Uncertainty Principles on Compact Riemannian Manifolds

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Based on a result of Rösler and Voit for ultraspherical polynomials, we derive an uncertainty principle for compact Riemannian manifolds M. The frequency variance of a function in $L^2(M)$ is therein defined by means of the radial part of the Laplace-Beltrami operator. The proof of the uncertainty rests upon Dunkl theory. In particular, a special differential-difference operator is constructed which plays the role of a generalized root of the radial Laplacian. Subsequently, we prove with a family of Gaussian-like functions that the deduced uncertainty is asymptotically sharp. Finally, we specify in more detail the uncertainty principles for well known manifolds like the d-dimensional unit sphere and the real projective space.

Keywords: Riemannian manifold, uncertainty principle, Dunkl operator.

1 Introduction

The most common mathematical description of the uncertainty principle is the following classical formulation, referred to as Heisenberg-Pauli-Weyl inequality (cf. [7], [8], [13]).

Theorem 1.1. *If $f \in L^2(\mathbb{R})$ with $xf(x), f', xf'(x) \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} (t-a)^2 |f(t)|^2 dt \cdot \int_{\mathbb{R}} (\omega - b)^2 |\hat{f}(\omega)|^2 d\omega \geq \frac{\|f\|^4}{16\pi^2}, \tag{1}$$

Equality is attained if and only if $f(t) = Ce^{2\pi i b(t-a)} e^{-\gamma(t-a)^2}$ for $C \in \mathbb{C}$ and $\gamma > 0$.

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In signal analysis, $f(t)$ denotes the amplitude of a signal at a point t and the Fourier transform $\hat{f}$ describes how the signal is build up from different frequencies. Inequality (1) states that a signal can not be well-localized simultaneously in the space and the frequency domain. The quantum mechanical interpretation of inequality (1) formulated in Heisenberg's pathbreaking work [14] is similar. In any quantum state, the values of two conjugate observables such as position and momentum can not both be precisely determined.

If the function f is defined on a manifold different from $\mathbb{R}^d$, the question of how to formulate an uncertainty principle like (1) becomes more difficult. On the unit circle $\mathbb{T}$, an interesting approach was pursued by Breitenberger in [2]. If one sets the frequency variance of a function $f \in L^2(\mathbb{T})$ as $\mathrm{var}_F(f) = \langle f', f' \rangle_{\mathbb{T}} - \langle f', f \rangle_{\mathbb{T}}^2$ and the mean value as $\varepsilon(f) = \frac{1}{2\pi} \int_0^{2\pi} e^{it} |f(e^{it})|^2 dt$, then it is possible to prove (cf. [2], [21], [23]) the following uncertainty principle:

Theorem 1.2. *If $f \in \mathrm{AC}(\mathbb{T}) \subset L^2(\mathbb{T})$ with $f' \in L^2(\mathbb{T})$ and $\|f\|_{\mathbb{T}} = 1$, then*

$$(1 - |\varepsilon(f)|^2) \cdot \mathrm{var}_F(f) \geq \frac{1}{4} |\varepsilon(f)|^2. \tag{2}$$

The constant $\frac{1}{4}$ on the right hand side is optimal.

As the Heisenberg-Pauli-Weyl inequality, also (2) has a physical interpretation. If one reads the value

$$\mathrm{var}_S(f) = \frac{1 - |\varepsilon(f)|^2}{|\varepsilon(f)|^2}$$

as the angular variance of a periodic function f (see Figure 1 for the geometric interpretation), then inequality (2) states that the values of the two observables angular position and angular momentum can not both be exactly determined at the same time.

Based on inequality (2), there have been similar attempts to construct uncertainty principles on the unit sphere S^d. Remarkable in this context are the papers of Rösler & Voit [27], Narcovich & Ward [20], Goh & Goodman [12] and Freeden & Windheuser [9]. Of special interest for the present article are the techniques developed in [27]. Therein, Rösler & Voit proved the following uncertainty principle for radial functions on the unit sphere.

Theorem 1.3. *If $f \in L^2(\mathrm{S}^d) \cap C^2(\mathrm{S}^d)$, $\|f\|_{\mathrm{S}^d} = 1$, is radial with respect to a point $p \in \mathrm{S}^d$, i.e. $f(x) = F(x \cdot p)$, define the spherical mean value by $\varepsilon(f,p) = \int_{\mathrm{S}^d} (x \cdot p) |f(x)|^2 d\mu(x)$ as well as the frequency variance by $\mathrm{var}_F(f) = \langle -\Delta_{\mathrm{S}^d} f, f \rangle_{\mathrm{S}^d}$, where Δ_{S^d} denotes the Laplace-Beltrami operator on S^d. Then*

$$(1 - |\varepsilon(f,p)|^2) \cdot \mathrm{var}_F(f) \geq \frac{d^2}{4} |\varepsilon(f,p)|^2, \tag{3}$$

and the constant $\frac{d^2}{4}$ on the right hand side is optimal.

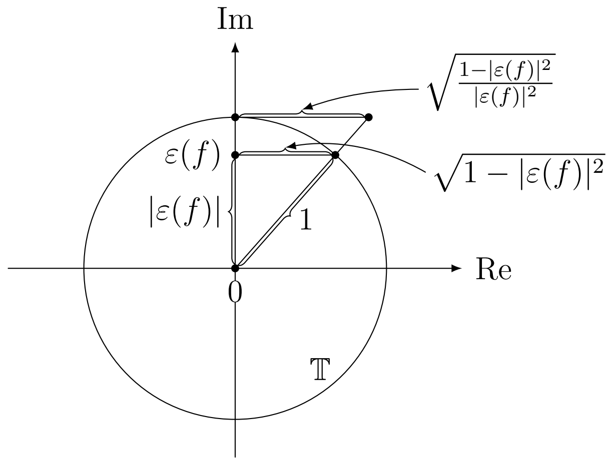

Figure 1: Geometric interpretation of the angular variance $\mathrm{var}_S(f)$ on the unit circle $\mathbb{T}$. The function f is chosen such that $\varepsilon(f) = \frac{3}{4}i$ and $\mathrm{var}_S(f) = \frac{7}{9}$.

In the present work, we are going to extend the uncertainty principle (3) to compact Riemannian manifolds M. The corresponding frequency variance of a function $f \in L^2(M)$ relies on the radial part of the Laplace-Beltrami operator Δ_M. To define a space variance on M, we use, similarly as in (2) and (3), an appropriately introduced mean value $\varepsilon(f,p)$. The proof of the uncertainty inequality itself is based on an operator theoretic approach as described in [7], [8] and [29]. For this approach to work, we need the root of the radial Laplace-Beltrami operator which can be obtained in a generalized form by means of Dunkl theory. In a further step, we are going to prove the asymptotic sharpness of the introduced uncertainty inequality. This is done by constructing an appropriate family of Gaussian-like functions on the manifold M. Finally, we discuss in more detail the uncertainty principle on some special manifolds like the unit sphere S^d and the real projective space $\mathbb{RP}^d_\pi$.

The paper is organized as follows. In Section 2, some preliminaries on compact Riemannian manifolds are introduced. The main result of the paper together with the formulation of the uncertainty principle can be found in Section 3. Herein, also the Dunkl operator, essentially for the proof of the uncertainty, is defined. In Section 4, we proof the asymptotic sharpness of the uncertainty inequality. In the final sections, we give some examples and additional information on special aspects of the uncertainty principle.

2 Preliminaries on Riemannian Manifolds

In this preliminary part, we summarize some basic facts about Riemannian manifolds and introduce the necessary notation for the upcoming sections. The details can be found among other standard references in [1], [4] and [10].

In the following, we denote by M a compact and connected Riemannian manifold without boundary and by M_p the tangent space at a point $p \in M$. A distance metric $d(p,q)$ between two points p and q on M can be introduced by setting

$$d(p,q) := \inf_{w} \int_a^b |w'(t)| dt,$$

where w ranges over all piecewise differentiable paths $w : [a,b] \to M$ satisfying $w(a) = p$ and $w(b) = q$. The metric $d(\cdot,\cdot)$ turns M into a metric space.

