Spectral Interpolation Schemes on Curves in Polar and Spherical Geometries

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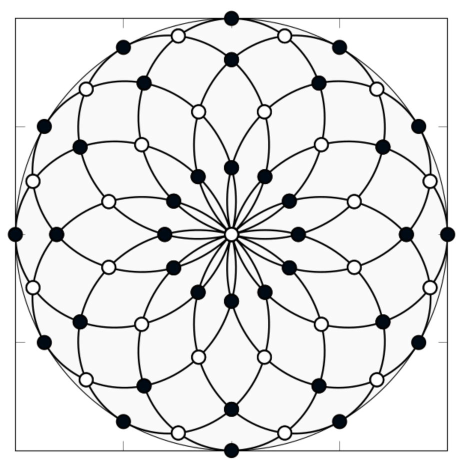

Introduction

Rhodonea curves and **spherical Lissajous curves** can be used as sampling trajectories to create novel nodes for **spectral interpolation** on the **disk** or the **unit sphere**. Using a parity-modified double Fourier basis as generating element for the interpolation space these nodes give a promising spectral interpolation scheme that can be implemented in a simple, efficient and numerically stable way.

Rhodonea nodes in the disk

For $\boldsymbol{m} = (m_1, m_2)$ with coprime $m_1, m_2 \in \mathbb{N}$, consider for $t \in \mathbb{R}$ the **rhodonea curve**

$$\rho^{(\boldsymbol{m})}(t) = \Big(\cos(m_2 t)\cos(m_1 t), \cos(m_2 t)\sin(m_1 t)\Big).$$

The **rhodonea nodes** $\mathbf{RD}^{(\boldsymbol{m})}$ are given as

$$\mathbf{RD}^{(\boldsymbol{m})} = \left\{\rho^{(\boldsymbol{m})}\left(\frac{l\pi}{2m_1 m_2}\right) \,\middle|\, l \in \{1, \ldots, 4m_1 m_2\}\right\}.$$

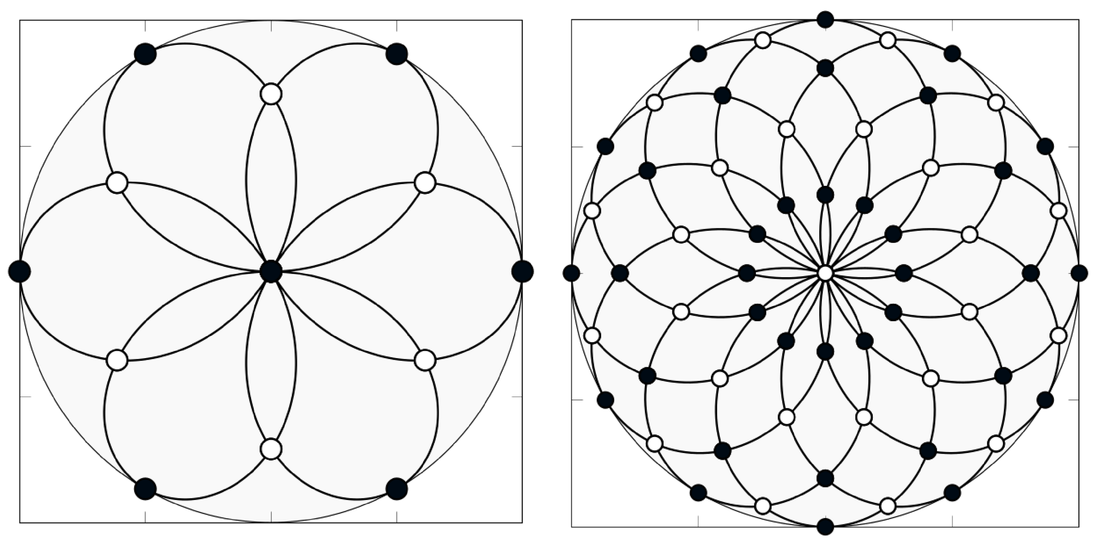

Fig 1. The nodes $\mathbf{RD}^{(2,3)}$ (left) and $\mathbf{RD}^{(5,6)}$ (right).

