Applications of Lissajous curves in MPI

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Tutorial talk given by W. Erb at the

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Outline of the talk

In this talk we present two applications for polynomial interpolation on Lissajous curves in Magnetic Particle Imaging (MPI).

1) Reduction of measurements using Lissajous node points [4].
2) Spectral filtering for polynomial reconstruction in MPI [5].

1) Reduction of measurements using the node points of non-degenerate Lissajous curves

Based on [4] by Kaethner, E., Ahlborg, Szwargulski, Knopp and Buzug

Idea of this work

- In order to reduce the number of system matrix measurements, reconstruct the concentration of the magnetic particles only at the node points of the generating Lissajous curves.
- All double points of the Lissajous curve are contained in these particular node sets, a large part of the information of the MPI signal is encoded in these node points.
- Use polynomial interpolation on the Lissajous node points to obtain the concentration on the entire field of view.

The path generating curves

For two relatively prime numbers n_1, n_2 the following non-degenerate Lissajous curves are used in MPI:

$$\gamma^{(2n_1,2n_2)} : [0,2\pi] \to [-1,1]^2 \ , \ \gamma^{(2\underline{\boldsymbol{n}})}(t) = (\sin n_2 t, \sin n_1 t) \ .$$

The set of Lissajous node points associated to $\gamma^{(2\underline{\boldsymbol{n}})}$ is given by

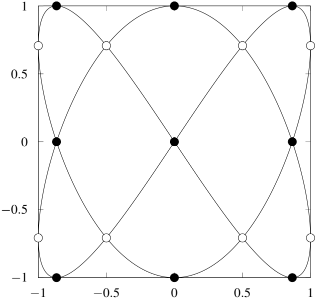
$$\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})} = \left\{ \gamma^{(2\underline{\boldsymbol{n}})}\left(\tfrac{\pi k}{2n_1 n_2}\right) : \qquad k = 0, \ldots, 4n_1 n_2 - 1 \right\} \ .$$

The spectral index set associated to the Lissajous nodes is given by

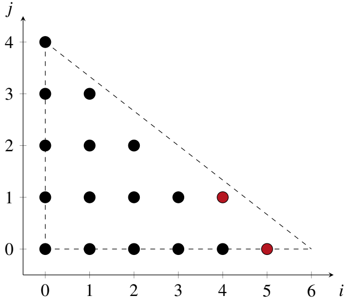
$$\underline{\boldsymbol{\Gamma}}^{(2\underline{\boldsymbol{n}})} = \left\{ (i,j) \in \mathbb{N}^2 \ : \ \frac{i}{2n_1} + \frac{j}{2n_2} < 1 \right\} \cup \{(0, 2n_2)\} \ .$$

The interpolation theory on $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$ is very close to the theory devoloped for the Padua points [1].

Example



(a) Nodes $\mathbf{LS}^{(6,4)}$



(b) Spectral index set $\boldsymbol{\Gamma}^{(6,4)}$

We know that there exists a unique bivariate interpolation polynomial

$$P_f^{(2\underline{\boldsymbol{n}})}(x,y) = \sum_{(i,j)\in\underline{\boldsymbol{\Gamma}}^{(2\underline{\boldsymbol{n}})}} c_{ij}(f)\,T_i(x)\,T_j(y)$$

that interpolates a given function f on the Lissajous nodes $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$ [3].

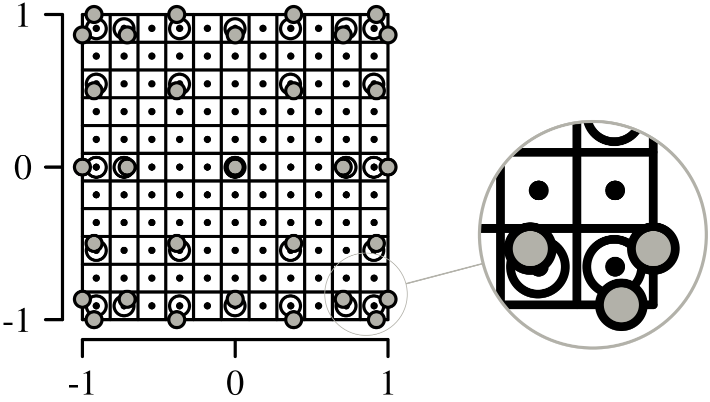
In order to obtain a reconstruction $\mathbf{c}$ of the measured frequencies $\widehat{\mathbf{u}}$ only on the set $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$, the following Tykhonov functional is minimized:

$$||\mathbf{SWc} - \widehat{\mathbf{u}}||_2^2 + \lambda ||\mathbf{c}||_2^2 \to \min! \qquad (1)$$

where

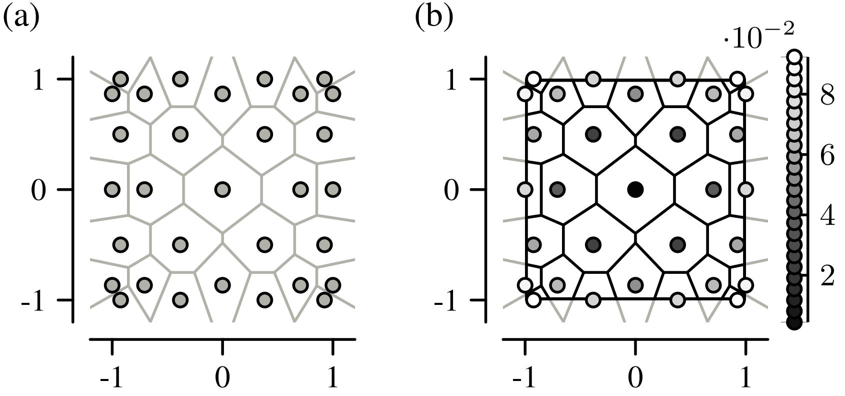
- $\mathbf{S}$ is a matrix collecting all measured system responds of delta samples on the nodes $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$.
- $\mathbf{W}$ describes a diagonal weight matrix to compensate for the non-uniform distribution of the nodes $\mathbf{LS}^{(2\underline{\boldsymbol{n}})}$.
- λ is the parameter of the Tykhonov regularization.

In practice, the measurements are given on a regular rectangular grid and we have to adapt them for the Lissajous nodes.

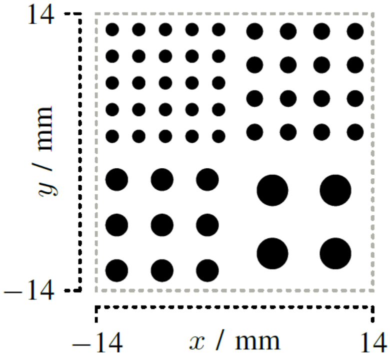


For a given Lissajous node point, we use the closest measurement on the rectangular grid. If more than 1 node point is given for one measurement we take only one of these node points into consideration.

The weights of the diagonal matrix **W** are determined by computing the area of the Voronoi tessellation of the Lissajous node points inside the square $[-1,1]^2$.



These weights can be determined also in other alternative ways.



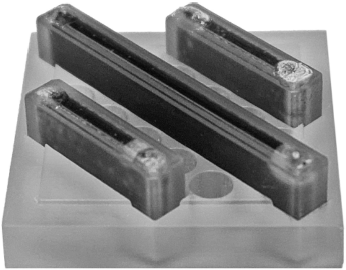


Figure: Illustrations of the used phantoms. Left: Simulated resolution phantom that consists of circles with a diameter of 3.2 mm, 2.3 mm, 1.7 mm, and 1.35 mm. Right: phantom used for the experiments. The phantom consists of three rectangular cuboids filled with Resovist©. The inner dimensions of the two shorter cuboids are $17.9 \times 2 \times 10 mm^3$, whereas the inner dimensions of the longer cuboid are $38.7 \times 2 \times 10 mm^3$ [4].

