Applications of the monotonicity of extremal zeros of orthogonal polynomials in interlacing and optimization problems

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We investigate monotonicity properties of extremal zeros of orthogonal polynomials depending on a parameter. Using a functional analysis method we prove the monotonicity of extreme zeros of associated Jacobi, associated Gegenbauer and q-Meixner-Pollaczek polynomials. We show how these results can be applied to prove interlacing of zeros of orthogonal polynomials with shifted parameters and to determine optimally localized polynomials on the unit ball.

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1 Introduction

Monotonicity properties of zeros of orthogonal polynomials have been extensively studied for several decades from different perspectives. The most well-known tool to prove the monotonicity of all the zeros with respect to a parameter is due to Markov (cf. for example [13], [6]). It only requires the derivative of the weight function and can be easily applied to the classical orthogonal polynomials. Other approaches include the Hellmann-Feynman theorem or using second-order ordinary linear differential equation techniques (for an overview see [9] and [6], Chapter 7).

Unfortunately a lot of non-classical orthogonal polynomials, like the associated or q-orthogonal polynomials often have complicated weight functions and differential equations which makes Markov's theorem and the differential equation methods difficult or impossible to apply. However, even in these cases one can obtain information about the monotonicity of at least the extreme zeros using an idea of Ismail, which is based on the Hellman-Feynman theorem and which uses only the coefficients of the three-term recurrence relation. In [5] the theorem is proved for birth and death process polynomials, here we state it generally for monic orthogonal polynomials.

Theorem 1.1. *Let $P_n(x,\tau)$, $n \geq 0$ be a family of monic orthogonal polynomials on $[a,b]$ ($-\infty \leq a \leq b \leq \infty$) depending on the parameter τ and fulfilling the three-term recursion formula*

$$
\begin{aligned}
(1) \qquad & xP_n(x,\tau) = P_{n+1}(x,\tau) + a_n(\tau)P_n(x,\tau) + b_n(\tau)P_{n-1}(x,\tau), \quad n \geq 1, \\
& P_0(x,\tau) = 1, \quad P_1(x,\tau) = x - a_0(\tau),
\end{aligned}
$$

with $b_n(\tau) > 0$. Assume that the coefficients $a_n(\tau)$ ($n \geq 0$) and $b_n(\tau)$ ($n \geq 1$) are differentiable monotone decreasing (increasing) functions of the parameter τ. Then the largest zero of the polynomial $P_n(x,\tau)$ is also a differentiable monotone decreasing (increasing) function of the parameter τ.
On the other hand, if the coefficients $b_n(\tau)$ are monotone decreasing (increasing) and the coefficients $a_n(\tau)$ are monotone increasing (decreasing) functions, then the smallest zero of $P_n(x,\tau)$ is differentiable monotone increasing (decreasing).

Using the symmetric three-term recurrence relation of the orthogonal polynomials, Theorem 1.1 was applied in a similar form in the articles [7], [10], [11], [12] and in some of the references therein to prove the monotonicity of the extremal zeros for a lot of well-known families of classical and q-orthogonal polynomials. For the sake of completeness, we give a short proof of Theorem 1.1 using a formula from [6].

Proof. Let $\lambda(\tau)$ be the largest zero of $P_n(x,\tau)$. Clearly, all zeros of $P_n(x,\tau)$ are differentiable functions of τ. Using the coefficients of the three-term recurrence (1) in formula (7.3.8) of [6], we get

$$
(2) \qquad \Big(\sum_{k=0}^{n-1} \frac{P_k^2(\lambda,\tau)}{\zeta_k}\Big)\frac{d\lambda(\tau)}{d\tau} = \sum_{k=0}^{n-1} \frac{P_k(\lambda,\tau)}{\zeta_k}\Big(a_k'(\tau)P_k(\lambda,\tau) + b_k'(\tau)P_{k-1}(\lambda,\tau)\Big)
$$

where a_k' and b_k' denote differentiation with respect to τ and $\zeta_k = \prod_{i=1}^k b_i(\tau)$. Since the polynomials $P_n(x,\tau)$ are monic, it is easy to show that $P_k(b,\tau) > 0$. Moreover, since $\lambda(\tau)$ is the largest zero of $P_n(x,\tau)$, we have due to the interlacing property of the polynomials $P_k(x,\tau)$ that $P_k(\lambda,\tau) > 0$ for $k = 0,\ldots,n-1$. Therefore, if $a_k(\tau)$ and $b_k(\tau)$ are decreasing (increasing) functions of the parameter τ, then the right hand side of equation (2) is negative (positive) and the first statement of the Theorem is shown. A similar argument for the smallest zero (keeping in mind that $\operatorname{sign}(P_k(a,\tau)) = (-1)^k$) implies the second statement. □

In the following, we apply this theorem to the associated Jacobi, associated Gegenbauer and q-Meixner-Pollaczek polynomials and show how the results can be used to prove interlacing of the zeros of different orthogonal polynomials with shifted parameter values. As another application, we use the monotonicity results of the associated Jacobi polynomials to determine higher-dimensional polynomials on the unit ball that are in a certain sense best localized at the center of the ball.

2 Monotonicity of the largest zero of the associated Jacobi polynomials

First of all, we use Theorem 1.1 to investigate the behavior of the largest zero of the associated Jacobi polynomials $P_n^{(\alpha,\beta)}(x,c)$, when the parameter α is altered. The following Theorem is an extension of [12], Corollary 10 to a larger parameter area.

Theorem 2.1. *Let, $c \geq 0$, $\alpha \geq 0$ and $\beta \geq -1/2$ and assume that $\beta \leq \max\{\frac{1}{2}, 2\alpha\}$ and $2c+\alpha+\beta > 0$ if $c > 0$. Then, the largest zero $\lambda(\alpha)$ of the associated Jacobi polynomials $P_n^{(\alpha,\beta)}(x,c)$ is a decreasing function of the parameter α.*

Proof. We consider the monic associated Jacobi polynomials $P_n^{(\alpha,\beta)}(x,c)$. The coefficients $a_n(\alpha)$ and $b_n(\alpha)$ of the three-term recurrence formula of the polynomials $P_n^{(\alpha,\beta)}(x,c)$ are given by (see, for instance, [4, p. 29])

$$\begin{aligned} a_n(\alpha) &:= \frac{\beta^2-\alpha^2}{(2n+2c+\alpha+\beta)\,(2n+2c+2+\alpha+\beta)}, \quad n \geq 0, \\ b_n(\alpha) &:= \frac{4(n+c)\,(n+c+\alpha)\,(n+c+\beta)\,(n+c+\alpha+\beta)}{(2n+2c+\alpha+\beta)^2\,(2n+2c+\alpha+\beta+1)\,(2n+2c+\alpha+\beta-1)}, \quad n \geq 1. \end{aligned}$$

To prove the statement of Theorem 2.1, we have to check that the assumptions of Theorem 1.1 hold. In particular, we have to check that the coefficients $a_n(\alpha)$ and $b_n(\alpha)$ are decreasing functions of the parameter α.