Theorem 1 (Characterization of nodes)

Let m_1, m_2 be coprime and $m_1 + m_2$ odd. Then, the set $\mathbf{RD}^{(\boldsymbol{m})}$ is the union of all self-intersection and all boundary points of the 2π periodic curve $\rho^{(\boldsymbol{m})}$. $\mathbf{RD}^{(\boldsymbol{m})}$ contains $2m_1 m_2 + 1$ nodes in the disk:

- The center $(0,0)$, traversed $2m_2$ times in one period,
- $2(m_1 - 1)m_2$ ordinary double points distinct from $(0,0)$,
- $2m_2$ points on the boundary circle.

Further properties of $\rho^{(\boldsymbol{m})}$ and $\mathbf{RD}^{(\boldsymbol{m})}$:

- $\rho^{(\boldsymbol{m})}$ is an algebraic variety of order $2m_1 + 2m_2$ given in polar coordinates as
$$\left\{\boldsymbol{x}(r,\theta) \,\middle|\, T_{m_1}(r)^2 = \cos^2(m_2\theta)\right\}.$$
- $\mathbf{RD}^{(\boldsymbol{m})}$ is given as
$$\left\{\boldsymbol{x}(r,\theta) \,\middle|\, T_{m_1}(r)^2 = \cos^2(m_2\theta) \in \{0,1\}\right\}.$$
- $\mathbf{RD}^{(\boldsymbol{m})}$ is the union of two disjoint Chebyshev grids in polar coordinates.

References

[1] Boyd, J. P., and Yu, F. Comparing seven spectral methods for interpolation and for solving the Poisson equation in a disk: Zernike polynomials, Logan-Shepp ridge polynomials, Chebyshev-Fourier series, cylindrical Robert functions, Bessel-Fourier expansions, square-to-disk conformal mapping and radial basis functions. *J. Comput. Phys. 230* (2011), 1408–1438.

[2] Erb, W. A spectral interpolation scheme on the unit sphere based on the nodes of spherical Lissajous curves. *IMA J. Numer. Anal. (in press)* (2018).

[3] Erb, W. Rhodonea curves as sampling trajectories for spectral interpolation on the unit disk. *arXiv:1812.00437 [math.NA]* (2018).

Spectral interpolation scheme for rhodonea nodes

Using a **parity-modified Chebyshev-Fourier basis** in polar coordinates $(r,\theta) \in [0,1] \times [0, 2\pi)$ [1]

$$X_{\boldsymbol{\gamma}}(r,\theta) = \begin{cases} T_{\gamma_1}(r)\cos(\gamma_2\theta), & \boldsymbol{\gamma} \in \boldsymbol{\Gamma}^{(\boldsymbol{m})}, \gamma_2 \geq 0, \\ T_{\gamma_1}(r)\sin(\gamma_2\theta), & \boldsymbol{\gamma} \in \boldsymbol{\Gamma}^{(\boldsymbol{m})}, \gamma_2 < 0, \end{cases}$$

we generate the interpolation spaces as

$$\Pi^{(\boldsymbol{m})} = \operatorname{span}\left\{X_{\boldsymbol{\gamma}} \,\middle|\, \boldsymbol{\gamma} \in \boldsymbol{\Gamma}^{(\boldsymbol{m})}\right\}^{(\ddagger)}.$$

Here, $T_\gamma(r)$ denotes a Chebyshev polynomial of degree $\gamma \in \mathbb{N}_0$ and $\boldsymbol{\Gamma}^{(\boldsymbol{m})} \subset \mathbb{Z}^2$ a **spectral index set**. In order to obtain a unique interpolant for the nodes $\mathbf{RD}^{(\boldsymbol{m})}$, the set $\boldsymbol{\Gamma}^{(\boldsymbol{m})}$ has to be chosen properly. A suitable choice is

$$\boldsymbol{\Gamma}_\square^{(\boldsymbol{m})} = \left\{\boldsymbol{\gamma} \in \mathbb{Z}^2 \,\middle|\, \begin{array}{l} 0 \leq \gamma_1 \leq 2m_1 \\ -m_2 < \gamma_2 \leq m_2 \\ \gamma_1 + \gamma_2 \text{ is even} \end{array}\right\}.$$

