Shape-driven interpolation with discontinuous kernels

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Problem

Interpolating an image with **sharp edges** by smooth basis functions, as for instance radial basis functions (RBFs) or polynomials, results in overshoots and oscillations at the edges, commonly known as **Gibbs phenomenon**.
To reflect discontinuities in the interpolation of scattered data sets, we study techniques based on **variably scaled discontinuous kernels (VSDKs)** [1]. We obtained characterizations and theoretical **Sobolev type error estimates** for the **native spaces** of these kernels. Tests confirm the theoretical convergence rates and show that Gibbs artifacts can be avoided in a VSDK interpolation scheme. With a robust estimate of the discontinuities, the proposed method can be applied successfully to problems in **medical imaging**.

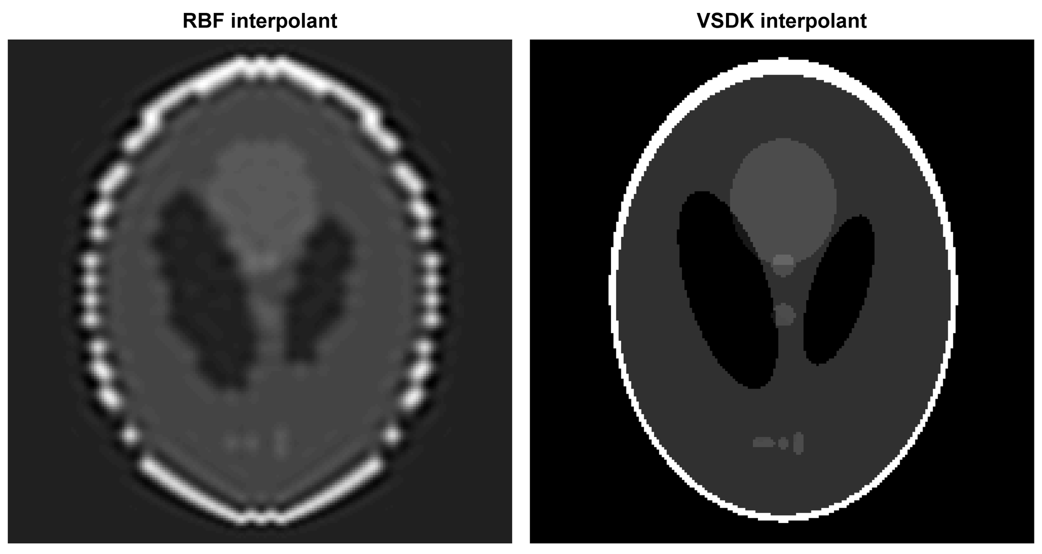

Fig 1. RBF and VSDK interpolation of the Shepp-Logan phantom on a **LS** node set.

Preliminaries on VSDK

Assumption 1 *We assume that:*

i) The bounded set $\Omega \subset \mathbb{R}^d$ is the union of n pairwise disjoint sets Ω_i, $i \in \{1,\dots,n\}$.

ii) The subsets Ω_i have a Lipschitz boundary.

*iii) The **scale function** $\psi : \Omega \to \Sigma$, $\Sigma \subset \mathbb{R}$ is constant on the subsets Ω_i, i.e., $\psi(\boldsymbol{x}) = \alpha_i$ with $\alpha_i \in \mathbb{R}$ for all $\boldsymbol{x} \in \Omega_i$.*

Definition 1 *Suppose that Assumption 1 holds and K is a continuous positive definite kernel on $\Omega \times \Sigma \subset \mathbb{R}^{d+1}$ based on the **radial basis function** ϕ. The **variably scaled discontinuous kernel** K_ψ on $\Omega \times \Omega$ is defined as*

$$K_\psi(\boldsymbol{x},\boldsymbol{y}) := K((\boldsymbol{x},\psi(\boldsymbol{x})),(\boldsymbol{y},\psi(\boldsymbol{y}))) = \phi(\sqrt{\|\boldsymbol{x}-\boldsymbol{y}\|_2^2 + |\psi(\boldsymbol{x})-\psi(\boldsymbol{y})|^2}).$$

The **native space** $\mathcal{N}_{K_\psi}(\Omega)$ of the kernel K_ψ is defined as the completion of the inner product space $H_{K_\psi}(\Omega) = \mathrm{span}\{K_\psi(\cdot,\boldsymbol{y}),\ \ \boldsymbol{y} \in \Omega\}$.