Since the compactness of M implies the geodesic completeness [4, Theorem 1.7.2], there exists for every $p \in M$ and $\xi \in M_p$ an unique geodesic $\gamma_\xi : \mathbb{R} \to M$ satisfying $\gamma_\xi(0) = p$ and $\gamma'_\xi(0) = \xi$. Moreover, the Hopf-Rinow-Theorem [4, Theorem 1.7.1] ensures that any two points $p, q \in M$ can be joined by a minimal geodesic with length $d(p,q)$. Through the geodesic γ_ξ, one can define the exponential map $\exp_p : M_p \to M$ by

$$\exp_p t\xi := \gamma_\xi(t),$$

for all $t \in \mathbb{R}$ and $\xi \in M_p$. For $p \in M$ and $\delta > 0$, we introduce on M the open ball and the sphere with center p as

$$\begin{aligned} B(p,\delta) &:= \{x \in M,\ d(x,p) < \delta\}, \\ S(p,\delta) &:= \{x \in M,\ d(x,p) = \delta\}. \end{aligned}$$

By the same token, we define on the tangent space M_p

$$\begin{aligned} \mathfrak{B}(p,\delta) &:= \{\xi \in M_p,\ \|\xi\| < \delta\}, \\ \mathfrak{S}(p,\delta) &:= \{\xi \in M_p,\ \|\xi\| = \delta\}, \\ \mathfrak{S}_p &:= \mathfrak{S}(p,1). \end{aligned}$$

Now, we turn to the notion of a cut point. For $\xi \in \mathfrak{S}_p$, we define

$$R(\xi) := \sup\{t > 0 :\ d(p, \gamma_\xi(t)) = t\}$$

as the maximal distance in direction ξ for which $\exp_p$ is isometric. The point $\gamma_\xi(R(\xi))$ is referred to as the cut point of the point p along the geodesic $\gamma_\xi(t)$. Since M is compact, one can show that the function $R(\xi)$ is Lipschitz continuous [15] and strictly positive on $\mathfrak{S}_p$. Thus, also the ratio

$$\kappa(\xi) := \frac{\pi}{R(\xi)}$$

is a well defined Lipschitz continuous function on $\mathfrak{S}_p$. The set $\mathfrak{C}_p := \{R(\xi)\xi :\ \xi \in \mathfrak{S}_p\}$ is called the tangential cut locus of p in M_p and $C_p := \exp_p \mathfrak{C}_p$ the cut locus of p in M. The point set $\{p\}$ has measure zero, but moreover, one can prove that also the cut locus C_p is a set of Riemannian measure zero. If we define the sets $\mathfrak{D}_p := \{t\xi \in M_p :\ 0 \le t < R(\xi),\ \xi \in \mathfrak{S}_p\}$ and $D_p := \exp_p \mathfrak{D}_p$, then, as a consequence of the Hopf-Rinow-Theorem, we get the decomposition (cf. [10, Prop. 2.113])

$$M = D_p \cup C_p.$$

Through the exponential map $\exp_p$, we can introduce the geodesic spherical coordinates on M (we use GSC as a shortcut). If a coordinate system $\xi = \xi(u)$ is given on the unit sphere $\mathfrak{S}_p$, where u varies over a domain in $\mathbb{R}^{d-1}$, then every point $q \in M$ can be described in the GSC as $q(t, \xi(u)) = \exp_p(t\xi(u))$, where $0 \leq t \leq R(\xi)$. In the geodesic spherical coordinates, the Riemannian measure on M can be determined as [4, III.3]

$$dV(\exp_p(t\xi)) = \Theta(t, \xi) dt d\mu_p(\xi),$$

where Θ is a well defined smooth weight function on $[0, R(\xi)] \times \mathfrak{S}_p$ and $d\mu_p$ denotes the standard surface measure on $\mathfrak{S}_p$. The weight function Θ is zero for $t = 0$ and strictly positive for all points $(t, \xi) \in (0, R(\xi)) \times \mathfrak{S}_p$.

For an integrable function f, the integration on M can be written in the GSC as

$$\int_M f dV = \int_{\mathfrak{D}_p} f(\exp_p t\xi)\Theta(t, \xi) dt d\mu_p(\xi) = \int_{\mathfrak{S}_p} \int_0^{R(\xi)} f(\exp_p t\xi)\Theta(t, \xi) dt d\mu_p(\xi).$$

To simplify the notation, we write $f(t, \xi)$ instead of $f(\exp_p t\xi)$. To get rid of the term $R(\xi)$ in the integral boundaries, we modify the GSC through the coordinate transform $\tau = \kappa(\xi)t$. In this modified version of the GSC (denoted as MGSC), every point $q \in M$ can be written in the form $q(\tau, \xi) = \exp_p(\frac{\tau}{\kappa(\xi)}\xi(u))$, where $(\tau, \xi) \in [0, \pi] \times \mathfrak{S}_p$. The points $q(\tau, \xi)$ with $\tau = 0$ represent the point set $\{p\}$, and the points $q(\tau, \xi)$ with $\tau = \pi$ describe the cut locus C_p of p. In the MGSC, the integration on M reads as

$$\int_M f dV = \int_{\mathfrak{S}_p} \int_0^{\pi} f\Big(\frac{\tau}{\kappa(\xi)}, \xi\Big)\Theta\Big(\frac{\tau}{\kappa(\xi)}, \xi\Big)\frac{1}{\kappa(\xi)} d\tau d\mu_p(\xi).$$

To switch easily between the two coordinate systems GSC and MGSC, we introduce the functions $\tilde{f}$ and $\tilde{\Theta}$ on $[0, \pi] \times \mathfrak{S}_p$ by

$$\tilde{f}(\tau, \xi) := f\Big(\frac{\tau}{\kappa(\xi)}, \xi\Big) \quad \text{and} \quad \tilde{\Theta}(\tau, \xi) = \frac{1}{\kappa(\xi)}\Theta\Big(\frac{\tau}{\kappa(\xi)}, \xi\Big).$$

Finally, we define the space of square integrable functions on M as

$$L^2(M) := \Big\{f : M \to \mathbb{C} : \int_M |f|^2 dV < \infty\Big\}.$$

Endowed with the scalar product $\langle f, g\rangle_M := \int_M f\bar{g} dV$, the space $L^2(M)$ is a Hilbert space with the norm $\|f\|_M := \sqrt{\langle f, f\rangle_M}$. In the GSC and the MGSC, the scalar product reads as

$$\begin{aligned}\langle f, g\rangle_M &= \int_{\mathfrak{S}_p} \int_0^{R(\xi)} f(t, \xi)\overline{g(t, \xi)}\Theta(t, \xi) dt d\mu_p(\xi) \\ &= \int_{\mathfrak{S}_p} \int_0^{\pi} \tilde{f}(\tau, \xi)\overline{\tilde{g}(\tau, \xi)}\tilde{\Theta}(\tau, \xi) d\tau d\mu_p(\xi).\end{aligned}$$

3 Radial Uncertainty Principles on Compact Riemannian Manifolds

A function F on M is called radial with respect to a point p if, in the GSC centered at p, it depends solely on the distance variable t. Radial functions of some Riemannian manifolds are deeply linked to special functions and orthogonal polynomials (see [30] for a general overview). On the unit sphere S^d, for instance, the radial functions can be written in terms of Gegenbauer polynomials. Exactly for these radial functions having an expansion in Gegenbauer polynomials, Rösler & Voit [27] proved the uncertainty principle (3). Later on, these results were extended to spherical Bessel functions [28], to Jacobi polynomials [17], and to Laguerre and Hermite polynomials [18]. In the present work, we will adopt the theory developed in those papers, especially the Dunkl theory used therein, to prove a radial uncertainty principle on compact Riemannian manifolds.

To define a frequency variance, Rösler & Voit used in [27] the second order differential operator of the Gegenbauer polynomials. The analog on a Riemannian manifold M is the Laplace-Beltrami Operator Δ_M. For a radial function F on M, the Laplace-Beltrami Operator Δ_M assumes locally at p the particular form [1, Proposition G.V.3]

$$(\Delta_M F)(t,\xi) = \frac{d^2}{dt^2}F(t) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\frac{d}{dt}F(t),$$

where $\partial_t\Theta$ denotes the partial derivative of the weight function Θ with respect to the variable t. This operator can be extended to the whole manifold M and used globally for functions f on M. In the GSC, we define

$$(\Delta_{p,t} f)(t,\xi) := \frac{\partial^2}{\partial t^2}f(t,\xi) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\frac{\partial}{\partial t}f(t,\xi). \tag{4}$$

For radial functions, the operator $\Delta_{p,t}$ corresponds locally with the Laplace-Beltrami operator Δ_M. Therefore, the operator $\Delta_{p,t}$ is referred to as radial Laplace operator. As a domain of the radial Laplace operator, we use the set

$$\mathcal{D}(\Delta_{p,t}) := \Big\{ f \in C^2(M) :\ \frac{\partial}{\partial t}f(0,\xi) = \frac{\partial}{\partial t}f(R(\xi),\xi) = 0,\ \xi \in \mathfrak{S}_p \Big\}. \tag{5}$$

Since M is compact, $\mathcal{D}(\Delta_{p,t})$ is a dense subset of $L^2(M)$. If we switch to the MGSC, the radial Laplacian reads as

$$(\Delta_{p,t} f)^{\sim}(\tau,\xi) = \kappa(\xi)^2\Big(\frac{\partial^2}{\partial \tau^2}\tilde{f}(\tau,\xi) + \frac{\partial_\tau\tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)}\frac{\partial}{\partial \tau}\tilde{f}(\tau,\xi)\Big).$$

With these preliminaries, we introduce the (radial) frequency variance $\mathrm{var}_{F,p}(f)$ of a function $f \in \mathcal{D}(\Delta_{p,t}) \subset L^2(M)$ as

$$\mathrm{var}_{F,p}(f) := \langle -\Delta_{p,t} f, f\rangle_M. \tag{6}$$

Since in (6) only the radial frequencies of the function f are used to determine the frequency variance, the subsequent uncertainty principle will also have a predominant radial character.