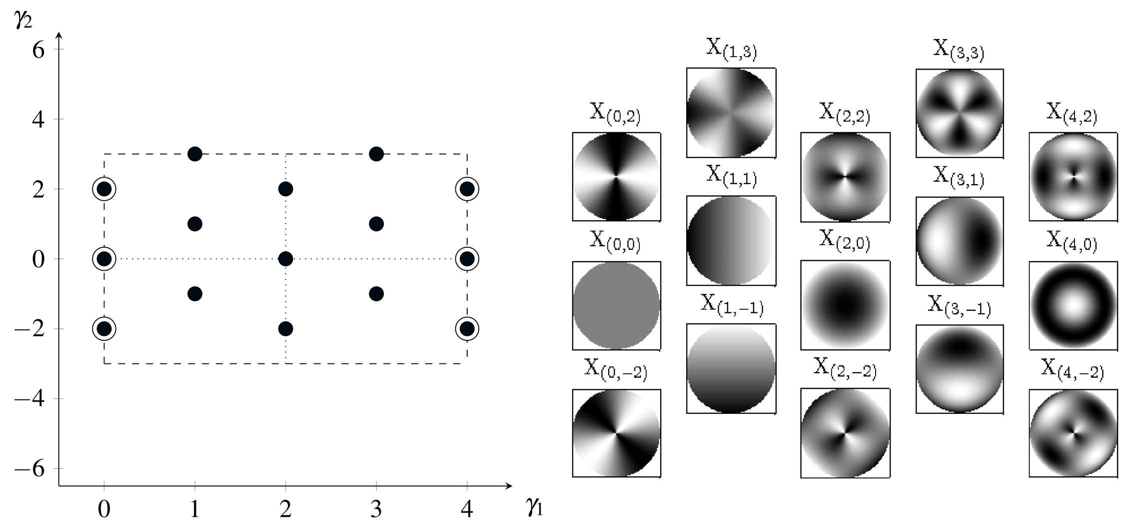

Fig 2. The spectral set $\boldsymbol{\Gamma}_\square^{(2,3)}$ (left) and the corresponding basis for the space $\Pi_\square^{(2,3)}$ (right).

(‡) For a few $\boldsymbol{\gamma}$ at the border of $\boldsymbol{\Gamma}^{(\boldsymbol{m})}$ it is necessary to use $X_{(\gamma_1,-\gamma_2)}$ instead of $X_{(\gamma_1,\gamma_2)}$, see [3].

Theorem 2 (Uniqueness of interpolation)

Let f be a continuous function on the disk. There exists a unique continuous function

$$P_f^{(\boldsymbol{m})}(\boldsymbol{x}) = \sum_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_\square^{(\boldsymbol{m})}} c_{\boldsymbol{\gamma}}(f) X_{\boldsymbol{\gamma}}(\boldsymbol{x}) \qquad (1)$$

in the space $\Pi_\square^{(\boldsymbol{m})}$ with the interpolation property

$$P_f^{(\boldsymbol{m})}(\boldsymbol{z}) = f(\boldsymbol{z}) \quad \text{for all} \quad \boldsymbol{z} \in \mathbf{RD}^{(\boldsymbol{m})}.$$

- It is not trivial that $P_f^{(\boldsymbol{m})}$ is continuous: the basis functions $X_{\boldsymbol{\gamma}}$ have in general a discontinuity at the center $(0,0)$.
- The coefficients $c_{\boldsymbol{\gamma}}(f)$ in (1) can be calculated in $\mathcal{O}(m_1 m_2 \ln m_1 \ln m_2)$ arithmetic steps from the samples $f(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbf{RD}^{(\boldsymbol{m})}$, using a 2D fast Fourier transform.

Theorem 3 (Numerical condition)

For the space $\Pi_\square^{(\boldsymbol{m})}$, the Lebesgue constant $\Lambda_\square^{(\boldsymbol{m})} = \sup_{\|f\|_\infty \leq 1} \|P_f^{(\boldsymbol{m})}\|_\infty$ is bounded by

$$\Lambda_\square^{(\boldsymbol{m})} \leq C_\square \ln(m_1 + 1)\ln(m_2 + 1)$$

with a constant $C_\square$ independent of $\boldsymbol{m}$.

Spectral interpolation of images

To test the scheme, we interpolate data of the shadow of the black hole at the center of the galaxy M87. We use the dataset "eso1907a" provided by the Event Horizon Telescope (EHT) collaboration. We sample this dataset in a circular region at the rhodonea nodes $\mathbf{RD}^{(\boldsymbol{m})}$ for $\boldsymbol{m} = (8,9)$, $\boldsymbol{m} = (21,22)$ and $\boldsymbol{m} = (61,62)$. The interpolation results as well as the corresponding relative interpolation errors are displayed in Fig 3.