For a node set $\mathcal{X} = \{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\}$ and values $f_k \in \mathbb{R}$, the **VSDK interpolant** V_f on Ω,

$$V_f(\boldsymbol{x}) = \sum_{k=1}^{N} c_k K_\psi(\boldsymbol{x},\boldsymbol{x}_k), \quad \boldsymbol{x} \in \Omega,$$

is determined by $V_f(\boldsymbol{x}_k) = f_k$, $k \in \{1,\dots,N\}$.

References

[1] Bozzini, Lenarduzzi, Rossini, Schaback, *Interpolation with variably scaled kernels*, IMA J. Numer. Anal. **35** (2015), 199–219.

[2] Fuselier, Wright, *Scattered Data Interpolation on Embedded Submanifolds with Restricted Positive Definite Kernels: Sobolev Error Estimates*, SIAM J. Numer. Anal., **50**, 3 (2012), 1753–1776.

[3] Narcowich, Ward, Wendland, *Sobolev bounds on functions with scattered zeros, with applications to radial basis function surface fitting*. Math. Comp. **74**, 250 (2005), 743–763.

Error estimates for interpolation with VSDK

Based on the decomposition Ω in Assumption 1 we define for $s \geq 0$ and $1 \leq p \leq \infty$ the following spaces of piecewise smooth functions on Ω:

$$\mathrm{WP}_p^s(\Omega) := \{ f \mid f_{\Omega_i} \in \mathrm{W}_p^s(\Omega_i),\ i \in \{1,\dots,n\}\}.$$

f_{Ω_i} denotes the restriction of f to the subdomain Ω_i and $\mathrm{W}_p^s(\Omega_i)$ denotes the standard Sobolev spaces on Ω_i. As norm on $\mathrm{WP}_p^s(\Omega)$ we set

$$\|f\|_{\mathrm{WP}_p^s}^p = \sum_{i=1}^{n} \|f_{\Omega_i}\|_{\mathrm{W}_p^s(\Omega_i)}^p.$$

In the following we assume that the Fourier transform of $\phi(\|\cdot\|)$ has an algebraic Fourier decay:

$$\widehat{\phi(\|\cdot\|)}(\boldsymbol{\omega}) \sim (1+\|\boldsymbol{\omega}\|_2^2)^{-s-\frac{1}{2}}, \quad s > \frac{d-1}{2}. \quad (1)$$

Theorem 1 (Characterization of spaces)

Suppose that the decay condition (1) holds true. Then, the **native space of the discontinuous kernel** K_ψ satisfies

$$\mathcal{N}_{K_\psi}(\Omega) = \mathrm{WP}_2^s(\Omega),$$

with the two norms being equivalent.

Based on a Sobolev sampling inequality for the single subregions Ω_i developed in [2, 3], we obtain a Sobolev error estimate for the interpolation error in the spaces $\mathrm{WP}_p^s(\Omega)$. For this we define the regional mesh norms h_i on Ω_i:

$$h_i = \sup_{\boldsymbol{x}\in\Omega_i} \inf_{\boldsymbol{x}_i \in \mathcal{X}\cap\Omega_i} \|\boldsymbol{x}-\boldsymbol{x}_i\|_2.$$

and $h = \max_{i\in\{1,\dots,n\}} h_i$.

Theorem 2 (Error estimates)

Let Assumption 1 be satisfied. Let $s > 0$, $1 \leq q \leq \infty$, $m \in \mathbb{N}_0$ such that $\lfloor s \rfloor > m + \frac{d}{2}$. Additionally, suppose that ϕ satisfies the Fourier decay (1). Then, for $f \in \mathrm{WP}_2^s(\Omega)$, we obtain for $h \leq h_0$ the **error estimate**

$$\|f - V_f\|_{\mathrm{WP}_q^m} \leq C h^{s-m-d(1/2-1/q)_+} \|f\|_{\mathrm{WP}_2^s}.$$

The constant $C > 0$ is independent of h.