For the proof of the uncertainty principle, we will use an operator theoretic approach. For this purpose, we have to express the frequency variance (6) as the squared norm of an operator acting on f. Hence, we are searching for the root of the operator $-\Delta_{p,t}$. In a generalized form, such a root can be obtained by means of Dunkl theory.

In [27], Rösler & Voit extended L^2-functions on $[0,\pi]$ to even periodic functions on $(-\pi,\pi]$ and used the resulting symmetry to define a differential-difference operator on $(-\pi,\pi]$. This so called Dunkl operator turned out to be a generalized root of the second order differential operator of the Gegenbauer polynomials.

Proceeding in a similar way, we extend a function f on M onto a twofold copy X of M. Using the MGSC, this is done by doubling the range of τ and considering τ as a periodic variable. The set X is defined as $X = (-\pi,\pi] \times \mathfrak{S}_p$, where the points (π,ξ) and $(-\pi,\xi)$ are identified with each other for all $\xi \in \mathfrak{S}_p$. The weight function $\tilde{\Theta}$ is extended symmetrically onto X, i.e.

$$\tilde{\Theta}(\tau,\xi) = \tilde{\Theta}(|\tau|,\xi), \quad (\tau,\xi) \in (-\pi,\pi] \times \mathfrak{S}_p. \tag{7}$$

The extension of the derivative $\partial_\tau \tilde{\Theta}$ onto X is defined such that it is odd in τ, i.e.

$$\partial_\tau \tilde{\Theta}(-\tau,\xi) = -\partial_\tau \tilde{\Theta}(\tau,\xi), \quad (\tau,\xi) \in (-\pi,\pi] \times \mathfrak{S}_p. \tag{8}$$

In this way, $\partial_\tau \tilde{\Theta}$ can be seen as the Radon-Nikodym derivative of the symmetric extension $\tilde{\Theta}$ with respect to the variable τ. Next, we define a volume element on X by $dV := \frac{1}{2}\tilde{\Theta}(\tau,\xi) d\tau d\mu_p(\xi)$. Moreover, we introduce the Hilbert space of square integrable functions on X as

$$L^2(X) := \Big\{ g : X \to \mathbb{C} : \int_X |g|^2 dV < \infty \Big\}$$

with scalar product $\langle g_1, g_2 \rangle_X := \int_X g_1 \overline{g_2} dV$ and the subspace of even functions as

$$L^2_e(X) := \Big\{ g \in L^2(X) : \; g(\tau,\xi) = g(-\tau,\xi) \text{ a.e.} \Big\}.$$

For a function $f \in L^2(M)$ and an even function $g \in L^2_e(X)$, we can define in the MGSC the even extension operator and the restriction operator as

$$e : L^2(M) \to L^2(X), \quad e(f)(\tau,\xi) := \tilde{f}(|\tau|,\xi), \tag{9}$$

$$r : L^2_e(X) \to L^2(M), \quad r(g)^{\sim}(\tau,\xi) := g(\tau,\xi). \tag{10}$$

In particular, these operators constitute isometric isomorphisms between $L^2(M)$ and $L^2_e(X)$. Similar as in [27], we introduce now the following differential-difference operator on $L^2(X)$, referred to as Dunkl operator:

$$(T_\tau g)(\tau,\xi) := \kappa(\xi) \Big(\frac{\partial}{\partial \tau} g(\tau,\xi) + \frac{\partial_\tau \tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)} \frac{\big(g(\tau,\xi) - g(-\tau,\xi)\big)}{2} \Big), \tag{11}$$

$(\tau,\xi)\in(-\pi,\pi]\times\mathfrak{S}_p$, defined on the domain

$$\mathcal{D}(T_\tau):=\Big\{g\in C(X):\ \frac{\partial}{\partial\tau}g\in C(X)\Big\}.$$

By (9) and (10), the Dunkl operator T_τ and the radial Laplacian $\Delta_{p,t}$ are related by

$$-\Delta_{p,t}f=r((iT_\tau)^2e(f)),\quad \text{for } f\in\mathcal{D}(\Delta_{p,t}). \tag{12}$$

Thus, the operator iT_τ is the desired generalized root of $-\Delta_{p,t}$. For the proof of the uncertainty principle, it is essential that iT_τ is symmetric.

Lemma 3.1. *The operator iT_τ is symmetric and densely defined on $L^2(X)$.*

Proof. We essentially follow the proof of Lemma 3.1 in [27]. To check the symmetry of iT_τ, we take $f,g\in\mathcal{D}(T_\tau)$. Integration by parts with respect to the variable τ yields

$$\begin{aligned}\int_{\mathfrak{S}_p}\int_{-\pi}^{\pi}\frac{\partial}{\partial\tau}f(\tau,\xi)\overline{g(\tau,\xi)}\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi)&=-\int_{\mathfrak{S}_p}\int_{-\pi}^{\pi}f(\tau,\xi)\frac{\partial}{\partial\tau}\Big(\overline{g(\tau,\xi)}\tilde\Theta(\tau,\xi)\Big)d\tau d\mu_p(\xi)\\&=-\int_{\mathfrak{S}_p}\int_{-\pi}^{\pi}f(\tau,\xi)\Big(\overline{\frac{\partial}{\partial\tau}g(\tau,\xi)}+\overline{g(\tau,\xi)}\frac{\partial_\tau\tilde\Theta(\tau,\xi)}{\tilde\Theta(\tau,\xi)}\Big)\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi).\end{aligned}$$

Now, we get by definition of the operator T_τ:

$$\begin{aligned}&\int_{\mathfrak{S}_p}\int_{-\pi}^{\pi}(iT_\tau f)(\tau,\xi)\overline{g(\tau,\xi)}\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi)=\\&=-i\int_{\mathfrak{S}_p}\kappa(\xi)\int_{-\pi}^{\pi}\Big(f(\tau,\xi)\overline{\frac{\partial}{\partial\tau}g(\tau,\xi)}+f(\tau,\xi)\overline{g(\tau,\xi)}\frac{\partial_\tau\tilde\Theta(\tau,\xi)}{\tilde\Theta(\tau,\xi)}\Big)\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi)\\&\quad+i\int_{\mathfrak{S}_p}\kappa(\xi)\int_{-\pi}^{\pi}\frac{f(\tau,\xi)-f(-\tau,\xi)}{2}\overline{g(\tau,\xi)}\frac{\partial_\tau\tilde\Theta(\tau,\xi)}{\tilde\Theta(\tau,\xi)}\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi)\\&=-i\int_{\mathfrak{S}_p}\kappa(\xi)\int_{-\pi}^{\pi}\Big(f(\tau,\xi)\overline{\frac{\partial}{\partial\tau}g(\tau,\xi)}+\frac{f(\tau,\xi)+f(-\tau,\xi)}{2}\overline{g(\tau,\xi)}\frac{\partial_\tau\tilde\Theta(\tau,\xi)}{\tilde\Theta(\tau,\xi)}\Big)\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi)\\&=-i\int_{\mathfrak{S}_p}\kappa(\xi)\int_{-\pi}^{\pi}\Big(f(\tau,\xi)\overline{\frac{\partial}{\partial\tau}g(\tau,\xi)}+f(\tau,\xi)\overline{\frac{g(\tau,\xi)-g(-\tau,\xi)}{2}}\frac{\partial_\tau\tilde\Theta(\tau,\xi)}{\tilde\Theta(\tau,\xi)}\Big)\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi)\\&=\int_{\mathfrak{S}_p}\int_{-\pi}^{\pi}f(\tau,\xi)\overline{(iT_\tau g)(\tau,\xi)}\tilde\Theta(\tau,\xi)d\tau d\mu_p(\xi).\end{aligned}$$

□

Uncertainty principles in a Hilbert space can, in general, be formulated by using the commutator of two densely defined operators (cf. [7], [8], [29]). As underlying Hilbert space we consider the space $L^2(X)$. As a position operator $A: L^2(X)\to L^2(X)$, we fix an arbitrary function $h\in\mathcal{D}(T_\tau)$ and set $Ag:=h\,g$ for $g\in L^2(X)$. As frequency operator $B: L^2(X)\to L^2(X)$, we take the Dunkl operator iT_τ, i.e. $Bg:=iT_\tau g$. Clearly, A is a normal operator and B is symmetric due to Lemma 3.1. Also, both operators are densely defined in $L^2(X)$. Therefore, we can use an operator theoretic approach involving a symmetric and a normal operator (cf. [29, Theorem 5.1]) to prove the following uncertainty:

