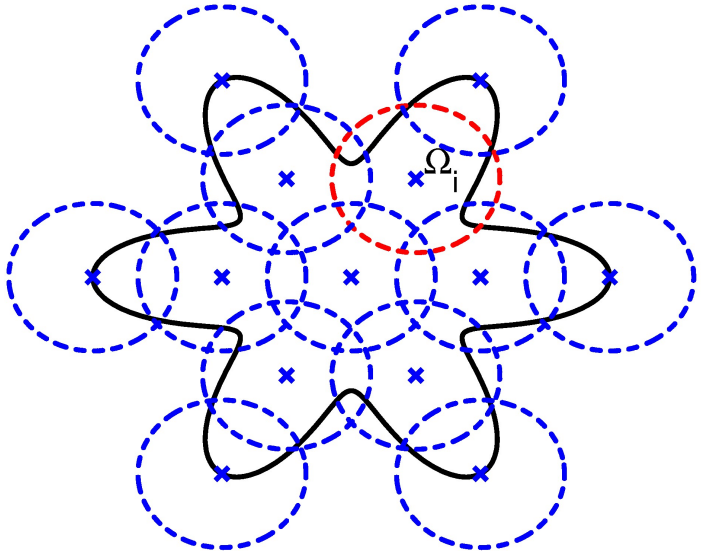
Scaling in a partition of unity RBF collocation approach for PDEs

Global approximant:

$$\tilde{u}(\underline{x}) = \sum_{i=1}^{M} w_i(\underline{x}) u_i(\underline{x}),$$

where $w_i(\underline{x})$ are weight functions.



Local RBF approximants:

$$u_i(\underline{x}) = \sum_{j=1}^{N_i} \lambda_j^i \phi_j(\underline{x}).$$

Poisson problem:

$$\begin{cases} \Delta u(\underline{x}) &= f(\underline{x}), & \underline{x} \in \Omega, \\ u(\underline{x}) &= g(\underline{x}), & \in \partial\Omega. \end{cases}$$

Continuity of the global approximant is achieved is by requiring $u_i(\underline{x}_k) = u_j(\underline{x}_k)$, if $\underline{x}_k \in \Omega_i \cap \Omega_j$.

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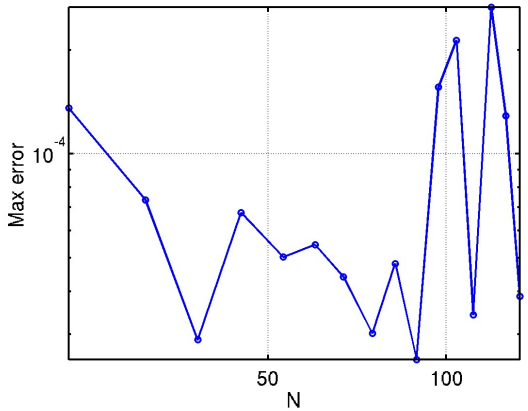
Global and local scaling effects

The relevant quantity for scaling of RBFs is $q = \varepsilon r$.

As we increase the number of subdomains, the support radius r_s decreases.

If $\varepsilon = \varepsilon_0 / r_s$, q is constant corresponding to stationary interpolation and no convergence.



For a constant global ε, q decreases. This allows convergence, but leads to ill-conditioning.

Solution: The RBF-QR method allows stable computation for small values of q.

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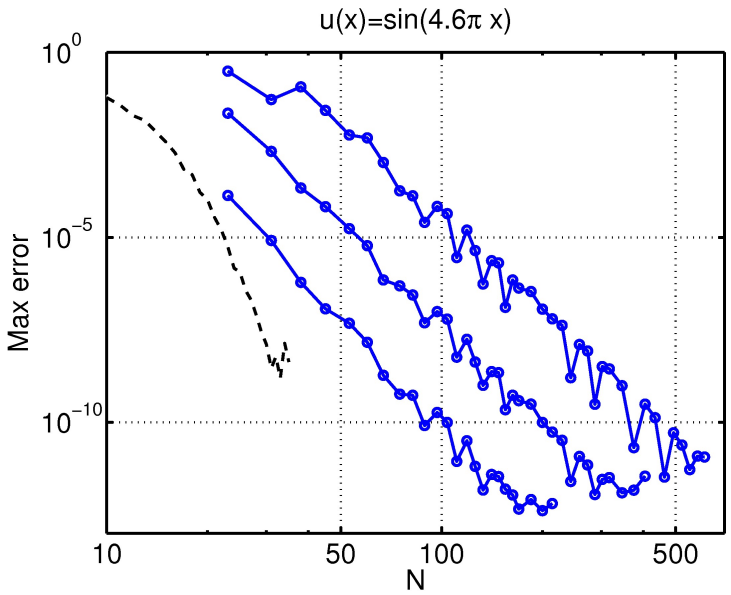
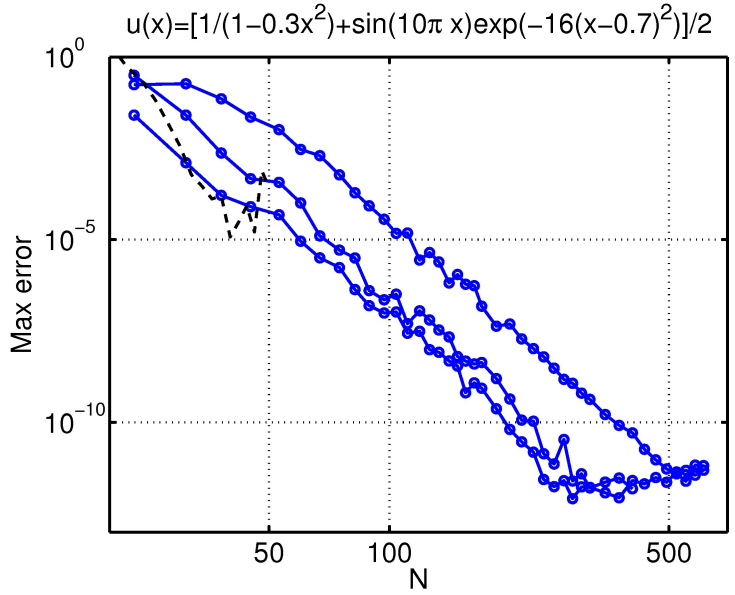
Convergence with # subdomains M

points/subdomain $= 12$ $\quad M = 3, \ldots, 80$

Overlap $=$ support radius/2 $\quad \varepsilon = 4,\ 8,\ 16$





Blue Partition of unity RBF collocation.

Dashed Global RBF collocation.

Slope is approximately 9 $=$
polynomial degree($N = 12$) -2 (second order PDE).

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Summary

- Scaling is too important to neglect.
- With the RBF-QR method, scaling strategies need to be reconsidered.
- Spectral convergence is available for real in 2D.
- When ill-conditioning due to scaling is removed, partitioning strategies become very interesting to pursue.