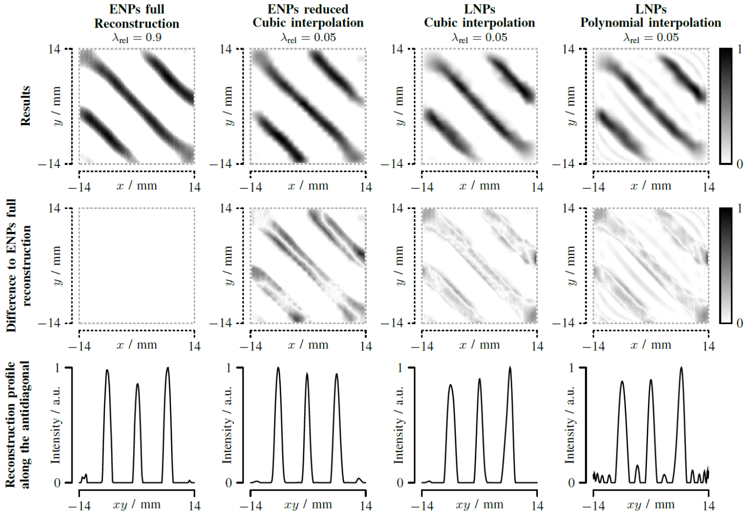


Figure: Reduced reconstruction based on equidistant and Lissajous grids. For the Lissajous grids we used cubic and polynomial interpolation [4].

Spectral filters for interpolation in MPI

Based on the work [5, 6] of S. De Marchi, W. Erb and F. Marchetti.

Spectral interpolation or approximation methods get problematic if the underlying function is not smooth or even not continuous.

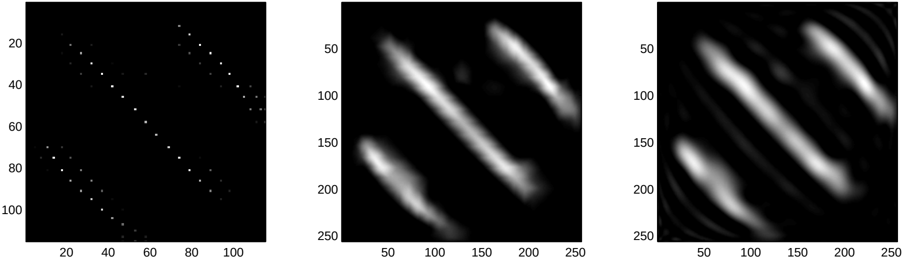


Figure: Reconstruction using cubic and polynomial interpolation on the Lissajous nodes in MPI. On the right hand side, a Gibbs artifact can be observed for the polynomial interpolation [4].

We recapitulate the interpolation scheme on the nodes of non-degenerate Lissajous curves. For relatively prime numbers n_1, n_2 we consider

$$\gamma^{(2n_1,2n_2)} : [0, 2\pi] \to [-1, 1]^2 \; , \; \gamma^{(2\underline{\boldsymbol{n}})}(t) = (\sin n_2 t, \sin n_1 t) \; .$$

The set of Lissajous node points associated to $\gamma^{(2\underline{\boldsymbol{n}})}$ is given by

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The spectral index set associated to the Lissajous nodes is given by

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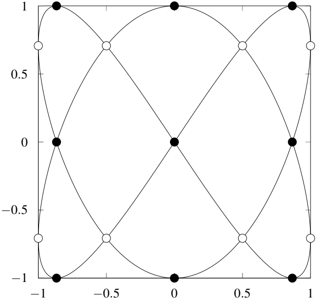
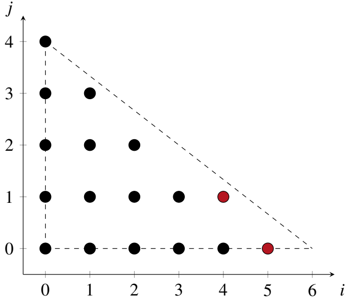
Example



(a) Nodes $\mathbf{LS}^{(6,4)}$



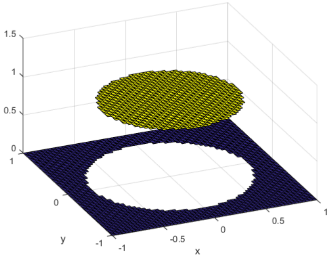
(b) Spectral index set $\boldsymbol{\Gamma}^{(6,4)}$