Theorem 3.2. *For an even function $g \in L^2_e(X) \cap \mathcal{D}(T_\tau)$, a function $h \in \mathcal{D}(T_\tau)$, and $a \in \mathbb{C}$, $b \in \mathbb{R}$, the following uncertainty principle holds:*

$$\|(h-a)g\|_X \cdot \|(iT_\tau - b)g\|_X \geq \frac{1}{2}|\langle g \cdot T_\tau h, g\rangle_X|. \tag{13}$$

Proof. Since A is a normal operator and B is symmetric, we have due to [29, Theorem 5.1]

$$\|(A-a)g\|_X \cdot \|(B-b)g\|_X \geq \frac{1}{2}|\langle [A,B]g, g\rangle_X|$$

for all functions in $\mathcal{D}(AB) = \mathcal{D}(BA) = \mathcal{D}(T_\tau)$. For the commutator of A and B defined on $\mathcal{D}(T_\tau)$, we get

$$[A,B]g(\tau,\xi) = -i\kappa(\xi)\Big(\frac{\partial}{\partial\tau}h(\tau,\xi)g(\tau,\xi) + \frac{\partial_\tau\tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)}\frac{\big(h(\tau,\xi)-h(-\tau,\xi)\big)}{2}g(-\tau,\xi)\Big).$$

Thus, for an even function $g \in L^2_e(X)$, we have $[A,B]g = -g \cdot iT_\tau h$. □

The minimum of $\|(A-a)g\|_X$ and $\|(B-b)g\|_X$ is attained at (cf. [8], [29])

$$a = \frac{\langle Ag, g\rangle_X}{\|g\|_X^2} \quad \text{and} \quad b = \frac{\langle Bg, g\rangle_X}{\|g\|_X^2}, \tag{14}$$

respectively . Since the derivative $\frac{\partial}{\partial\tau}g$ of an even function $g \in \mathcal{D}(T_\tau) \cap L^2_e(X)$ is odd, i.e. it satisfies $\frac{\partial}{\partial\tau}g(\tau,\xi) = -\frac{\partial}{\partial\tau}g(-\tau,\xi)$ a.e., we get $\langle Bg, g\rangle_X = i\langle T_\tau g, g\rangle_X = 0$. Hence, the minimum of $\|(B-b)g\|_X$ is attained at $b = 0$. For the special values (14) of a and b, the uncertainty product (13) reads as

$$\Big(\|hg\|_X^2 - \frac{|\langle hg, g\rangle_X|^2}{\|g\|_X^2}\Big) \cdot \|T_\tau g\|_X^2 \geq \frac{1}{4}|\langle g \cdot T_\tau h, g\rangle_X|^2. \tag{15}$$

For a function $f \in L^2(M)$, we take now the even extension $e(f) \in L^2_e(X)$ and use inequality (15) to get an uncertainty principle for compact Riemannian manifolds. For the function h characterizing the position operator, we set, similarly as in [27] and [17], $h(\tau,\xi) = e^{i\tau}$. Of course, also other choices for h are possible (cf. [29]), but in general $e^{i\tau}$ is a reasonable option. In fact, the function $e^{i\tau}$ is well defined on X, is periodic in the variable τ and lies in the domain of the Dunkl operator T_τ. Next, we define the generalized mean value as

$$\begin{aligned}\varepsilon(f,p) := \langle e^{i\tau}e(f), e(f)\rangle_X &= \frac{1}{2}\int_{\mathfrak{S}_p}\int_{-\pi}^{\pi} e^{i\tau}|e(f)(\tau,\xi)|^2\tilde{\Theta}(\tau,\xi)d\tau d\mu_p(\xi)\\ &= \int_{\mathfrak{S}_p}\int_0^{\pi}\cos(\tau)|\tilde{f}(\tau,\xi)|^2\tilde{\Theta}(\tau,\xi)d\tau d\mu_p(\xi)\\ &= \int_{\mathfrak{S}_p}\int_0^{R(\xi)}\cos(\kappa(\xi)t)|f(t,\xi)|^2\Theta(t,\xi)dtd\mu_p(\xi).\end{aligned} \tag{16}$$

If the function f is normalized such that $\|f\|_M = 1$, then the value $\varepsilon(f,p)$ lies between -1 and 1. The generalized mean value is an indication on how well the function f is localized at a point p. The closer $\varepsilon(f,p)$ gets to 1, the better f is localized at p. We formulate now our main result.

Theorem 3.3. *If $f \in L^2(M) \cap \mathcal{D}(\Delta_{p,t})$ with $\|f\|_M = 1$, then the following uncertainty principle holds:*

$$\Big(1 - \varepsilon(f,p)^2\Big) \cdot \langle -\Delta_{p,t} f, f\rangle_M \geq \frac{1}{4}\Big|\langle \Big(\kappa(\xi)\cos\big(\kappa(\xi)t\big) + \frac{\partial_t \Theta(t,\xi)}{\Theta(t,\xi)} \sin\big(\kappa(\xi)t\big)\Big) f, f\rangle_M\Big|^2. \tag{17}$$

Proof. If $f \in L^2(M) \cap \mathcal{D}(\Delta_{p,t})$, then the even extension $e(f) \in \mathcal{D}(T_\tau)$ is an element of the domain of the Dunkl operator T_τ. As a multiplier function h in inequality (15), we choose $h(t,\xi) = e^{i\tau}$. Then, $h \in \mathcal{D}(T_\tau)$ and in inequality (15), we get

$$\begin{aligned}
&\|he(f)\|_X^2 = \|e(f)\|_X^2 = \|f\|_M^2 = 1,\\
&|\langle he(f), e(f)\rangle_X|^2 = |\langle e^{i\tau} e(f), e(f)\rangle_X|^2 = \varepsilon(f,p)^2,\\
&iT_\tau h(\tau,\xi) = \kappa(\xi)\Big(- e^{i\tau} - \frac{\partial_\tau \tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)} \sin(\tau)\Big).
\end{aligned}$$

Further, since $\frac{\partial_\tau \tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)} \sin(\tau)$ is an even function in τ, we conclude

$$\begin{aligned}
\langle e(f) \cdot iT_\tau h, e(f)\rangle_X &= -\langle \kappa(\xi)\Big(e^{i\tau} + \frac{\partial_\tau \tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)} \sin(\tau)\Big) e(f), e(f)\rangle_X\\
&= -\frac{1}{2}\int_{\mathfrak{S}_p} \kappa(\xi) \int_{-\pi}^{\pi} \Big(e^{i\tau} + \frac{\partial_\tau \tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)} \sin(\tau)\Big) |e(f)(\tau,\xi)|^2 \tilde{\Theta}(\tau,\xi) d\tau d\mu_p(\xi)\\
&= -\int_{\mathfrak{S}_p} \kappa(\xi) \int_0^{\pi} \Big(\cos(\tau) + \frac{\partial_\tau \tilde{\Theta}(\tau,\xi)}{\tilde{\Theta}(\tau,\xi)} \sin(\tau)\Big) |\tilde{f}(\tau,\xi)|^2 \tilde{\Theta}(\tau,\xi) d\tau d\mu_p(\xi)\\
&= -\int_{\mathfrak{S}_p} \int_0^{R(\xi)} \Big(\kappa(\xi)\cos\big(\kappa(\xi)t\big) + \frac{\partial_t \Theta(t,\xi)}{\Theta(t,\xi)} \sin\big(\kappa(\xi)t\big)\Big) |f(t,\xi)|^2 \Theta(t,\xi) dt d\mu_p(\xi)\\
&= -\langle \Big(\kappa(\xi)\cos\big(\kappa(\xi)t\big) + \frac{\partial_t \Theta(t,\xi)}{\Theta(t,\xi)} \sin\big(\kappa(\xi)t\big)\Big) f, f\rangle_M.
\end{aligned}$$

Finally, using the symmetry of iT_τ and relation (12), we get for $f \in \mathcal{D}(\Delta_{p,t})$:

$$\langle -\Delta_{p,t} f, f\rangle_M = -\langle (iT_\tau)^2 e(f), e(f)\rangle_X = \|T_\tau e(f)\|_X^2.$$

□

For even and normalized functions in the weighted Hilbert space $L^2([-\pi,\pi], w)$, the following uncertainty principle was shown in [12] by Goh and Goodman:

$$\frac{1 - \big(\int_0^\pi \cos(t)|f(t)|^2 w(t) dt\big)^2}{\big(\int_0^\pi |f(t)|^2 (\cos(t)w(t) + \sin(t)w'(t)) dt\big)^2} \cdot \int_0^\pi |f'(t)|^2 w(t) dt \geq \frac{1}{4}. \tag{18}$$

Our uncertainty inequality (17) presented above resembles this weighted uncertainty (18). This is not surprising, since in both cases the theory and the techniques used are conceptually the same. In contrast to [12], we considered even L^2-functions defined on the higher dimensional compact set $X = (-\pi, \pi] \times \mathfrak{S}_p$ and a weight function $\tilde{\Theta}$ which depends both on the variable τ and the direction ξ. Therefore, (17) can be considered as an extension of (18). Moreover, in our case the weight function $\tilde{\Theta}$ plays a more substantial role since it contains implicitly information on the geometry of the Riemannian manifold M. Similar to the inequalities (17) and (18) is also the uncertainty principle [18, Corollary 7] developed by Li and Liu in which the weight function w is linked to a Sturm-Liouville operator.

