Cohomology theories I

Bruno Chiarellotto

Università di Padova

Sep 11th, 2011

Road Map

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Let X be a topological space. One has singular cohomology/homology. All is encoded in the Eleienberg-Steenrod axioms: excision, Mayer-Vietoris... We know that suach axioms together with the dimension axiom characterize the coh/hom theory. We have also CW simplicial,....$H^i_{sing}(X, A)$ A is a ring.

If X is locally contractible, we may calculate singular coh. using sheaf theory. We consider the category Sh_X of sheaves in groups. It is abelian, enough injectives. If we have a functor $F : Sh_X \to \mathcal{B}$ left exact then we may define its (right) derived functors

$$\mathbb{R}^i F : Sh_X \to \mathcal{B}, i \geq 0$$

In particular if $F = \Gamma(X, -) : Sh_X \to Ab$, global sections

Then if X is locally contractible we have

$$H^i(X, A_X) = H^i_{sing}(X, A)$$

$\mathbb{R}^i\Gamma(X, -)(A_X)$ is denoted by $H^i(X, A_X)$. Where A_X is a locally constant sheaf with values in A, $A = \mathbb{Z}, \mathbb{R}, \mathbb{C} ... \mathbb{Z}/(n).\mathbb{Z}_l, \mathbb{Q}_l$, l a prime.

Remark

Another way of calculating sheaf cohomology will be using Cech methods.

To any open covering $\mathcal{U}$ of X and any sheaf $\mathcal{F}$ one may associate a complex $\mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$.

And then define the Cech coh. of $\mathcal{F}$ on $\mathcal{U}$ as

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(\mathscr{C}^\bullet(\mathcal{U}, \mathcal{F}))$$

EX. $X = \mathbb{P}^1_{\mathbb{C}}, \mathcal{U} = \{\mathbb{A}^1, \mathbb{A}^1_\infty\}, \mathcal{F} = \mathcal{O}_X$ the structural sheaf. Find the Cech coh.

We define the Cech coh. of $\mathcal{F}$ the

$$\lim_{\rightarrow} \check{H}^i(\mathcal{U}, \mathcal{F}) = \check{H}^i(X, \mathcal{F})$$

If X is paracompact , then for any sheaf $\mathcal{F}$ we have

$$H^i(X, \mathcal{F}) = \check{H}^i(X, \mathcal{F})$$

if the sheaf is the locally constant A_X. Then we have another way of calculating sing coh.

Remark

In general, if $\mathscr{U} = \{U_j\}_{j\in J}$ is given by opens such that for any finite $S \subset J$

$$H^k(\cap_{s\in S} U_s, \mathcal{F}) = 0, k > 0$$

then

$$\check{H}^i(\mathscr{U}, \mathcal{F}) = H^i(X, \mathcal{F})$$

" $\mathscr{U}$ is an acyclic covering for $\mathcal{F}$". If there exists such a covering for $\mathcal{F}$, then Čech cohomology for $\mathcal{F}$ coincides with the sheaf cohomolology for $\mathcal{F}$.

If, moreover, X has also a structure of differential manifold over $\mathbb{R}$ ....$\mathscr{C}^\infty$...
then one knows that it is locally contractible...sheaf (cech) coh. is sing. coh.
But then one can take the sheaf of $\mathscr{C}^\infty_{\mathbb{R}}$ $i-th$–forms on X : $\mathscr{E}^i$. And consider the complex of the global sections

$$0 \to \mathscr{E}^0(X) \xrightarrow{d^0} \mathscr{E}^1(X) \xrightarrow{d^1} \mathscr{E}^2(X) \to ...$$

" the de Rham Complex = $\mathcal{DR}(X, \mathbb{R})$". . We denote by $\mathcal{DR}_X$ the associated complex of sheaves. We may then define

$$H^i_{dR}(X) = H^i(\mathcal{DR}(X, \mathbb{R})) = \text{ Closed i-forms/Exact i-forms}$$

Usually we denote $H^i_{dR}(X) = H^i_{dR}(X, \mathbb{R})$ because if we take the i-forms with $\mathbb{C}$-coefficients we will get $H^i_{dR}(X, \mathbb{C}) = H^i_{dR}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.
The **de Rham theorem** then states
$\mathcal{DR}_X$ is an acyclic (for the global sections functor-abelian sheaves) resolution of the constant $\mathbb{R}_X$ sheaf on X.