We know that there exists a unique bivariate interpolation polynomial

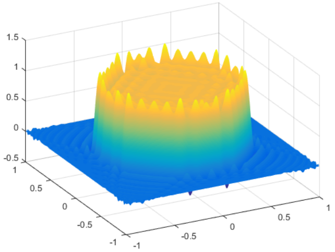
$$P_f^{(2\underline{\boldsymbol{n}})}(x,y) = \sum_{(i,j)\in\underline{\boldsymbol{\Gamma}}^{(2\underline{\boldsymbol{n}})}} c_{ij}(f)\, T_i(x)\, T_j(y)$$

that interpolates a given function f on the Lissajous nodes $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$ [3].

If we apply this scheme to a function with discontinuities, we can observe a Gibbs phenomenon.



(a) $f(x,y) = \mathbb{1}_{[0,0.75]}(x^2 + y^2)$.



(b) Interpolant $P_f^{(66,64)}$ illustrating the Gibbs phenomenon.

Goal: Construction of spectral filters for polynomial interpolation schemes on Lissajous curves in order to reduce the Gibbs phenomenon.

Spectral filters: For a bivariate function f having an expansion in terms of Chebyshev polynomials, we consider tensor product filters of the form

$$S_N^\sigma f(x,y) = \sum_{(i,j)\in\mathbb{N}_0^2} \sigma_i \sigma_j c_{ij}(f) T_i(x) T_j(y).$$

where $\sigma_i = \sigma(i/N)$ for some even filter function σ satisfying

1. $\sigma \in C^{p-1}$ for some order $p \in \mathbb{N}$.
2. $\sigma(0) = 1$ and $\sigma^{(l)}(0) = 0$ for $1 \leq l \leq p-1$.
3. $\sigma(\eta) = 0$ for $|\eta| \geq 1$.

Examples of filter functions

- The Fejér filter (first order)

$$\sigma(\eta) = 1 - \eta \; .$$

- The Lanczos or sinc filter (first order)

$$\sigma(\eta) = \frac{\sin(\pi\eta)}{\pi\eta} \; .$$

- The *raised cosine* filter (second order)

$$\sigma(\eta) = \frac{1}{2}(1 + \cos(\pi\eta)) \; .$$

- The exponential filter of order p (p even)

$$\sigma(\eta) = e^{-\alpha|\eta|^p} \; .$$

An spatial adaptive spectral filter

As a spatially adaptable spectral filter we consider

$$\sigma^p(\eta) = \begin{cases} \exp\left(\frac{|\eta|^p}{\eta^2-1}\right) & |\eta| < 1 \ , \\ 0 & |\eta| \geq 1 \ , \end{cases}$$

where $p : \mathbb{R} \to \mathbb{R}_+$ is now a function depending on the position (x, y). The parameter function $p = p(x, y)$ is our key for adaptivity. In this way, we get the filtered function

$$S_N^{\sigma^{\mathbf{p}}} f(x,y) = \sum_{(i,j) \in \mathbb{N}_0^2} \sigma_i^{p_1} \sigma_j^{p_2} \, c_{ij}(f) \, T_i(x) T_j(y).$$

where both p_1 and p_2 depend on x and y.

Let (ξ_x, ξ_y) be the closest point of discontinuity of the function f with respect to (x, y) in the euclidean norm. We set

$$d_1(x,y) = |x - \xi_x| \quad \text{and} \quad d_2(x,y) = |y - \xi_y|.$$

Theorem

Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a piecewise analytic function on some open bounded Lipschitz domain Ω. Then, setting

$$p_1(x_1, x_2) = (N\eta_1^* d_1(x,y))^{1/2}, \qquad p_2(x_1, x_2) = (N\eta_2^* d_2(x,y))^{1/2},$$

with suitable choosen parameters η_1^*, η_2^*, the error $|f - S_N^{\sigma^{\mathbf{p}}} f|$ decays with an asymptotic exponential rate away from the points of discontinuity of f.

Proof. Based on a tensor product construction of a 1D idea developed by Tadmor and Tanner [7].