Another interesting uncertainty principle for compact Riemannian manifolds based on a different approach can be found in the work [19] of Martini. Here, it is shown that for all $\alpha, \beta > 0$ and $f \in L^2(M)$ with null mean value the following inequality holds:

$$\|f\|_M \leq C_{\alpha,\beta} \|t^\alpha f\|_M^{\frac{\beta}{\alpha+\beta}} \cdot \|(-\Delta_M)^{\frac{\beta}{2}} f\|_M^{\frac{\alpha}{\alpha+\beta}}.$$

This inequality is a special case of a more general theory treating uncertainty principles on abstract measure spaces (see also [5] and [26]). The proof of this inequality is mainly based on the spectral theorem and on estimates involving the heat semigroup generated by the Laplace-Beltrami operator Δ_M. In contrast to (17), the constant $C_{\alpha,\beta}$ in the above inequality is not explicitly known.

Turning back to the frequency variance in inequality (17), formula (12) implies that

$$\langle -\Delta_{p,t} f, f \rangle_M = \|T_\tau e(f)\|_X^2 = \Big\| \kappa(\xi) \frac{\partial}{\partial \tau} e(f) \Big\|_X^2 = \Big\| \frac{\partial}{\partial t} f \Big\|_M^2$$

for $f \in \mathcal{D}(\Delta_{p,t})$. So, instead of (6) we could have defined the frequency variance also as

$$\mathrm{var}_{F,p}(f) = \Big\| \frac{\partial}{\partial t} f \Big\|_M^2. \tag{19}$$

This formula illustrates that the frequency variance in our notion is completely determined by the radial derivative of the function f. Many authors (see for instance [12] or [19]) prefer to use the full Laplace-Beltrami operator for the frequency variance (i.e. $\mathrm{var}_F(f) = \langle -\Delta_M f, f \rangle_M$) instead of the radial approach (6). However, since $\mathrm{var}_{F,p}(f) \leq \langle -\Delta_M f, f \rangle_M$ for functions f that are locally supported at $p \in M$, we get a more stringent inequality in (17) if we use the radial Laplacian.

Using the alternative definition (19) of the frequency variance, inequality (17) can be proven for a larger class of functions, namely for

$$\mathcal{D}(\tfrac{\partial}{\partial t}) := \Big\{ f \in C(M) : \ \frac{\partial}{\partial t} f \in C(M), \ \frac{\partial}{\partial t} f(0) = \frac{\partial}{\partial t} f(R(\xi)) = 0, \ \xi \in \mathfrak{S}_p \Big\}.$$

Similar as in the case of the Breitenberger uncertainty principle, Theorem 3.3 motivates the definition

$$\mathrm{var}_{S,p}(f) := d^2 \frac{1 - \varepsilon(f,p)^2}{|\langle \big(\kappa(\xi) \cos(\kappa(\xi) t) + \frac{\partial_t \Theta(t,\xi)}{\Theta(t,\xi)} \sin(\kappa(\xi) t)\big) f, f \rangle_M|^2} \tag{20}$$

for the position variance of $f \in L^2(M)$ at $p \in M$. Then, we get the uncertainty inequality

$$\operatorname{var}_{S,p}(f) \cdot \operatorname{var}_{F,p}(f) \geq \frac{d^2}{4} \tag{21}$$

for all normalized functions $f \in \mathcal{D}(\Delta_{p,t})$, provided that the right hand side of inequality (17) is not zero.

The expectation value of a density $f \in L^2(M)$, $\|f\|_M = 1$, can be found by means of the generalized mean value $\varepsilon(f,p)$. We have already remarked that for all points p the value $\varepsilon(f,p)$ is a measure on how well the function f is localized at p. The closer $\varepsilon(f,p)$ gets to 1, the more the mass of f is concentrated at p. The point at which f is localized best is then the point p_f where $\varepsilon(f,p)$ gets maximal, i.e.

$$p_f = \arg\sup_{p \in M} \varepsilon(f,p).$$

If p_f is uniquely determined, we call it the expectation value of f.

4 Sharpness of the Uncertainty Principle

In this section, we show that the uncertainty inequalities (17) and (21) are asymptotically sharp. In particular, we construct a family H_λ of Gaussian-like functions on the manifold M such that, for $\lambda \to 0$, we attain equality in (17) and (21). For this purpose, we need some properties of the Gaussian bell. First of all, we have for $k \in \mathbb{N}_0$ and $\sigma \in \mathbb{R}_+$ the well known moment formulas (cf. [22, p. 110])

$$\int_0^\infty t^{2k} e^{-\frac{t^2}{\sigma^2}} dt = \frac{\sqrt{\pi}}{2} \frac{(2k)!}{4^k k!} \sigma^{2k+1}, \tag{22}$$

$$\int_0^\infty t^{2k+1} e^{-\frac{t^2}{\sigma^2}} dt = \frac{k!}{2} \sigma^{2k+2}. \tag{23}$$

On $[0,\infty)$, we define now for $d \in \mathbb{N}$, $d \geq 1$, the Gaussians

$$G_{d,\sigma}(t) := \begin{cases} \sqrt{\frac{2}{\sqrt{\pi}} \frac{4^k k!}{(2k)!} \frac{1}{\sigma^{k+1/2}}} e^{-\frac{t^2}{2\sigma^2}} & \text{if } d = 2k+1, \\ \sqrt{\frac{2}{k!} \frac{1}{\sigma^{k+1}}} e^{-\frac{t^2}{2\sigma^2}} & \text{if } d = 2k+2. \end{cases}$$

The moment formulas (22) and (23) imply that $G_{d,\sigma}$ is a function in the weighted Hilbert space $L^2_d = L^2([0,\infty), t^{d-1}dt)$ normalized such that $\|G_{d,\sigma}\|_{L^2_d} = 1$. Moreover, we get the following result:

Lemma 4.1. *Consider $G_{d,\sigma}$ as an element of the Hilbert space L^2_d. Then*

$$\langle\Big(-\frac{d^2}{dt^2}-\frac{d-1}{t}\frac{d}{dt}\Big)G_{d,\sigma},G_{d,\sigma}\rangle_{L^2_d}=\frac{d}{2}\frac{1}{\sigma^2},\tag{24}$$

$$\langle t^2G_{d,\sigma},G_{d,\sigma}\rangle_{L^2_d}=\frac{d}{2}\sigma^2,\tag{25}$$

$$\langle\frac{d}{dt}G_{d,\sigma},\frac{d}{dt}G_{d,\sigma}\rangle_{L^2_d}=\frac{d}{2}\frac{1}{\sigma^2}.\tag{26}$$

Proof. We prove equation (24) by direct calculation and using the formulas (22) and (23). For d odd and $k=\frac{d-1}{2}$, we get

$$\begin{aligned}\langle(-\tfrac{d^2}{dt^2}-\tfrac{d-1}{t}\tfrac{d}{dt})G_{d,\sigma},G_{d,\sigma}\rangle_{L^2_d}&=\int_0^\infty\frac{2}{\sqrt{\pi}}\frac{4^kk!}{(2k)!}\frac{1}{\sigma^{2k+1}}\Big(-\frac{d^2}{dt^2}-\frac{2k}{t}\frac{d}{dt}\Big)e^{-\frac{t^2}{2\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}t^{2k}dt\\&=\int_0^\infty\frac{2}{\sqrt{\pi}}\frac{4^kk!}{(2k)!}\frac{1}{\sigma^{2k+1}}\Big(-\frac{t^2}{\sigma^4}+(2k+1)\frac{1}{\sigma^2}\Big)e^{-\frac{t^2}{\sigma^2}}t^{2k}dt\\&=-\frac{1}{4}\frac{k!}{(2k)!}\frac{(2k+2)!}{(k+1)!}\frac{1}{\sigma^2}+(2k+1)\frac{1}{\sigma^2}=\frac{d}{2}\frac{1}{\sigma^2}.\end{aligned}$$

On the other hand, for d even and $k=\frac{d-2}{2}$, we have

$$\begin{aligned}\langle(-\tfrac{d^2}{dt^2}-\tfrac{d-1}{t}\tfrac{d}{dt})G_{d,\sigma},G_{d,\sigma}\rangle_{L^2_d}&=\int_0^\infty\frac{2}{k!}\frac{1}{\sigma^{2k+2}}\Big(-\frac{d^2}{dt^2}-\frac{2k+1}{t}\frac{d}{dt}\Big)e^{-\frac{t^2}{2\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}t^{2k+1}dt\\&=\int_0^\infty\frac{2}{k!}\frac{1}{\sigma^{2k+2}}\Big(-\frac{t^2}{\sigma^4}+(2k+2)\frac{1}{\sigma^2}\Big)e^{-\frac{t^2}{\sigma^2}}t^{2k+1}dt\\&=-\frac{(k+1)!}{k!}\frac{1}{\sigma^2}+(2k+2)\frac{1}{\sigma^2}=\frac{d}{2}\frac{1}{\sigma^2}.\end{aligned}$$