First, we consider the derivative $a_n'(\alpha)$. For $n \geq 0$, $c > 0$, or $n > 0$, $c \geq 0$, we have

$$\begin{aligned} a_n'(\alpha) &= \frac{\left(-2\alpha - \frac{\beta^2-\alpha^2}{(2n+2c+\alpha+\beta)} - \frac{\beta^2-\alpha^2}{(2n+2c+2+\alpha+\beta)}\right)}{(2n+2c+\alpha+\beta)\,(2n+2c+2+\alpha+\beta)} \\ &= -2\,\frac{\left(4\alpha\,(n+c)^2 + 2(n+c)(2\alpha+(\alpha+\beta)^2) + (1+\beta)(\alpha+\beta)^2\right)}{(2n+2c+\alpha+\beta)^2\,(2n+2c+2+\alpha+\beta)^2}. \end{aligned}$$

Since $\alpha \geq 0$, $\beta \geq -1/2$ and $2c+\alpha+\beta > 0$ if $c > 0$, the term on the right hand side is always nonpositive. It remains to check the case $n = 0$, $c = 0$. In this case, we have

$$a_0'(\alpha) = -\frac{2(1+\beta)}{(\alpha+\beta+2)^2} < 0.$$

Thus, $a_n(\alpha)$, $n \geq 0$, is a monotone decreasing function of the parameter α if the assumptions of the theorem are satisfied.

Next, we examine the derivative $b_n'(\alpha)$. For $n \geq 1$, we get

$$b_n'(\alpha) = b_n(\alpha)\left(\frac{1}{n+c+\alpha} + \frac{1}{n+c+\alpha+\beta} - \frac{2}{2n+2c+\alpha+\beta}\right.$$
$$\left. - \frac{1}{2n+2c+\alpha+\beta+1} - \frac{1}{2n+2c+\alpha+\beta-1}\right).$$

We consider first the case when $\alpha \geq \frac{1}{2}$ and $-\frac{1}{2} \leq \beta \leq 2\alpha$. We get the upper bound

$$\begin{aligned} b_n'(\alpha) &\leq b_n(\alpha)\left(\frac{1}{n+c+\alpha} + \frac{1}{n+c+\alpha+\beta} - \frac{4}{2n+2c+\alpha+\beta}\right) \\ &= b_n(\alpha)\frac{-2(n+c)\alpha + (\beta+\alpha)(\beta-2\alpha)}{(n+c+\alpha)(n+c+\alpha+\beta)(2n+2c+\alpha+\beta)} \leq 0. \end{aligned}$$

Hence, $b_n'(\alpha)$ is negative if $\alpha \geq \frac{1}{2}$, $-\frac{1}{2} \leq \beta \leq 2\alpha$, $c \geq 0$ and $n \geq 1$. Next, we consider the case $0 \leq \alpha \leq \frac{1}{2}$ and $-\frac{1}{2} \leq \beta \leq \frac{1}{2}$. In this case, we get the estimate

$$\begin{aligned} b_n'(\alpha) = b_n(\alpha) &\left(\frac{2}{(n+c+\alpha+\frac{\beta}{2}) - \frac{\beta^2}{4n+4c+4\alpha+2\beta}}\right. \\ &\left. - \frac{2}{2n+2c+\alpha+\beta} - \frac{2}{2n+2c+\alpha+\beta-\frac{1}{2n+2c+\alpha+\beta}}\right) \\ = b_n(\alpha) &\left(\frac{2}{(n+c+\alpha+\frac{\beta}{2}) - \frac{\beta^2}{4n+4c+4\alpha+2\beta}}\right. \\ &- \frac{2}{(2n+2c+\alpha+\beta-\frac{1}{4n+4c+2\alpha+2\beta}) + \frac{1}{4n+4c+2\alpha+2\beta}} \\ &\left. - \frac{2}{(2n+2c+\alpha+\beta-\frac{1}{4n+4c+2\alpha+2\beta}) - \frac{1}{4n+4c+2\alpha+2\beta}}\right) \\ \leq b_n(\alpha) &\left(\frac{2}{(n+c+\alpha+\frac{\beta}{2}) - \frac{\beta^2}{4n+4c+4\alpha+2\beta}} - \frac{2}{(n+c+\frac{\alpha}{2}+\frac{\beta}{2}) - \frac{1}{8n+8c+4\alpha+4\beta}}\right). \end{aligned}$$

Since $(n+c+\alpha+\frac{\beta}{2}) \geq (n+c+\frac{\alpha}{2}+\frac{\beta}{2})$ and $\frac{\beta^2}{4n+4c+4\alpha+2\beta} \leq \frac{1}{8n+8c+4\alpha+4\beta}$, we get also in this case $b_{n+c}'(\alpha) \leq 0$.

In total, $b_n'(\alpha) \leq 0$ and $b_n(\alpha)$ is a monotone decreasing function of the parameter α if the conditions on α and β are fulfilled. Hence, Theorem 1.1 yields the statement for the associated Jacobi polynomials $P_n^{(\alpha,\beta)}(x,c)$. □

Remark 2.2. Let $c \geq 0$, $\alpha \geq -1/2$, $\beta \geq 0$, $\alpha \leq \max\{\frac{1}{2}, 2\beta\}$ and $2c+\alpha+\beta > 0$ if $c > 0$. Then, since $P_n^{(\alpha,\beta)}(-x,c) = (-1)^n P_n^{(\beta,\alpha)}(x,c)$, Theorem 2.1 implies that the smallest zero of $P_n^{(\alpha,\beta)}(x,c)$ is an increasing function of the parameter β.

Remark 2.3. Similar to Theorem 2.1, it can be shown that the largest zero $\lambda(\nu)$ of the associated Gegenbauer polynomials $C_n^{(\nu)}(x,c)$, $c \geq 0$, $\nu \geq \frac{1}{2}$, is a decreasing function of the parameter ν. This was first proven in [11].

The monotonicity of the extreme zeros also holds when two parameters are shifted simultaneously.

Theorem 2.4. *For $\nu \geq 1/2$ and $0 \leq \tau \leq c$, the largest zero $\lambda(\tau)$ of the associated Gegenbauer polynomials $C_n^{(\nu+\tau)}(x, c-\tau)$ is a decreasing function of the parameter τ, while the smallest zero is an increasing function of τ.*

Similarly, for $\alpha \geq 0$, $\beta > -1$, $0 \leq \sigma \leq c$ and $2c+\alpha+\beta > 0$, the largest zero $\lambda(\sigma)$ of the associated Jacobi polynomials $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$ is a decreasing function of the parameter σ.