A heuristic approach

The statement of this theorem is a nice theortic result. In practice however the filtered function $S_N^{\sigma^{\mathbf{p}}} f$ has discontinuities. The following usage of the parameters p_1 and p_2 turned out be more suitable:

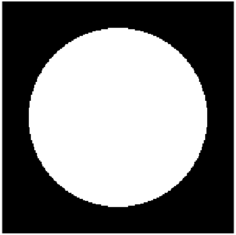
$$p_1(x,y) = p_2(x,y) = (N\eta^*)^{1/2} d(x,y)^{\beta/2},$$

where

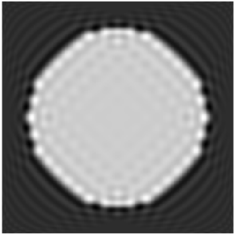
$$d(x,y) = \sqrt{d_1(x,y)^2 + d_2(x,y)^2}, \quad \text{and} \quad \beta > 0.$$

For our experiments, $\beta = 1/2$ turned out to be a good choice.

Example



(a) The original function g.

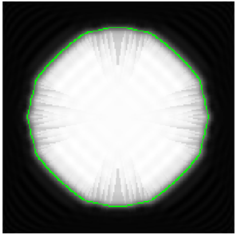


(b) Polynomial approximation $P_g^{(66,64)}$.

Figure: Example of the Gibbs phenomenon for polynomial interpolation of the indicator function for a disk.

Example

(a) Spectral filtered polynomial $P_g^{(66,64)}$ using the raised cosine filter.



(b) Spectral filtered polynomial $P_g^{(66,64)}$ with the adaptive filter from the theorem.

Figure: Approximation of the function g with filtered polynomial interpolants. In this case we used the adaptivity strategy from the theorem. The edges were detected using the Canny edge detector [2].

Example

(a) Spectral filtered polynomial $P_g^{(66,64)}$ using the raised cosine filter.

(b) Spectral filtered polynomial $P_g^{(66,64)}$ using the adaptive filter with $\beta = 1/2$ and $\eta^* = 0.0224$.

Figure: Approximation of the function g with filtered polynomial interpolants. In this case we used the heuristic adaptive strategy with $\beta = 1/2$. The edges were detected using the Canny edge detector [2].

Strategy for adaptive filtering in MPI

1. Take the reconstructed particle density on the node points $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$ of the Lissajous curve.
2. Obtain a first reconstruction by interpolating the function values on the Lissajous nodes $\underline{\mathbf{LS}}^{(2\underline{\boldsymbol{n}})}$ using the polynomial interpolation.
3. Apply a first non-adaptive spectral filtering process (for instance the cosine filter).
4. Use an edge-detector (in our case the Canny edge detector) on the filtered reconstruction to find the edges and the distances required for the adaptivity.
5. Apply the final adaptive filtering procedure on the first reconstruction.

Simulated MPI example 1

Figure: Left: First reconstruction obtained by interpolating along the Lissajous curve. Right: Final reconstruction using the adaptive filtering method using a raised cosine filter for the first filtering and the parameters $\beta = 1/2, \eta^* = 0.0224$ for the final adaptive filtering [5]. The edges detected by the Canny edge detector are marked in green.

Simulated MPI example 2

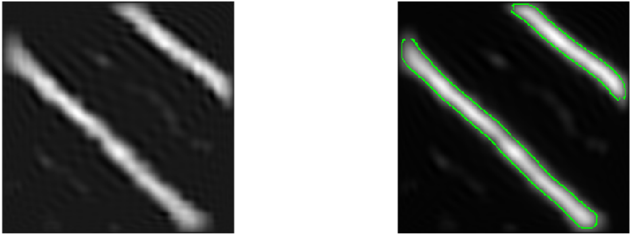


Figure: Left: First reconstruction obtained by interpolating along the Lissajous curve. Right: Final reconstruction using the adaptive filtering method using a raised cosine filter for the first filtering and the parameters $\beta = 1/2, \eta^* = 0.0224$ for the final adaptive filtering [5]. The edges detected by the Canny edge detector are marked in green.

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