Similarly, equations (25) and (26) follow by direct calculation. □

Now, we choose $\delta>0$ small enough such that $B(p,\delta)\subset D_p\subset M$ and introduce a smooth cut-off function $\varphi_\delta:[0,\infty)\to[0,1]$ with $\varphi_\delta(t)=1$ for $0\le t\le\frac{\delta}{2}$, $0\le\varphi_\delta(t)\le1$ for $\frac{\delta}{2}\le t\le\delta$, and $\varphi_\delta(t)=0$ for $t\ge\delta$. Further, we set $c_\xi=\kappa(\xi)^{-\frac{1}{2}}$ and define for $\lambda\in]0,\infty[$ the following function in the GSC at $p\in M$:

$$H_\lambda(t,\xi):=\frac{G_{d,c_\xi\lambda}(t)\varphi_\delta(t)}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M}.\tag{27}$$

The function H_λ is compactly supported in $B(p,\delta)$ an element of the domain of the operator $\Delta_{p,t}$. If $|\mathfrak{S}_p|$ denotes the surface volume of the unit sphere $\mathfrak{S}_p$, then we have:

Proposition 4.2.

$$\lim_{\lambda\to 0}\frac{(1-\varepsilon(H_\lambda,p)^2)}{\lambda^2} = \frac{d}{2|\mathfrak{S}_p|}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi), \tag{28}$$

$$\lim_{\lambda\to 0}\lambda^2\langle\Delta_{p,t}H_\lambda,H_\lambda\rangle_M = \frac{d}{2|\mathfrak{S}_p|}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi), \tag{29}$$

$$\lim_{\lambda\to 0}\langle\Big(\kappa(\xi)\cos(\kappa(\xi)t)+\frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t)\Big)H_\lambda,H_\lambda\rangle_M = \frac{d}{|\mathfrak{S}_p|}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi). \tag{30}$$

In particular, the uncertainty inequalities (17) *and* (21) *are asymptotically sharp.*

Proof. Beside Lemma 4.1, we need two facts for the proof of Proposition 4.2. The first one is a property of the weight function Θ. If $\delta > 0$ is chosen small enough, we have for $t \leq \delta$ the Taylor expansion (cf. [3, XII 8])

$$\Theta(t,\xi) = t^{d-1} - \frac{\mathrm{Ric}(\xi,\xi)}{6}t^{d+1} + \mathrm{O}(t^{d+2}), \tag{31}$$

$$\partial_t\Theta(t,\xi) = (d-1)t^{d-2} - \frac{(d+1)\,\mathrm{Ric}(\xi,\xi)}{6}t^{d} + \mathrm{O}(t^{d+1}), \tag{32}$$

where $\mathrm{Ric}(\cdot,\cdot)$ denotes the Ricci tensor on $M_p \times M_p$. The second fact concerns the Gaussian $G_{d,c_\xi\lambda}$. Since the term $c_\xi = \kappa(\xi)^{-\frac{1}{2}}$ is uniformly bounded above and below by positive constants, there exists for $\delta > 0$ and $\epsilon > 0$ a $\lambda_{\delta,\epsilon}$ such that for all $\lambda < \lambda_{\delta,\epsilon}$ and $\xi \in \mathfrak{S}_p$ we have

$$\int_{\delta/2}^{\infty} G_{d,c_\xi\lambda}(t)^2 t^{d-1}dt < \epsilon. \tag{33}$$

We consider now the L^2-norm of the function $G_{d,c_\xi\lambda}\varphi_\delta$ on M. Using the Taylor expansion (31) of the weight function Θ and property (25) of Lemma 4.1, we get the estimate

$$\begin{aligned}
\lim_{\lambda\to 0}\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2 &= \lim_{\lambda\to 0}\int_{\mathfrak{S}_p}\int_0^\delta G_{d,c_\xi\lambda}(t)^2\varphi_\delta(t)^2\Theta(t,\xi)dtd\mu_p(\xi)\\
&= \lim_{\lambda\to 0}\int_{\mathfrak{S}_p}\int_0^\delta G_{d,c_\xi\lambda}(t)^2\varphi_\delta(t)^2\big(t^{d-1}+\mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi)\\
&\leq \lim_{\lambda\to 0}\int_{\mathfrak{S}_p}\int_0^\infty G_{d,c_\xi\lambda}(t)^2\big(t^{d-1}+\mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi)\\
&= \lim_{\lambda\to 0}|\mathfrak{S}_p| + \mathrm{O}(\lambda^2) = |\mathfrak{S}_p|,
\end{aligned}$$

where $|\mathfrak{S}_p|$ denotes the volume of the $d-1$ dimensional unit sphere $\mathfrak{S}_p$ in the tangent space M_p. Using property (33), we get for an arbitrary $\epsilon > 0$ and $\lambda < \lambda_{\delta,\epsilon}$

$$\begin{aligned}\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2 &= \int_{\mathfrak{S}_p}\int_0^\delta G_{d,c_\xi\lambda}(t)^2\varphi_\delta(t)^2\big(t^{d-1}+\mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi)\\ &\geq \int_{\mathfrak{S}_p}\int_0^\infty G_{d,c_\xi\lambda}(t)^2\big(t^{d-1}+\mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi) - \epsilon|\mathfrak{S}_p|\\ &= (1-\epsilon)|\mathfrak{S}_p| + \mathrm{O}(\lambda^2).\end{aligned}$$

Therefore,

$$\lim_{\lambda\to 0}\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2 = |\mathfrak{S}_p|. \tag{34}$$

We consider now equation (28). Using the Taylor expansion (31) of the weight function Θ and equation (25), we get the upper estimate

$$\begin{aligned}1-\varepsilon(H_\lambda,p) &= \frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\int_0^\delta \big(1-\cos(\kappa(\xi)t)\big)G_{d,c_\xi\lambda}(t)^2\varphi_\delta(t)^2\Theta(t,\xi)dtd\mu_p(\xi)\\ &\leq \frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\kappa(\xi)^2\int_0^\delta \frac{t^2}{2}G_{d,c_\xi\lambda}(t)^2\varphi_\delta(t)^2\Theta(t,\xi)dtd\mu_p(\xi)\\ &\leq \frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\kappa(\xi)^2\int_0^\infty \frac{t^2}{2}G_{d,c_\xi\lambda}(t)^2\big(t^{d-1}+\mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi)\\ &= \frac{d}{4}\lambda^2\frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi) + \mathrm{O}(\lambda^4).\end{aligned}$$

Further, since $\varepsilon(H_\lambda,p)\leq 1$, we have $(1+\varepsilon(H_\lambda,p))\leq 2$. In total, we get

$$\lim_{\lambda\to 0}\frac{1-\varepsilon(H_\lambda,p)^2}{\lambda^2} \leq 2\lim_{\lambda\to 0}\frac{1-\varepsilon(H_\lambda,p)}{\lambda^2} \leq \frac{d}{2|\mathfrak{S}_p|}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi). \tag{35}$$

Next, we turn to equation (29). For the following estimate, we use the Taylor expansion (31) and equation (26) of Lemma 4.1.

$$\begin{aligned}\Big\|\frac{\partial}{\partial t}H_\lambda\Big\|_M^2 &= \frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\int_0^\delta \Big|\frac{\partial}{\partial t}\big(G_{d,c_\xi\lambda}(t)\varphi_\delta(t)\big)\Big|^2\Theta(t,\xi)dtd\mu_p(\xi)\\ &\leq \frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\int_0^\infty \Big[\big|\partial_t G_{d,c_\xi\lambda}(t)\big|^2 + 2|\partial_t G_{d,c_\xi\lambda}(t)|\|\partial_t\varphi_\delta\|_\infty +\\ &\qquad + |G_{d,c_\xi\lambda}(t)|\|\partial_t\varphi_\delta\|_\infty^2\Big]\big(t^{d-1}+\mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi)\\ &= \frac{d}{2}\frac{1}{\lambda^2}\frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi) + \mathrm{O}\Big(\frac{1}{\lambda}\Big).\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0}\lambda^2\langle\Delta_{p,t}H_\lambda,H_\lambda\rangle_M = \lim_{\lambda\to 0}\lambda^2\Big\|\frac{\partial}{\partial t}H_\lambda\Big\|_M^2 \leq \frac{d}{2|\mathfrak{S}_p|}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi). \tag{36}$$

Finally, we take a look at equation (30). Due to (31) and (32), the function $\frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t)$ has locally at p the Taylor expansion

$$\frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t) = (d-1)\kappa(\xi) + \mathrm{O}(t^2). \tag{37}$$