Proof. We consider first the associated Gegenbauer polynomials $C_n^{(\nu+\tau)}(x, c-\tau)$. The coefficients $a_n(\tau)$ and $b_n(\tau)$ of the three-term recursion are given by

$$\begin{aligned} a_n(\tau) &:= 0, \quad n \geq 0, \\ b_n(\tau) &:= \frac{(n+c-\tau)(n+c+2\nu+\tau-1)}{4\,(n+c+\nu)\,(n+c+\nu-1)}, \quad n \geq 1. \end{aligned}$$

For the derivatives, we get

$$a_n'(\tau) = 0, \qquad b_n'(\tau) = b_n(\tau)\left(\frac{1}{n+c+2\nu-1+\tau} - \frac{1}{n+c-\tau}\right) \leq 0 \qquad \text{if} \quad \nu \geq \frac{1}{2}.$$

Thus, due to Theorem 1.1, the largest zero of $C_n^{(\nu+\tau)}(x, c-\tau)$ is a decreasing function and the smallest zero is an increasing function of the parameter τ.

Next, we consider the associated Jacobi polynomials $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$. The coefficients of the three-term recursion formula are given by

$$\begin{aligned} a_n(\sigma) &:= \frac{\beta^2 - (\alpha+2\sigma)^2}{(2n+2c+\alpha+\beta)\,(2n+2c+2+\alpha+\beta)}, \quad n \geq 0, \\ b_n(\sigma) &:= \frac{4(n+c-\sigma)\,(n+c+\alpha+\sigma)\,(n+c+\beta-\sigma)\,(n+c+\alpha+\beta+\sigma)}{(2n+2c+\alpha+\beta)^2\,(2n+2c+\alpha+\beta+1)\,(2n+2c+\alpha+\beta-1)}, \quad n \geq 1. \end{aligned}$$

So, for the derivatives, we get

$$\begin{aligned} a_n'(\sigma) &= \frac{-4(\alpha+2\sigma)}{(2n+2c+\alpha+\beta)\,(2n+2c+2+\alpha+\beta)}, \\ b_n'(\sigma) &= b_n(\sigma)\cdot\left(-\frac{1}{n+c-\sigma} + \frac{1}{n+c+\alpha+\sigma} - \frac{1}{n+c+\beta-\sigma} + \frac{1}{n+c+\alpha+\beta+\sigma}\right) \\ &= -b_n(\sigma)\left(\frac{\alpha+2\sigma}{(n+c-\sigma)(n+c+\alpha+\sigma)} + \frac{\alpha+2\sigma}{(n+c+\alpha+\beta+\sigma)(n+c+\beta-\sigma)}\right). \end{aligned}$$

Both $a_n'(\sigma)$ and $b_n'(\sigma)$ are nonpositive, if $\alpha \geq 0$, $\beta > -1$, $0 \leq \sigma \leq c$ and $2c + \alpha + \beta > 0$. Thus, the largest zero of the polynomial $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$ is a decreasing function of the parameter σ. □

If $\beta \leq \frac{1}{2}$ and $c \geq \frac{1}{2}$, we can have a stronger statement about the monotonicity of the zeros of the associated Jacobi polynomials $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$. Using the Hellmann-Feynman approach, we can show that all the zeros are decreasing with respect to σ. For this end, we need the following auxiliary result concerning a particular chain sequence. For the definition of a chain sequence see [1], Definition 5.1.

Lemma 2.5. *The sequence $(a_k)_{k=1}^\infty$ defined by $a_k = \frac{1}{4} + \frac{1}{16}\frac{1}{(k-\frac{1}{2})(k+\frac{1}{2})}$ is a chain sequence.*

Proof. We use as parameter sequence $(h_k)_{k=0}^\infty$ with coefficients $h_k = \frac{1}{2}\Big(1 - \frac{1}{2k+1}\Big)$. Then, $h_0 = 0$, $0 < h_k \leq \frac{1}{2}$ for $k \geq 1$ and we get

$$a_k = (1 - h_{k-1})h_k = \frac{k^2}{(2k-1)(2k+1)} = \frac{1}{4} + \frac{1}{16}\frac{1}{(k-\frac{1}{2})(k+\frac{1}{2})}.$$

□

Theorem 2.6. *For $\alpha \geq 0$, $-1/2 \leq \beta \leq 1/2$, $2c \geq 1$, $0 \leq \sigma \leq c$ and $\alpha + \sigma > 0$ consider the family $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$ of associated Jacobi polynomials depending on the parameter σ. Then, all the zeros of $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$ are decreasing functions of the parameter σ.*

Proof. We use the same notations as in Theorem 2.4. Moreover, we introduce the matrix

$$\mathbf{J}_n(\sigma) = \begin{pmatrix} a_0(\sigma) & \sqrt{b_1(\sigma)} & 0 & 0 & \cdots & 0 \\ \sqrt{b_1(\sigma)} & a_1(\sigma) & \sqrt{b_2(\sigma)} & 0 & \cdots & 0 \\ 0 & \sqrt{b_2(\sigma)} & a_2(\sigma) & \sqrt{b_3(\sigma)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{b_{n-2}(\sigma)} & a_{n-1}(\sigma) & \sqrt{b_{n-1}(\sigma)} \\ 0 & \cdots & \cdots & 0 & \sqrt{b_{n-1}(\sigma)} & a_n(\sigma) \end{pmatrix}.$$

According to [9], all the zeros of the polynomial $P_n^{(\alpha+2\sigma,\beta)}(x, c-\sigma)$ are decreasing, if the derivative $\frac{d}{d\sigma}\mathbf{J}_n(\sigma)$ is a negative definite matrix. According to Theorem 6 of [9], the matrix $\frac{d}{d\sigma}\mathbf{J}_n(\sigma)$ is negative definite if and only if the diagonal coefficients $a_n'(\sigma)$ are negative and

$$\left(\frac{\left(\frac{d}{d\sigma}\sqrt{b_1(\sigma)}\right)^2}{a_1'(\sigma)a_0'(\sigma)}, \frac{\left(\frac{d}{d\sigma}\sqrt{b_2(\sigma)}\right)^2}{a_2'(\sigma)a_1'(\sigma)}, \quad \ldots \quad, \frac{\left(\frac{d}{d\sigma}\sqrt{b_n(\sigma)}\right)^2}{a_n'(\sigma)a_{n-1}'(\sigma)}, \quad \ldots \right)$$

defines a chain sequence. We have already seen in Theorem 2.4 that $a_n'(\sigma) < 0$. For the coefficients in the sequence, we get

$$\begin{aligned}
\frac{\left(\frac{d}{d\sigma}\sqrt{b_n(\sigma)}\right)^2}{a_n'(\sigma)a_{n-1}'(\sigma)} &= \frac{(2n+2c+\alpha+\beta-2)\,(2n+2c+\alpha+\beta+2)}{16\,(2n+2c+\alpha+\beta+1)\,(2n+2c+\alpha+\beta-1)} \\
&\times\left(\frac{(n+c-\sigma)(n+c+\alpha+\sigma)}{(n+c+\alpha+\beta+\sigma)(n+c+\beta-\sigma)}+2+\frac{(n+c+\alpha+\beta+\sigma)(n+c+\beta-\sigma)}{(n+c-\sigma)(n+c+\alpha+\sigma)}\right) \\
&\leq \frac{1}{16}\left(\Big(1-\frac{\beta}{n+c+\beta-\sigma}\Big)\Big(1-\frac{\beta}{(n+c+\alpha+\beta+\sigma)}\Big)+2\right. \\
&\qquad\left.+\Big(1+\frac{\beta}{n+c-\sigma}\Big)\Big(1+\frac{\beta}{(n+c+\alpha+\sigma)}\Big)\right) \\
&= \frac{1}{4}+\frac{\beta^2(2n+2c+\alpha+\beta)^2}{16(n+c-\sigma)(n+c+\alpha+\sigma)(n+c+\beta-\sigma)(n+c+\alpha+\beta+\sigma)}.
\end{aligned}$$