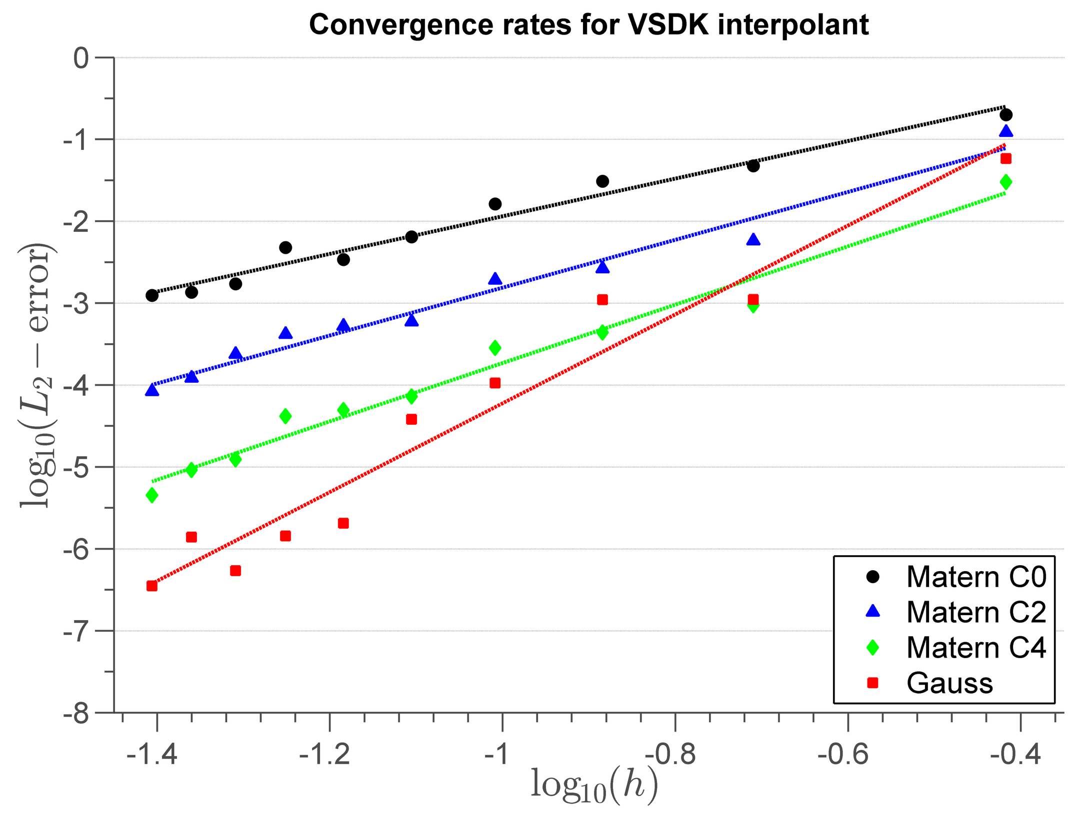

Fig 2. Convergence rates of VSDK-interpolation for the Shepp-Logan phantom on **LS** node sets.

Shape-driven interpolation of images

To test the scheme, we use the painting "Composition en rouge, jaune, bleu et noir", of P. Mondrian, 1921. We sample it at the Lissajous nodes **LS** $= \{(\sin(\frac{\pi k}{2(m+1)}), \sin(\frac{\pi k}{2m})) :\ k = 1,\dots,4m(m+1)\}$. As RBF for the kernel K we use the C^0-Matèrn function $\phi(r) = e^{-r}$. The results are shown in Fig 3.