Using (37) and property (33), we derive for an arbitrary $\epsilon > 0$ and $\lambda < \lambda_{\delta,\epsilon}$

$$\begin{aligned}
&\langle\Big(\kappa(\xi)\cos(\kappa(\xi)t) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t)\Big)H_\lambda, H_\lambda\rangle_M = \\
&\quad = \frac{1}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\int_0^\delta \Big(\kappa(\xi)\cos(\kappa(\xi)t) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t)\Big)\cdot \\
&\qquad\qquad \cdot \big(G_{d,c_\xi\lambda}(t)\varphi_\delta(t)\big)^2\Theta(t,\xi)dtd\mu_p(\xi) \\
&\quad = \frac{d}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\kappa(\xi)\int_0^\delta \big(G_{d,c_\xi\lambda}(t)\varphi_\delta(t)\big)^2\big(t^{d-1} + \mathrm{O}(t^{d+1})\big)dtd\mu_p(\xi) \\
&\quad \geq \frac{d}{\|G_{d,c_\xi\lambda}\varphi_\delta\|_M^2}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi)(1-\epsilon) + \mathrm{O}(\lambda^2)
\end{aligned}$$

Thus, we conclude

$$\lim_{\lambda\to 0}\langle\Big(\kappa(\xi)\cos(\kappa(\xi)t) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t)\Big)H_\lambda, H_\lambda\rangle_M \geq \frac{d}{|\mathfrak{S}_p|}\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi). \tag{38}$$

Now, inserting the inequalities (35), (36) and (38) in the uncertainty inequality (17), we get the same value on both sides, namely $\frac{d^2}{4|\mathfrak{S}_p|^2}\big(\int_{\mathfrak{S}_p}\kappa(\xi)d\mu_p(\xi)\big)^2$. Thus, inequalities (35), (36) and (38) are in fact equalities and the statement is proven. □

5 Uncertainty Principles on the Unit Sphere and the Real Projective Space

In this section, we consider two important examples of compact Riemannian manifolds, the unit sphere S^d and the real projective space $\mathbb{RP}^d_\pi$. For both, we derive an uncertainty principle from the general inequality (17) and relate it to uncertainties known from the literature.

We start with the d-dimensional unit sphere S^d. If $p \in \mathrm{S}^d$, we identify the tangent space $(\mathrm{S}^d)_p$ at p with the orthogonal complement $p^\perp$ of the linear vector space $\mathbb{R}p$ in $\mathbb{R}^{d+1}$. An arbitrary point $x \in \mathrm{S}^d$ can be represented as

$$x = x(t,\xi) = \cos(t)p + \sin(t)\xi,$$

where $t \in [0,\pi]$ and $\xi \in \mathfrak{S}_p$ is a unit vector in the hyperplane $p^\perp$. Since for fixed ξ the functions $\gamma_\xi(t) = x(t,\xi)$ describe the geodesics on S^d (see [4, II.3]), the coordinates (t,ξ)

correspond exactly with the GSC at p. The cut locus C_p of p consists of the single point $\{-p\}$. Further, $R(\xi) = \pi$ for all $\xi \in \mathrm{S}^{d-1}$, and the weight function Θ can be determined as [1, C.III]

$$\Theta(t,\xi) = \sin(t)^{d-1}. \tag{39}$$

Moreover, the Laplace-Beltrami operator on S^d is given as (cf. [3, II.5])

$$\Delta_{\mathrm{S}^d} f(t,\xi) = \frac{\partial^2}{\partial t} f(t,\xi) + (d-1)\frac{\cos(t)}{\sin(t)}\frac{\partial}{\partial t} f(t,\xi) + \frac{\Delta_{\mathrm{S}^{d-1}}(f(t,\xi)|_{\mathrm{S}^{d-1}})}{\sin^2(t)}. \tag{40}$$

and the radial Laplace operator as

$$\Delta_{p,t} f(t,\xi) = \frac{\partial^2}{\partial t} f(t,\xi) + (d-1)\frac{\cos(t)}{\sin(t)}\frac{\partial}{\partial t} f(t,\xi). \tag{41}$$

The uncertainty principle on S^d can now be formulated as follows:

Corollary 5.1. *If $f \in L^2(\mathrm{S}^d) \cap \mathcal{D}(\Delta_{p,t})$ and $\|f\|_{\mathrm{S}^d} = 1$, then the following uncertainty principle holds:*

$$\Big(1 - \varepsilon(f,p)^2\Big) \cdot \langle -\Delta_{p,t} f, f\rangle_{\mathrm{S}^d} \geq \frac{d^2}{4}\, \varepsilon(f,p)^2. \tag{42}$$

The constant $\frac{d^2}{4}$ on the right hand side is optimal.

Proof. If we apply Theorem 3.3 to the unit sphere S^d and use the respective weight function (39), the only thing that remains to validate is the right hand side of inequality (17). This can be done by the following simple calculation:

$$\begin{aligned}\langle\Big(\cos(t) + \frac{\partial_t \Theta(t,\xi)}{\Theta(t,\xi)}\sin(t)\Big) f, f\rangle^2_{\mathrm{S}^d} &= \Big(\varepsilon(f,p) + \langle (d-1)\frac{\cos(t)}{\sin(t)}\sin(t) f, f\rangle_{\mathrm{S}^d}\Big)^2 \\ &= (d\,\varepsilon(f,p))^2.\end{aligned}$$

The optimality of the constant $\frac{d^2}{4}$ is a consequence of Proposition 4.2. □

If we consider as a special case the radial functions on S^d, inequality (42) corresponds exactly with the uncertainty principle (3) proven in [27] for functions having an expansion in terms of the Gegenbauer polynomials $C_n^{(\frac{d-1}{2})}$. This is not a surprising result, since the polynomials $C_n^{(\frac{d-1}{2})}$ constitute a basis for the radial, square integrable functions on S^d and the radial Laplacian (41) corresponds to the second order differential operator of the corresponding Gegenbauer polynomials. The sharpness of inequality (42) is therefore also a consequence of the sharpness of inequality (3).

Other works treating uncertainty principles on the unit sphere attained similar results, but worked with slightly different techniques. In [20], Narcovich and Ward used a vector valued differential operator to split the Laplace-Beltrami operator on S^2. Also Goh and Goodman [11], [12] worked with a vector valued differential operator to prove a similar uncertainty principle on S^d.

If we adopt the general definition (20) of the space variance to the unit sphere, we get

$$\mathrm{var}_{S,p}(f) = \frac{1-\varepsilon(f,p)^2}{\varepsilon(f,p)^2}.$$

In [6], [24] and [25], this definition of space variance was used to determine optimally space localized band-limited polynomials and wavelets on the torus $\mathbb{T}$ and on the unit sphere S^d.

Let us now turn to the real projective space $\mathbb{RP}^d_\pi$. We consider the sphere $\mathrm{S}^d(2)$ with radius 2 and define the antipodal map $A : \mathrm{S}^2(2) \to \mathrm{S}^2(2)$ as $Ax = -x$. The real projective space $\mathbb{RP}^d_\pi$ with diameter π is then defined as the quotient of $\mathrm{S}^d(2)$ under the antipodal map. The identification of $\mathbb{RP}^d_\pi$ with $\mathrm{S}^d(2)/A$ allows the introduction of geodesic spherical coordinates as in the case of the unit sphere. In this way, the volume element on $\mathbb{RP}^d_\pi$ can be deduced from (39) as

$$dV = 2^{d-1}\sin\left(\frac{t}{2}\right)^{d-1} dt d\mu_p(\xi), \tag{43}$$

where the geodesic length t varies between 0 and π. The cut locus C_p on $\mathbb{RP}^d_\pi$ corresponds to the set of points lying on the equator of $\mathrm{S}^d(2)$ with respect to the point p. Due to our special construction, the distance $R(\xi)$ from p to the cut locus C_p is, independently of ξ, equal to π. Due to (43), we have $\Theta(t,\xi) = 2^{d-1}\sin(\frac{t}{2})^{d-1}$, and the radial part of the Laplacian is given as

$$\Delta_{p,t} f(t,\xi) = \frac{\partial^2}{\partial t^2} f(t,\xi) + \frac{(d-1)}{2}\frac{\cos(\frac{t}{2})}{\sin(\frac{t}{2})}\frac{\partial}{\partial t} f(t,\xi).$$

Now, an uncertainty principle for the real projective space can be formulated as follows:

Corollary 5.2. *Let $f \in L^2(\mathbb{RP}^d_\pi) \cap \mathcal{D}(\Delta_{p,t})$ and $\|f\|_{\mathbb{RP}^d_\pi} = 1$, then the following uncertainty principle holds:*

$$\Big(1-\varepsilon(f,p)^2\Big) \cdot \langle -\Delta_{p,t} f, f\rangle_{\mathbb{RP}^d_\pi} \geq \Big(\frac{(d-1)}{4} + \frac{(d+1)}{4}\varepsilon(f,p)\Big)^2. \tag{44}$$

The constants on the right hand side are optimal.