Since $0 \leq c-\sigma \leq 2c+\alpha$, it is easy to see that

$$\begin{aligned}
&(n+c-\sigma)(n+c+\alpha+\sigma) \geq n(n+2c+\alpha), \\
&(n+c+\beta-\sigma)(n+c+\alpha+\beta+\sigma) \geq (n+\beta)(n+2c+\alpha+\beta).
\end{aligned}$$

Thus, we get

$$\frac{\left(\frac{d}{d\sigma}\sqrt{b_n(\sigma)}\right)^2}{a_n'(\sigma)a_{n-1}'(\sigma)} \leq \frac{1}{4}+\frac{\beta^2(2n+2c+\alpha+\beta)^2}{16n(n+2c+\alpha)(n+\beta)(n+2c+\alpha+\beta)}.$$

Now, for $-1/2 \leq \beta \leq 0$ and since $2c+\alpha \geq 1$, one can check that

$$(2n+2c+\alpha+\beta)^2(n+1/2) \leq 4n(n+2c+\alpha)(n+2c+\alpha+\beta), \quad n \geq 1. \tag{3}$$

Indeed, setting $y := 2c+\alpha$, (3) is equivalent to

$$2(y-1)n^2 + y(2n^2+3yn+2\beta n-2n-\frac{1}{2}y-\beta) - \beta(\beta n+2n+\frac{1}{2}\beta^2) \geq 0,$$

and here all three terms are nonnegative, since $y \geq 0$ and $-1/2 \leq \beta \leq 0$.

On the other hand, if $0 \leq \beta \leq 1/2$, one can also see that

$$(2n+2c+\alpha+\beta)^2 \leq 4(n+2c+\alpha)(n+2c+\alpha+\beta).$$

In total, we get for $-1/2 \leq \beta \leq 1/2$ the estimate

$$\frac{\left(\frac{d}{d\sigma}\sqrt{b_n(\sigma)}\right)^2}{a_n'(\sigma)a_{n-1}'(\sigma)} \leq \frac{1}{4}+\frac{1}{16(n-\frac{1}{2})(n+\frac{1}{2})}.$$

Since the coefficients $\frac{\left(\frac{d}{d\sigma}\sqrt{b_n(\sigma)}\right)^2}{a'_n(\sigma)a'_{n-1}(\sigma)}$ are dominated by the coefficients of the chain sequence of Lemma 2.5, they also form a chain sequence (see [1], Theorem 5.7). Therefore the matrix $\frac{d}{d\sigma}\mathbf{J}_n(\sigma)$ is negative definite. □

Corollary 2.7. *For $-1/2 \leq \beta \leq 1/2$, $\alpha \geq 0$, the zeros of $P_n^{(\alpha+2\sigma,\beta)}(x, 1-\sigma)$ are interlacing with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$ for all $0 \leq \sigma \leq 1$.*

Proof. By [2], we know that the zeros of $P_n^{(\alpha+2,\beta)}(x)$ are interlacing with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$. Moreover, it is a classical result (see [1], p. 86, Theorem 4.1) that the zeros of the associated polynomial $P_n^{(\alpha,\beta)}(x, 1)$ are also interlacing with the zeros of $P_{n+1}^{(\alpha,\beta)}(x)$. Due to the monotonicity proven in Theorem 2.6 the zeros of $P_n^{(\alpha+2\sigma,\beta)}(x, 1-\sigma)$ are lying in between the zeros of $P_n^{(\alpha+2,\beta)}(x)$ and $P_n^{(\alpha,\beta)}(x, 1)$ for $0 \leq \sigma \leq 1$. (The particular case $\alpha = \sigma = 0$ not covered by Theorem 2.6 can be easily verified separately.) Therefore, the Corollary is proven. □

3 Extreme zeros and interlacing of the q-Meixner-Pollaczek polynomials

Theorem 1.1 can be used to establish interlacing results for other classes of orthogonal and q-orthogonal polynomials as well. In particular, it can help in determining the exact order of zeros in an interlacing pattern, which is especially useful in cases where the weight function is complicated, hence Markov's theorem on monotonicity is difficult or impossible to apply. Here we present the q-Meixner-Pollaczek polynomials as an example.

The q-Meixner-Pollaczek polynomials are defined by

$$P_n(x; a|q) = \frac{(a^2; q)_n}{a^n e^{in\theta}(q; q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n},\ ae^{i(\theta+2\phi)},\ ae^{-i\theta} \\ a^2,\ 0 \end{matrix} \;\middle|\; q; q \right)$$

where $x = \cos(\theta + \phi)$, and they are orthogonal on $[-1,\ 1]$ when $0 < a < 1$.

The three-term recurrence relation for the monic polynomials $p_n(x) := p_n(x; a|q)$ is

$$xp_n(x) = p_{n+1}(x) + aq^n \cos\phi p_n(x) + \frac{1}{4}(1 - q^n)(1 - a^2 q^{n-1})p_{n-1}(x).$$

Set $a = q^\alpha$ and $p_n^\alpha(x) := p_n(x; q^\alpha|q)$.

Lemma 3.1. *Let $0 < q < 1$ and $\alpha > 0$. If $\cos\phi > 0$ then the smallest zero of the q-Meixner-Pollaczek polynomials is a decreasing function of α. Similarly, if $\cos\phi < 0$, then the largest zero is an increasing function of α.*

Proof. The coefficients are

$$a_n(\alpha) = q^{n+\alpha}\cos\phi, \quad n \geq 0$$
$$b_n(\alpha) = \frac{1}{4}(1-q^n)(1-q^{n+2\alpha-1}), \quad n \geq 1.$$

While $b_n(\alpha)$ is always an increasing function of α, $a_n(\alpha)$ is decreasing if $\cos\phi > 0$ and increasing if $\cos\phi < 0$. Thus, the statement of the Lemma follows from Theorem 1.1. □

Now set $P_n^\alpha = P_n(x; q^\alpha|q)$.