Hence for a $\mathscr{C}^\infty$-manifold X:

$$H^i_{dR}(X, \mathbb{R}) = H^i(X, \mathbb{R}_X) = H^i_{sing}(X, \mathbb{R})$$

Of course

$$H^i_{dR}(X, \mathbb{C}) = H^i(X, \mathbb{C}_X) = H^i_{sing}(X, \mathbb{C}).$$

Remark. $\mathscr{E}^i$ is a fine sheaf. Hence acyclic.

Remark $H^i_{sing}(X, \mathbb{R})$ is the dual of $H_{i,sing}(X, \mathbb{R})$, while the de Rham coh. is defined using differentials.

In order to understand the de Rham Coh. we should see it as a "hypercohomology" or better as in the derived category. Suppose to have a category $\mathscr{A}$ having enough injective then if one takes a complex $A^\bullet = ...A^i \to A^{i+1} \to ...$

(it is bounded in the negative index...to avoid problems). The one can find a double complex $I^{\bullet,\bullet}$ made by injectives

Such that $I^{i,\bullet}$ is an injective resolution of A^i $\forall i$ and compatible with the maps $A^i \to A^{i+1}$.

Then one can make a single complex $C^\bullet$ out of it . (note: the sign!!)

If we start with a left exact functor F from $\mathscr{A}$, then we define $\mathbb{R}F(A^\bullet)$ as

$$\mathbb{R}^i F(A^\bullet) = H^i(FC^\bullet).$$

If the complex is made by F acyclic objects, then $\mathbb{R}^i F(A^\bullet) = H^i(FA^\bullet)$.

This is the case for the global sections functor and then a complex of sheaves. Hence for the differentiable de Rham complex $\mathscr{E}^\bullet$ and for the global section functor.
We indicate it as hypercohomology and we denote as $\mathbb{H}^i(X, A^\bullet)$ for a complex of sheaves on X.

$\mathscr{E}^\bullet$ is a resolution (acyclic for the global section functor) of the sheaf $\mathbb{R}_X$!

The complex with $\mathbb{C}$-valued $\mathscr{C}^\infty$-diff. form is denoted by $\mathscr{E}^\bullet_{\mathbb{C}}$ and it is a resolution (acyclic) of $\mathbb{C}_X$.

Analytic

We may even specialize our setting asking that X is also an analytic manifold. It implies it has even dimension and a complex structure in such a way the glueing functions are analytic. We may define the sheaf of holomorphic functions $\mathcal{O}_X$ and we know that $d = \partial + \overline{\partial}$.

Then we have that $\mathscr{E}^i_{\mathbb{C}} = \oplus_{k+j=i} \mathscr{E}^{j,k}_{\mathbb{C}}$. If we denote by Ω^i the holomorphic i-forms, then

$$0 \to \Omega^i \to \mathscr{E}^{i,0}_{\mathbb{C}} \xrightarrow{\overline{\partial}} \mathscr{E}^{i,1}_{\mathbb{C}} \to ...$$

is exact . Hence $\mathscr{E}^{i,\bullet}_{\mathbb{C}}$ is an acyclic resolution of Ω^i.
The Doulbeault theorem says that:

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{\partial} \Omega^1 \to \Omega^2 \to .... = 0 \to \mathbb{C}_X \to \Omega^\bullet$$

is exact complex of sheaves!

It follows that, for X an analytic manifold then $\mathscr{E}_{\mathbb{C}}^{\bullet,\bullet}$ is an acyclic resolution of $\Omega^{\bullet}$. Its simple complex is $\mathscr{E}_{\mathbb{C}}^{\bullet}$. Hence

$$H^i(X, \mathbb{C}_X) = H^i_{dR}(X, \mathbb{C}) = H^i(\mathscr{E}_{\mathbb{C}}^{\bullet}(X)) = \mathbb{H}^i(X, \Omega_X^{\bullet})$$

I.e. For an analytic manifold its de Rham, singular cohomology can be calculated using only holomorphic differential which can be at max. n-differentials if the $dim_{\mathbb{C}} X = n$.