Proof. If we apply Theorem 3.3 to the space $\mathbb{RP}^d_\pi$, we get on the right hand side of inequality (17):

$$\begin{aligned}\langle\Big(\cos(t) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(t)\Big)f, f\rangle^2_{\mathbb{RP}^d_\pi} &= \Big(\varepsilon(f,p) + \langle\frac{d-1}{2}\frac{\cos(\frac{t}{2})}{\sin(\frac{t}{2})}\sin(t) f, f\rangle_{\mathbb{RP}^d_\pi}\Big)^2 \\ &= \Big(\varepsilon(f,p) + \langle\frac{d-1}{2}\big(1+\cos(t)\big)f, f\rangle_{\mathbb{RP}^d_\pi}\Big)^2 \\ &= \Big(\frac{d-1}{2} + \frac{d+1}{2}\varepsilon(f,p)\Big)^2.\end{aligned}$$

The optimality of the constants in (44) follows from Proposition 4.2. □

Through the relation [16, page 41]

$$C_{2n}^{(\frac{d-1}{2})}(\cos(\tfrac{t}{2})) = \frac{(\frac{d-1}{2})_n}{(\frac{1}{2})_n} P_n^{(\frac{d-2}{2},-\frac{1}{2})}(\cos(t))$$

between the Gegenbauer polynomials $C_{2n}^{(\frac{d-1}{2})}$ of even order and the Jacobi polynomials $P_n^{(\frac{d-2}{2},-\frac{1}{2})}$, one can check easily that the radial functions on $\mathbb{RP}_\pi^d$ are exactly the functions that have an expansion in terms of the Jacobi polynomials $P_n^{(\frac{d-2}{2},-\frac{1}{2})}$. So, inequality (44) restricted to radial functions on $\mathbb{RP}_\pi^d$ is precisely the same as the uncertainty principle proven in [17] for the Jacobi polynomials $P_n^{(\frac{d-2}{2},-\frac{1}{2})}$.

6 Uncertainty Principles on Curves

If the manifold M is a one dimensional curve, we can simplify inequality (17) considerably. We consider a C^∞-differentiable Jordan curve $\gamma : (-R, R] \to \mathbb{R}^d$, naturally parameterized such that $|\gamma'(t)| = 1$ for every $t \in (-R, R]$. The geodesic distance on the curve is then given as

$$d(\gamma(t_1), \gamma(t_2)) = \left| \int_{t_1}^{t_2} |\gamma'(t)| dt \right| = |t_1 - t_2|$$

and the length of the whole curve is $2R$. Now, for the formulation of the uncertainty principle, we adopt the notation of the previous chapters. Without loss of generality we can assume that the point p where the uncertainty is referred to corresponds to $\gamma(0)$. Then the cut locus corresponds to the point $\gamma(R)$ and the weight function Θ satisfies $\Theta(t, \xi) = |\gamma'(\xi t)| = 1$ for all $t \in [0, R]$ and $\xi \in \{\pm 1\}$. The integration along the curve γ can be written in the GSC as

$$\int_\gamma f d\gamma = \sum_{\xi \in \{\pm 1\}} \int_0^R f(\gamma(\xi t)) dt$$

and the Laplacian Δ_γ translates to

$$\Delta_\gamma f(\gamma(\xi t)) = \Delta_{p,t} f(\gamma(\xi t)) = \frac{d^2}{dt^2} f(\gamma(\xi t)).$$

If we use the definition (19) for the frequency variance, we can formulate the uncertainty principle (17) on the curve γ as follows.

Corollary 6.1. *If $f \in \mathcal{D}(\frac{d}{dt}) \cap L^2(\gamma)$ with $\|f\|_\gamma = 1$, then the following inequality holds:*

$$\left(1 - \varepsilon(f,p)^2\right) \cdot \left\| \frac{d}{dt} f \right\|_\gamma^2 \geq \frac{1}{4} \frac{\pi^2}{R^2} \varepsilon(f,p)^2, \tag{45}$$

where

$$\varepsilon(f,p) = \int_{-R}^{R} \cos\left(\frac{\pi}{R}t\right)|f(\gamma(t))|^2 dt.$$

The constant $\frac{1}{4}$ on the right hand side of inequality (45)is optimal.

We remark that this result can also be shown in a different way. Since a smooth Jordan curve γ with length $2R$ is isometric to the circle with radius $\frac{R}{\pi}$, the uncertainty for γ can directly be deduced from the Breitenberger uncertainty principle [2]. Further, this connection implies also the optimality of (45) (cf. [23] for the optimality on the torus).

7 Estimates of Uncertainty Principles using Comparison Principles

For a general Riemannian manifold M with dimension $d \geq 2$, the right hand side of the uncertainty product (17) is usually hard to determine. We can simplify this term if some further information on the curvature of the Riemannian manifold is given. In particular, if we assume that the Ricci curvature satisfies

$$\operatorname{Ric}(\xi,\xi) \geq \kappa_1^2(d-1)|\xi|^2$$

for a constant $\kappa_1 > 0$ and all tangent vectors ξ in the tangent bundle TM, then the Bonnet-Myers Theorem [4, Theorem II.6.1] states that the value $R(\xi)$ is bounded from above by $\frac{\pi}{\kappa_1}$. On the other hand, if we assume that all sectional curvatures are less than or equal to a given constant κ_2^2, $\kappa_2 \geq \kappa_1$, then Bishop's comparison Theorem [4, Theorem III.4.1] states that

$$\frac{\partial_t \Theta(t,\xi)}{\Theta(t,\xi)} \geq (d-1)\kappa_2 \frac{\cos(\kappa_2 t)}{\sin(\kappa_2 t)} \tag{46}$$

for all $\xi \in \mathfrak{S}_p$ and $0 < t < \frac{\pi}{\kappa_2}$. Moreover, the Morse-Schönberg Theorem [4, Theorem II.6.3] assures that in this case $R(\xi) \geq \frac{\pi}{\kappa_2}$. Combining (46) and $\kappa_1 \leq \kappa(\xi) \leq \kappa_2$, we get the estimate

$$\begin{aligned}\kappa(\xi)\cos(\kappa(\xi)t) + \frac{\partial_t\Theta(t,\xi)}{\Theta(t,\xi)}\sin(\kappa(\xi)t) &\geq \kappa_1\cos(\kappa_2 t) + (d-1)\kappa_2\frac{\cos(\kappa_2 t)}{\sin(\kappa_2 t)}\sin(\kappa_1 t)\\ &\geq d\kappa_1\cos(\kappa_2 t)\end{aligned}$$

for all $0 < t \leq \frac{\pi}{2\kappa_2}$. So, if we introduce

$$\varepsilon_{\kappa_2}(f,p) = \int_{\mathfrak{S}_p}\int_0^{\frac{\pi}{2\kappa_2}} \cos(\kappa_2 t)|f(t,\xi)|^2\Theta(t,\xi)dtd\mu_p(\xi)$$

as a modified mean value, then the above assumptions assure that

$$\varepsilon(f,p) \geq \varepsilon_{\kappa_2}(f,p)$$

holds for all functions $f \in L^2(M)$ having compact support in $\overline{B(p, \frac{\pi}{2\kappa_2})}$. Adopting Theorem 3.3, we immediately get the following local uncertainty principle:

Theorem 7.1. *Let M be a compact Riemannian manifold ($d \geq 2$) whose Ricci curvature fulfills*

$$\mathrm{Ric}(\xi,\xi) \geq \kappa_1^2(d-1)|\xi|^2$$

for all tangent vectors $\xi \in TM$, and all of whose sectional curvatures are less or equal to a constant κ_2^2, $\kappa_2 \geq \kappa_1 > 0$. If $f \in L^2(M) \cap \mathcal{D}(\Delta_{p,t})$, $\|f\|_M = 1$, has compact support in $\overline{B(p, \frac{\pi}{2\kappa_2})}$, then the following inequality holds:

$$\left(1 - \varepsilon_{\kappa_2}(f,p)^2\right) \cdot \langle -\Delta_{p,t} f, f\rangle_M \geq \kappa_1^2 \frac{d^2}{4} \varepsilon_{\kappa_2}(f,p)^2. \tag{47}$$

In the case that M is a d-dimensional sphere with radius $\frac{1}{\kappa}$, we have $\kappa_1 = \kappa_2 = \kappa$. Inequality (47) then reduces to the well known principle

$$\left(1 - \varepsilon(f,p)^2\right) \cdot \langle -\Delta_{p,t} f, f\rangle_M \geq \kappa^2 \frac{d^2}{4} \varepsilon(f,p)^2.$$

Thus, the point of Theorem 7.1 is that if M is a "sphere-like" manifold where the curvature κ^2 is not varying much, then also the resulting uncertainty principle is very similar to the uncertainty of a d-dimensional sphere with curvature κ^2. In contrast to the uncertainty principle (17), the sharpness of inequality (47) can not be guaranteed in general.

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