Corollary 3.2. *Let* $0 < q < 1$ *and* $\alpha > 0$. *If* $|\cos\phi| > q^\alpha$ *then the zeros of* P_n^α, $P_n^{\alpha+1}$ *and* $P_{n-1}^{\alpha+1}$ *interlace. More precisely, let*

$$\begin{aligned} 0 < x_1 < x_2 < \ldots < x_n \quad &\textit{be the zeros of} \quad P_n^{\alpha+1} \\ 0 < y_1 < y_2 < \cdots < y_{n-1} \quad &\textit{be the zeros of} \quad P_{n-1}^{\alpha+1} \quad \textit{and} \\ 0 < t_1 < t_2 < \ldots < t_n \quad &\textit{be the zeros of} \quad P_n^\alpha. \end{aligned}$$

Then

$$x_1 < t_1 < y_1 < x_2 < t_2 < \ldots < x_{n-1} < t_{n-1} < y_{n-1} < x_n < t_n \tag{4}$$

holds if $\cos\phi > 0$ *and*

$$t_1 < x_1 < y_1 < t_2 < x_2 < \ldots < t_{n-1} < x_{n-1} < y_{n-1} < t_n < x_n. \tag{5}$$

holds if $\cos\phi < 0$.

Proof. The (triple) interlacing property was proved in [8], Theorem 3.1. From this it is easy to see that either (4) or (5) holds. Therefore the corresponding order of the zeros follows from Lemma 3.1. □

Remark 3.3. The standard way of proving the monotonicity of (all) the zeros is applying Markov's Monotonicity Theorem (see e.g. [13], Theorem 6.12.2 and [6], Theorem 7.1.1). This would require determining the monotonicity of the quotient $\frac{w(x,\alpha)}{w(x,\alpha+1)}$ or that of the logarithmic derivative $\frac{\partial\{\ln w(x,\alpha)\}}{\partial\alpha}$ with respect to α, where $w(x,\alpha)$ is the weight function of the q-Meixner-Pollaczek polynomials:

$$w(x,\alpha) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,q^\alpha e^{i\phi})h(x,q^\alpha e^{-i\phi})},$$

where

$$h(x,t) = \prod_{k=0}^{\infty}(1 - 2txq^k + t^2q^{2k}).$$

However, with the method used above, the monotonicity of the zeros follows easily based on the information on the largest (or smallest) zeros.

Remark 3.4. The above method can be used for other classes of q-orthogonal polynomials as well, e.g. the Al-Salam-Chihara, q-Laguerre or Al-Salam-Carlitz II polynomials. Corresponding results for the monotonicity of the extremal zeros can be found in [10] and [12].

4 Optimally localized polynomials on the unit ball B^d

In this section, we adopt Theorem 2.1 and 2.4 to solve an optimization problem for orthogonal polynomials on the unit ball $\mathrm{B}^d = \{x \in \mathbb{R}^d : \ |x| \leq 1\}$. First, we need some preliminaries concerning orthonormal Jacobi polynomials and the corresponding Jacobi matrices.

To define orthonormal sets of polynomials on B^d, one needs the orthonormal Jacobi polynomials $p_l^{(\alpha,\beta)}(x)$, $x \in [-1,1]$, defined by the symmetric three-term recurrence relation (see [4], Table 1.1)

$$
\begin{aligned}
&\sqrt{b_{l+1}} p_{l+1}^{(\alpha,\beta)}(x) = (x - a_l) p_l^{(\alpha,\beta)}(x) - \sqrt{b_l} p_{l-1}^{(\alpha,\beta)}(x), \quad l = 0,1,2,3,\ldots \\
&p_{-1}^{(\alpha,\beta)}(x) = 0, \qquad p_0^{(\alpha,\beta)}(x) = \frac{1}{\sqrt{b_0}},
\end{aligned}
\tag{6}
$$

with the coefficients

$$
a_l = \frac{\beta^2 - \alpha^2}{(2l+\alpha+\beta)(2l+\alpha+\beta+2)}, \quad l = 0,1,2,\ldots \tag{7}
$$

$$
\begin{aligned}
b_l &= \frac{4l(l+\alpha)(l+\beta)(l+\alpha+\beta)}{(2l+\alpha+\beta)^2(2l+\alpha+\beta+1)(2l+\alpha+\beta-1)}, \quad l = 1,2,3,\ldots \\
b_0 &= 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.
\end{aligned}
\tag{8}
$$

Further, for $c \geq 0$ (if $c > 0$, assume that $\alpha+\beta \neq -2c$), the associated Jacobi polynomials $p_l^{(\alpha,\beta)}(x,c)$ are defined on the interval $[-1,1]$ by the shifted recurrence relation

$$
\begin{aligned}
&\sqrt{b_{c+l+1}}\, p_{l+1}^{(\alpha,\beta)}(x,c) = (x - a_{c+l})\, p_l^{(\alpha,\beta)}(x,c) - \sqrt{b_{c+l}}\, p_{l-1}^{(\alpha,\beta)}(x,c), \quad l = 0,1,2,\ldots, \\
&p_{-1}^{(\alpha,\beta)}(x,c) = 0, \qquad p_0^{(\alpha,\beta)}(x,c) = 1.
\end{aligned}
\tag{9}
$$

For $m \in \mathbb{N}$, the associated polynomials $p_l^{(\alpha,\beta)}(x,m)$ can be described with help of the symmetric Jacobi matrix $\mathbf{J}(\alpha,\beta)_n^m$, $0 \leq m \leq n$, defined by

$$
\mathbf{J}(\alpha,\beta)_n^m = \begin{pmatrix}
a_m & \sqrt{b_{m+1}} & 0 & 0 & \cdots & 0 \\
\sqrt{b_{m+1}} & a_{m+1} & \sqrt{b_{m+2}} & 0 & \cdots & 0 \\
0 & \sqrt{b_{m+2}} & a_{m+2} & \sqrt{b_{m+3}} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \sqrt{b_{n-1}} & a_{n-1} & \sqrt{b_n} \\
0 & \cdots & \cdots & 0 & \sqrt{b_n} & a_n
\end{pmatrix}. \tag{10}
$$

If $m = 0$, we write $\mathbf{J}_n(\alpha,\beta)$ instead of $\mathbf{J}_n^0(\alpha,\beta)$. Then, in view of the three-term recurrence formulas (6) and (9), the polynomials $p_l^{(\alpha,\beta)}(x)$ and $p_l^{(\alpha,\beta)}(x,m)$, $l \geq 1$, can be written as (this can be shown analogously as in [6], Theorem 2.2.4 for the monic polynomials)

$$p_l^{(\alpha,\beta)}(x) = \left(\prod_{k=0}^{l} b_k\right)^{-\frac{1}{2}} \det\left(x\mathbf{1}_l - \mathbf{J}(\alpha,\beta)_{l-1}\right), \tag{11}$$

$$p_l^{(\alpha,\beta)}(x,m) = \left(\prod_{k=1}^{l} b_k\right)^{-\frac{1}{2}} \det\left(x\mathbf{1}_l - \mathbf{J}(\alpha,\beta)_{m+l-1}^m\right), \tag{12}$$

where $\mathbf{1}_l$ denotes the l-dimensional identity matrix. Moreover, the zeros of the polynomials $p_l^{(\alpha,\beta)}(x)$ and $p_l^{(\alpha,\beta)}(x,m)$ correspond exactly with the eigenvalues of the matrices $\mathbf{J}(\alpha,\beta)_{l-1}$ and $\mathbf{J}(\alpha,\beta)_{l+m-1}^m$, respectively.