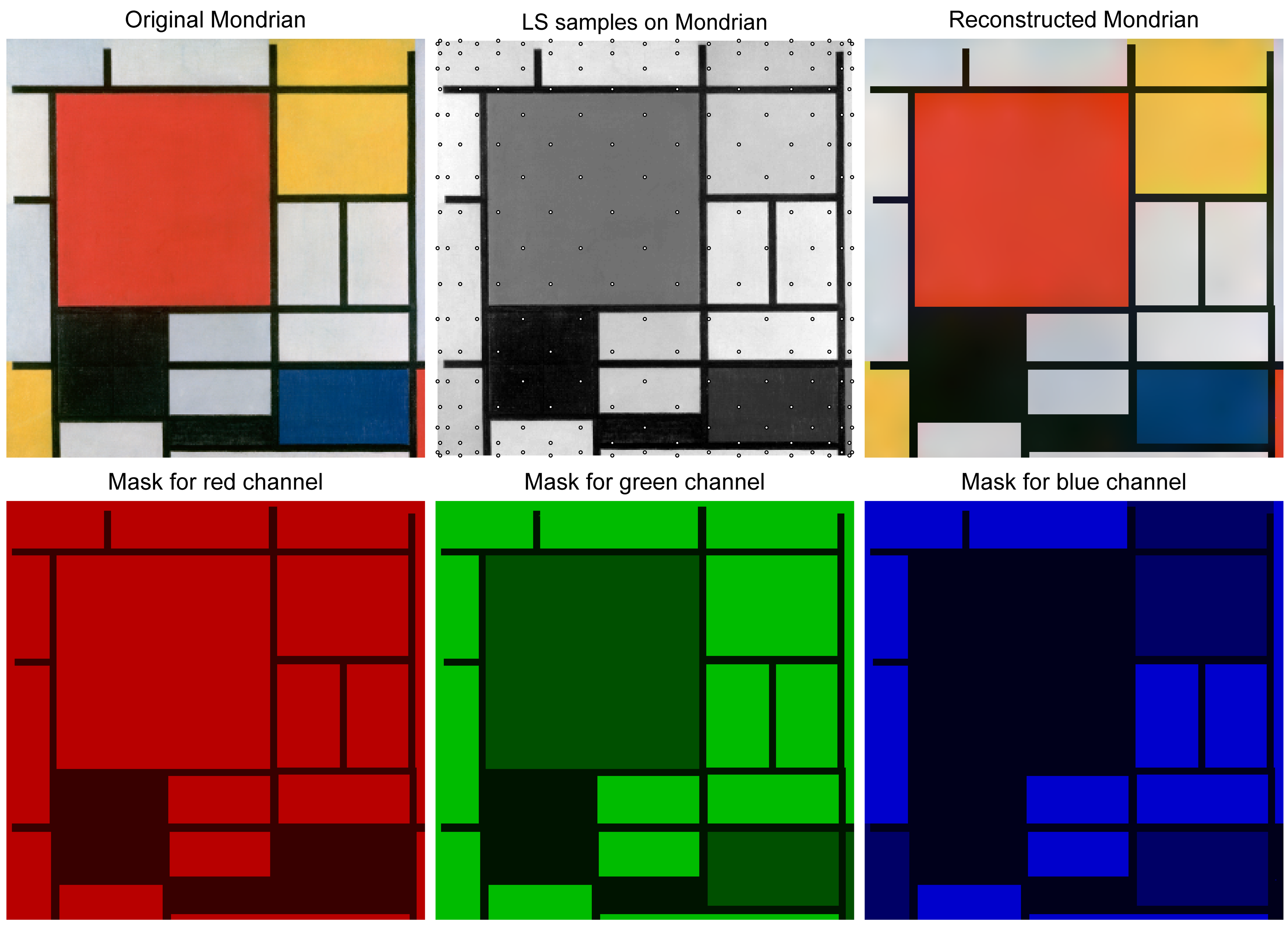

Fig 3. VSDK interpolation (up right) of a painting of P. Mondrian (up left) based on **LS** sampling nodes, $m = 9$ (up center). The scaling functions ψ for the three color channels are displayed below.

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