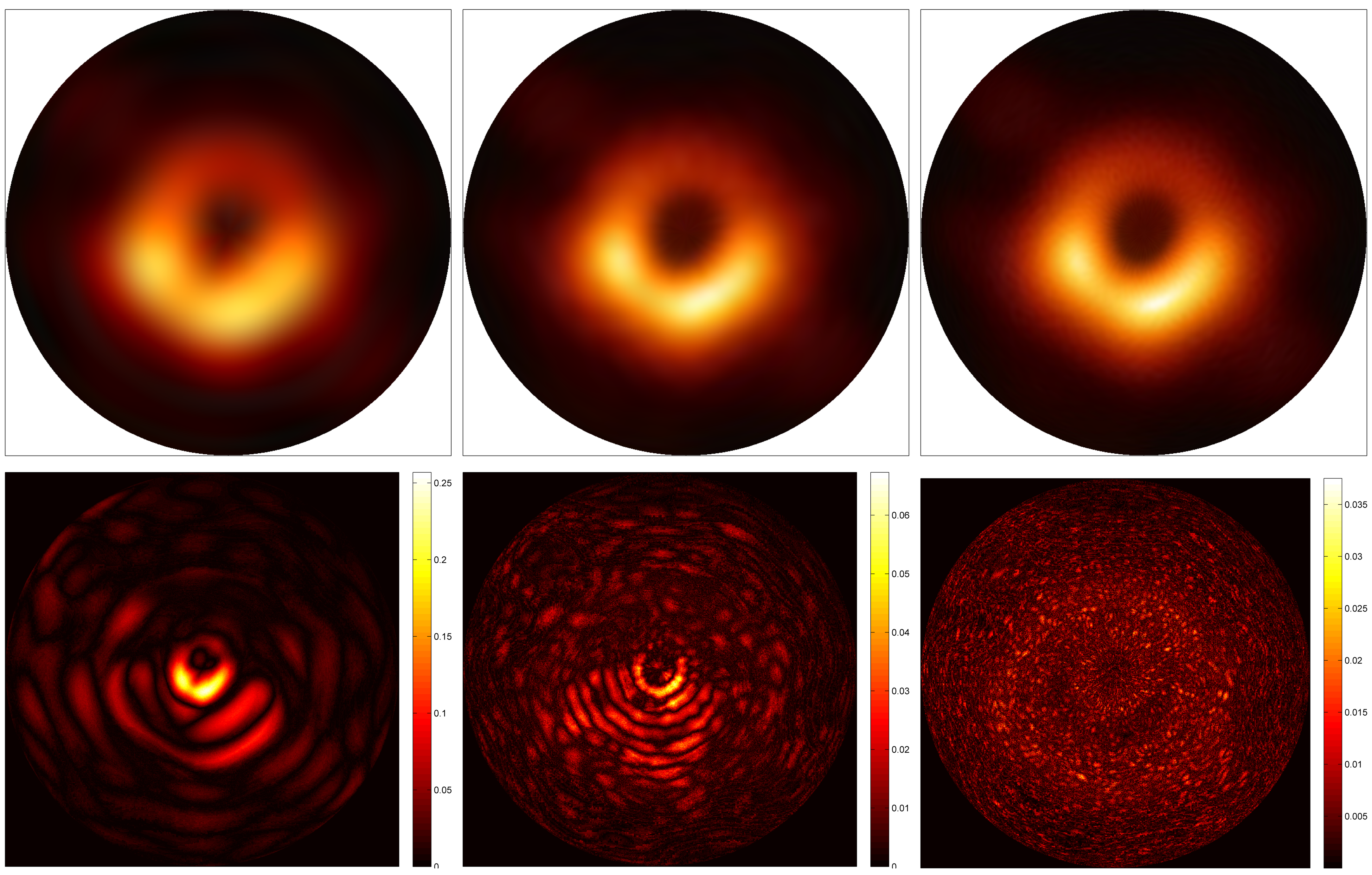

Fig 3. Spectral interpolation (upper part) of the black hole data based on samples at the rhodonea nodes $\mathbf{RD}^{(8,9)}$ (left), $\mathbf{RD}^{(21,22)}$ (middle) and $\mathbf{RD}^{(61,62)}$ (right). In the lower part, the relative interpolation error is illustrated. Credit for the orignal data: EHT collaboration et. al.

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