Now, turning back to the unit ball B^d, we introduce on the interval $[0,1]$ the weight function $w_\beta(r)$ by

$$w_\beta(r) := (1-r^2)^\beta, \quad \beta \geq -\frac{1}{2},$$

and denote by $L^2(\mathrm{B}^d, w_\beta)$ the space of functions with finite weighted L^2-norm $\|f\|_\beta := \left(\int_{\mathrm{B}^d} |f(x)|^2 w_\beta(|x|)dx\right)^{\frac{1}{2}}$. The space $L^2(\mathrm{B}^d, w_\beta)$ endowed with the inner product

$$\langle f,g\rangle_\beta := \int_{\mathrm{B}^d} f(x)\overline{g(x)}w_\beta(|x|)dx \tag{13}$$

is a Hilbert space.

By $\Pi_l(\mathrm{B}^d)$ we denote the subspace of functions on B^d which are polynomials of degree less or equal to l in x, and $\mathcal{H}_l(\mathrm{B}^d)$ the orthogonal complement of $\Pi_{l-1}(\mathrm{B}^d)$ in $\Pi_l(\mathrm{B}^d)$ with respect to the inner product $\langle\cdot,\cdot\rangle_\beta$. Then, it is well known (see [3], Proposition 2.3.1) that the polynomials

$$P_{l,k,j}^{\mathrm{B}^d}(x) = 2^{\frac{d+2k+2\beta+2}{4}} |x|^k p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(1-2|x|^2) Y_{k,j}\Big(\frac{x}{|x|}\Big), \qquad \begin{array}{l} 0 \leq k \leq l, \quad l-k \text{ even}, \\ 0 \leq j \leq N(d-1,k), \end{array}$$

form an orthonormal basis of the space $\mathcal{H}_l(\mathrm{B}^d)$, where the functions $Y_{k,j}$ denote the $(d-1)$-dimensional orthonormal spherical harmonics of order k and the value $N(d-1,k)$ denotes the dimension of the space spanned by the spherical harmonics $Y_{k,j}$ of order k.

The aim of this section is to find those polynomials $P \in \Pi_l(\mathrm{B}^d)$ which localize the center of the unit ball in an optimal way. For a function $f \in L^2(\mathrm{B}^d, w_\beta)$, we define the position variance with respect to the center 0 of B^d as

$$\mathrm{var}_S^{\mathrm{B}^d}(f) = \int_{\mathrm{B}^d} |x|^2 |f(x)|^2 w_\beta(|x|)dx. \tag{14}$$

Further, we introduce the polynomial spaces

$$\Pi_n^m(\mathrm{B}^d) := \bigoplus_{l=m}^{n} \mathcal{H}_l(\mathrm{B}^d), \quad \text{for even } m,n \in \mathbb{N},\ m \leq n,$$
$$\Pi_n(\mathrm{B}^d) := \Pi_n^0(\mathrm{B}^d), \quad \text{for even } n \in \mathbb{N},$$

with the unit spheres

$$\mathbb{S}_n^m := \left\{ P \in \Pi_n^m(\mathrm{B}^d) :\ \|P\|_\beta = 1 \right\}.$$

Now, the optimization problem we want to solve is the following:

$$\mathcal{P}_n^m := \arg\min_{P \in \mathbb{S}_n^m} \mathrm{var}_S^{\mathrm{B}^d}(P). \tag{15}$$

To find the optimal polynomials $\mathcal{P}_n^m$, we need first of all an auxiliary result concerning the position variance $\mathrm{var}_S^{\mathrm{B}^d}(P)$ of a polynomial $P \in \mathbb{S}_n^m$. From now on we assume that m and n are even.

Lemma 4.1. *Let $P \in \mathbb{S}_n^m$ be a polynomial given by*

$$P(x) = \sum_{l=m}^{n} \sum_{\substack{k=0 \\ l-k \ \text{even}}}^{l} \sum_{j=1}^{N(d-1,k)} c_{l,k,j} P_{l,k,j}^{\mathrm{B}^d}(x). \tag{16}$$

Then, the position variance $\mathrm{var}_S^{\mathrm{B}^d}(P)$ can be written as

$$\begin{aligned}\mathrm{var}_S^{\mathrm{B}^d}(P) = \frac{1}{2} - \frac{1}{2} \Bigg(&\sum_{k=0}^{m} \sum_{j=1}^{N(d-1,k)} \mathbf{c}_{k,j}^H \left[\mathbf{J}\left(\tfrac{d-2}{2}+k, \beta\right)_{\frac{n}{2}-\lceil \frac{k}{2} \rceil}^{\frac{m}{2}-\lfloor \frac{k}{2} \rfloor} \right] \mathbf{c}_{k,j} \\ &+ \sum_{k=m+1}^{n} \sum_{j=1}^{N(d-1,k)} \mathbf{c}_{k,j}^H \left[\mathbf{J}\left(\tfrac{d-2}{2}+k, \beta\right)_{\frac{n}{2}-\lceil \frac{k}{2} \rceil} \right] \mathbf{c}_{k,j} \Bigg)\end{aligned} \tag{17}$$

with the coefficient vectors

$$\begin{aligned}
\mathbf{c}_{k,j} &= (c_{m,k,j}, c_{m+2,k,j}, \ldots, c_{n,k,j})^T, \quad 0 \leq k \leq \tfrac{m}{2}, \quad k \ \textit{even}, \\
\mathbf{c}_{k,j} &= (c_{m+1,k,j}, c_{m+3,k,j}, \ldots, c_{n-1,k,j})^T, \quad 0 \leq k \leq \tfrac{m}{2}, \quad k \ \textit{odd}, \\
\mathbf{c}_{k,j} &= (c_{2k,k,j}, c_{2k+2,k,j}, \ldots, c_{n,k,j})^T, \quad \tfrac{m}{2}+1 \leq k \leq \tfrac{n}{2}, \quad k \ \textit{even}, \\
\mathbf{c}_{k,j} &= (c_{2k+1,k,j}, c_{2k+3,k,j}, \ldots, c_{n-1,k,j})^T, \quad \tfrac{m}{2}+1 \leq k \leq \tfrac{n}{2}, \quad k \ \textit{odd},
\end{aligned}$$

and the Jacobi matrices $\mathbf{J}\left(\frac{d-2}{2}+k, \beta\right)_{\frac{n}{2}-\lceil \frac{k}{2} \rceil}^{\frac{m}{2}-\lfloor \frac{k}{2} \rfloor}$ and $\mathbf{J}\left(\frac{d-2}{2}+k, \beta\right)_{\frac{n}{2}-\lceil \frac{k}{2} \rceil}$ associated to the Jacobi polynomials $p_l^{(\frac{d-2}{2}+k,\beta)}(x, \frac{m}{2} - \lfloor \frac{k}{2} \rfloor)$ and $p_l^{(\frac{d-2}{2}+k,\beta)}(x)$.

Proof. Consider the polynomial $P \in \mathbb{S}_n^m$ given by (16). Using the definition (14) of the variance $\mathrm{var}_S^{\mathrm{B}^d}$ and changing to spherical coordinates $(r,\xi) = (|x|, \frac{x}{|x|})$, we get the formula

$$\mathrm{var}_S^{\mathrm{B}^d}(P) = \int_{\mathrm{B}^d} |x|^2 |P(x)|^2 w_\beta(|x|)dx = \int_{\mathbb{S}^{d-1}} \int_0^1 r^2 |P(r\xi)|^2 r^{d-1} w_\beta(r) dr d\mu(\xi),$$

where μ denotes the standard Riemannian measure on the unit sphere and r^{d-1} corresponds to the Jacobian determinant of the transform in spherical coordinates. Since P is an element of $\mathbb{S}_n^m$ and therefore normalized with respect to the norm $\|\cdot\|_\beta$, the above expression can be rewritten as

$$\mathrm{var}_S^{\mathrm{B}^d}(P) = \frac{1}{2} - \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_0^1 (1-2r^2) |P(r\xi)|^2 r^{d-1} w_\beta(r) dr d\mu(\xi).$$

Now, with the coordinate change $t = 1-2r^2$, $t \in [-1,1]$, we get

$$\mathrm{var}_S^{\mathrm{B}^d}(P) = \frac{1}{2} - \frac{1}{8} \int_{\mathbb{S}^{d-1}} \int_{-1}^1 t |P((\tfrac{1-t}{2})^{\frac{1}{2}}\xi)|^2 \left(\tfrac{1-t}{2}\right)^{\frac{d-2}{2}} \left(\tfrac{1+t}{2}\right)^{\beta} dt d\mu(\xi).$$

Next, we insert the explicit representation of the polynomials P with

$$P_{l,k,j}^{\mathrm{B}^d}((\tfrac{1-t}{2})^{\frac{1}{2}}\xi) = 2^{\frac{d+2k+2\beta+2}{4}} (\tfrac{1-t}{2})^{\frac{k}{2}} p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t) Y_{k,j}(\xi).$$

Then, using the orthonormality of the spherical harmonics $Y_{k,l}$ and rearranging the order of summation, we get

$$\begin{aligned}
\mathrm{var}_S^{\mathrm{B}^d}(P) = \frac{1}{2} - \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{-1}^1 & \left(\sum_{l=m}^{n} \sum_{\substack{k=0 \\ l-k \text{ even}}}^{l} \sum_{j=1}^{N(d-1,k)} c_{l,k,j} t (1-t)^{\frac{k}{2}} p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t) Y_{k,j}(\xi) \right) \\
& \times \overline{\left(\sum_{l=m}^{n} \sum_{\substack{k=0 \\ l-k \text{ even}}}^{l} \sum_{j=1}^{N(d-1,k)} c_{l,k,j} (1-t)^{\frac{k}{2}} p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t) Y_{k,j}(\xi) \right)} \\
& \times (1-t)^{\frac{d-2}{2}} (1+t)^{\beta} dt d\mu(\xi) \\
= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{n} \sum_{j=1}^{N(d-1,k)} \int_{-1}^1 & \left(\sum_{\substack{l=\max\{m,k\} \\ l-k \text{ even}}}^{n} c_{l,k,j}\, t\, p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t) \right) \\
& \times \overline{\left(\sum_{\substack{l=\max\{m,k\} \\ l-k \text{ even}}}^{n} c_{l,k,j}\, p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t) \right)} (1-t)^{\frac{d-2}{2}+k} (1+t)^{\beta} dt.
\end{aligned}$$

Finally, we use the three-term recurrence relation (6) and the orthonormality of the Jacobi polynomials $p_l^{(\frac{d-2}{2}+k,\beta)}$ to conclude

$$\begin{aligned}
\operatorname{var}_S^{\mathrm{B}^d}(P) &= \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{n}\sum_{j=1}^{N(d-1,k)} \int_{-1}^{1} \Big(\sum_{\substack{l=\max\{m,k\} \\ l-k \text{ even}}}^{n} c_{l,k,j} \Big(\sqrt{b_{\frac{l-k}{2}}}\, p_{\frac{l-k}{2}-1}^{(\frac{d-2}{2}+k,\beta)}(t) \\
&\quad + a_{\frac{l-k}{2}}\, p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t) \quad + \sqrt{b_{\frac{l-k}{2}+1}}\, p_{\frac{l-k}{2}+1}^{(\frac{d-2}{2}+k,\beta)}(t)\Big)\Big) \\
&\quad \times \Big(\sum_{\substack{l=\max\{m,k\} \\ l-k \text{ even}}}^{n} c_{l,k,j}\, p_{\frac{l-k}{2}}^{(\frac{d-2}{2}+k,\beta)}(t)\Big)(1-t)^{\frac{d-2}{2}+k}(1+t)^{\beta} dt \\
&= \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{m}\sum_{j=1}^{N(d-1,k)} \mathbf{c}_{k,j}^H \Big[\mathbf{J}\big(\tfrac{d-2}{2}+k,\beta\big)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}^{\frac{m}{2}-\lfloor\frac{k}{2}\rfloor}\Big] \mathbf{c}_{k,j} \\
&\qquad - \frac{1}{2}\sum_{k=m+1}^{n}\sum_{j=1}^{N(d-1,k)} \mathbf{c}_{k,j}^H \Big[\mathbf{J}\big(\tfrac{d-2}{2}+k,\beta\big)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}\Big] \mathbf{c}_{k,j}.
\end{aligned}$$

□

The optimization problem (15) can now be solved explicitly as follows.

Theorem 4.2. *For $-\frac{1}{2} \le \beta \le d-2$ and $P \in \mathbb{S}_n^m(\mathrm{B}^d)$, the minimum of $\operatorname{var}_S^{\mathrm{B}^d}(P)$ is attained for the polynomial*

$$\mathcal{P}_n^m(x) = \kappa \sum_{l=\frac{m}{2}}^{\frac{n}{2}} p_{l-\frac{m}{2}}^{(\frac{d-2}{2},\beta)}(\lambda_{\max}, \tfrac{m}{2})\, p_l^{(\frac{d-2}{2},\beta)}(1-2|x|^2),$$

where the value $\lambda_{\max}$ denotes the largest zero of the polynomial $p_{\frac{n-m}{2}+1}^{(\frac{d-2}{2},\beta)}(x, \frac{m}{2})$. The constant κ has absolute value

$$|\kappa| = \left(\sum_{l=\frac{m}{2}}^{\frac{n}{2}} \left| p_{l-\frac{m}{2}}^{(\frac{d-2}{2},\beta)}(\lambda_{\max}, \tfrac{m}{2}) \right|^2 \right)^{-\frac{1}{2}}.$$

The optimal polynomials are unique up to multiplication with a complex scalar of absolute value 1. The minimum of the variance $\operatorname{var}_S^{\mathrm{B}^d}(P)$ with respect to polynomials $P \in \mathbb{S}_n^m(\mathrm{B}^d)$ can be determined as

$$\min\left\{\operatorname{var}_S^{\mathrm{B}^d}(P);\ P \in \mathbb{S}_n^m(\mathrm{B}^d)\right\} = \frac{1-\lambda_{\max}}{2}.$$

Proof. We consider polynomials $P \in \mathbb{S}_n^m$ of the form (16). Then, by Lemma 4.1, the variance $\operatorname{var}_S^{\mathbb{B}^d}(P)$ can be written as (17).

Therefore, minimizing the variance $\operatorname{var}_S^{\mathbb{B}^d}(P)$ over all polynomials $P \in \mathbb{S}_n^m$ is equivalent to minimizing the quadratic functional (17) over all coefficients $c_{l,k,j}$ such that

$$\sum_{l=m}^{n} \sum_{\substack{k=0 \\ l-k \text{ even}}}^{l} \sum_{j=1}^{N(d-1,k)} |c_{l,k,j}|^2 = 1.$$

The quadratic functional (17) has a block matrix structure and the minimum is attained for the eigenvector corresponding to the largest eigenvalue $\lambda_{\max}$ taken over all the symmetric matrices $\mathbf{J}\big(\frac{d-2}{2}+k,\beta\big)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}^{\frac{m}{2}-\lfloor\frac{k}{2}\rfloor}$, $0 \leq k \leq m$, and $\mathbf{J}\big(\frac{d-2}{2}+k,\beta\big)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}$, $m+1 \leq k \leq n$. Moreover, by formulas (11) and (12), the largest eigenvalue of the matrix $\mathbf{J}(\frac{d-2}{2}+k,\beta)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}^{\frac{m}{2}-\lfloor\frac{k}{2}\rfloor}$ corresponds precisely to the largest root of the associated Jacobi polynomial $p_{\frac{n-m+(-1)^k+1}{2}}^{(\frac{d-2}{2}+k,\beta)}(x,\frac{m}{2}-\lfloor\frac{k}{2}\rfloor)$ and the largest eigenvalue of the matrix $\mathbf{J}(\frac{d-2}{2}+k,\beta)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}$ to the largest zero of the Jacobi polynomials $p_{\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{(\frac{d-2}{2}+k,\beta)}(x)$.

Now, the results of Theorems 2.1 and 2.4 on the monotonicity of the largest zero of the associated Jacobi polynomials come into play. For $-\frac{1}{2} \leq \beta \leq d-2$, Theorem 2.1 and the interlacing property of the zeros of the Jacobi polynomials (see [13], Theorem 3.3.2) state that the matrix $\mathbf{J}\big(\frac{d-2}{2}+k,\beta\big)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}$, $m \leq k \leq n$, with the largest eigenvalue is precisely the matrix $\mathbf{J}(\frac{d-2}{2}+m,\beta)_{\frac{n-m}{2}}$. To show this, one could alternatively also use the interlacing results of [2].
Further, by Theorem 2.4, the matrix $\mathbf{J}\big(\frac{d-2}{2}+k,\beta\big)_{\frac{n}{2}-\lceil\frac{k}{2}\rceil}^{\frac{m}{2}-\lfloor\frac{k}{2}\rfloor}$, $0 \leq k \leq m$, with the largest eigenvalue is the matrix $\mathbf{J}(\frac{d-2}{2},\beta)_{\frac{n}{2}}^{\frac{m}{2}}$ which appears only one time as a submatrix in (17). Thereby, one has to distinguish between even and odd k and use Theorem 2.1 and the interlacing property of consecutive polynomials to see that $\mathbf{J}(\frac{d-2}{2},\beta)_{\frac{n}{2}}^{\frac{m}{2}}$ has a larger maximal eigenvalue than $\mathbf{J}(\frac{d-2}{2}+1,\beta)_{\frac{n}{2}-1}^{\frac{m}{2}}$.

For the index $k=m$, the matrix $\mathbf{J}(\frac{d-2}{2}+m,\beta)_{\frac{n-m}{2}}$ coincides with $\mathbf{J}(\frac{d-2}{2}+m,\beta)_{\frac{n-m}{2}}^{\frac{m}{2}-\lfloor\frac{m}{2}\rfloor}$. Hence, combining the arguments above, the unique overall submatrix in (17) with the largest eigenvalue $\lambda_{\max}$ is the matrix $\mathbf{J}(\frac{d-2}{2},\beta)_{\frac{n}{2}}^{\frac{m}{2}}$. Moreover, by formula (12), $\lambda_{\max}$ corresponds to the largest zero of the associated Jacobi polynomial $p_{\frac{n-m}{2}+1}^{(\frac{d-2}{2},\beta)}(x,\frac{m}{2})$. Due to the three-term recurrence relation (9) of the polynomial $p_{\frac{n-m}{2}+1}^{(\frac{d-2}{2},\beta)}(x,\frac{m}{2})$, the coefficients of the corresponding eigenvector can be determined as

$$\begin{aligned} c_{2l,0,1} &= p_{l-\frac{m}{2}}^{\frac{d-2}{2},\beta}(\lambda_{\max}, \tfrac{m}{2}), && \text{if} \quad \tfrac{m}{2} \leq l \leq \tfrac{n}{2}, \\ c_{l,k,j} &= 0, && \text{otherwise.} \end{aligned}$$

Finally, the coefficients are normalized by the constant κ and the uniqueness of $\mathcal{P}_n^m$ (up to multiplication with a complex scalar of absolute value 1) follows from the fact that the largest eigenvalue $\lambda_{\max}$ of $\mathbf{J}\big(\frac{d-2}{2}, \beta\big)_{\frac{n}{2}}^{\frac{m}{2}}$ is simple and that the monotonicity of the zeros in Theorems 2.1 and 2.4 is strict. □

Remark 4.3. Theorem 4.2 is also valid for odd n, in that case only the calculation gets a bit more complicated due to the fact that one has to use Gauss-brackets throughout the proof.

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