Remark

We could ask moreover that X were compact and not only a complex manifold, but endowed with a Riemannian metric. Then one would have an operator called "Laplacian" : ∇, which operates on the $\mathscr{E}^i_{\mathbb{C}}(X)$.

The Hodge theorem says that the $\mathbb{C}$-vector space $\{\omega \in \mathscr{E}^i_{\mathbb{C}}(X) \mid \nabla\omega = 0\}$ is isomorphic to $H^i_{dR}(X, \mathbb{C})$. The de Rham cohomology is given by "harmonic forms".

If the manifold is even more sophisticated : it is Kahler and it is projective. Then we have a more subtle decomposition (Hodge decomposition) as $\mathbb{C}$-vector space

$$H^i(X, \mathbb{C}) = H^i_{dR}(X, \mathbb{C}) = \oplus_{k+l=i} H^{k,l}(X)$$

where $\overline{H}^{k,l} = H^{l,k}$.

of weight i...???

The Algebraic Setting

Of course one could have started with an algebraic variety over $\mathbb{C}$.

Think about $Spec\mathbb{C}[x, y]/(xy - 1) = \mathbb{A}^1 \setminus \{0\}$. It has Zariski topology. How to construct a “consistent" coh. only using Zariski topology?

Taking the cohomology of a constant (locally) sheaf on the Zariski top. is wrong: it is a flasque sheaf (in general) hence with no cohomology groups...

It is impossible to define paths. We are left with de Rham cohomology (so far)

So let's start with something which looks like a analytic manifold:

a smooth variety over $\mathbb{C}$. Suppose of $dimX = n$. Then we may

construct the Zariski sheaves of the algebraic Kahler differentials Ω^1_X on the algebraic functions on X(i.e.$\mathcal{O}_X$). Then we form a complex $\Omega^\bullet_X$.

ex. $\mathcal{O}_X(\mathbb{A}^1 \setminus \{0\}) = \mathbb{C}[x, \frac{1}{x}]$. But $\mathbb{A}^1 \setminus \{0\} = \mathbb{C} \setminus \{0\}$ with the trascendental topology. An analytic manifold has some highly trascendental functions.... *exp x*.

Definition

Suppose X algebraic variety smooth over $\mathbb{C}$, then we define

$$H^\bullet_{dR}(X) = \mathbb{H}^\bullet(X, \Omega^\bullet_X)$$

all in algebraic (Zariski topology).

If X is an algebraic $\mathbb{C}$-variety, we may associate to it an analytic variety X^{an}.

Remark If X were smooth, then X^{an} is an analytic manifold. If X is projective then X^{an} is a compact Kahler manifold.

Because the Zariski topology on X is given by complement of the zeroes of polynomials which are closed for the trascendental topology in X^{an} We do have a continuos map

$$\iota : X^{an}_{tr} \to X_{Zar}$$

which is the identity at level of points. It is a map of ringed spaces: rational $\mathcal{O}_X$ for Zariski and holomorphic $\mathcal{O}_{X^{an}}$ for trascendental. Polynomials functions are holomorphic functions.

hence if $\mathcal{F}$ is a (Zar)-sheaf of $\mathcal{O}_X$-modules then we may associate

$$\mathcal{F}^{an} = \iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{X^{an}}$$

a sheaf of $\mathcal{O}_{X^{an}}$-modules. We then have

Theorem(Serre-GAGA)

If X is projective and $\mathcal{F}$ is coherent, then

$$H^\bullet(X, \mathcal{F}) = H^\bullet(X^{an}, \mathcal{F}^{an}).$$

(sheaf cohomology)
Moreover suppose X smooth then we have $\Omega^i_{X^{an}} = (\Omega^i_X)^{an}$, $\forall i$, they are coherent. If X is **projective**, too, then by GAGA

$H(X, \Omega^i_X) = H(X^{an}, \Omega^i_{X^{an}})$. By a spectral-sequence argument we then conclude, if X is smooth and projective:

$$H^\bullet_{dR}(X) = H^\bullet_{dR}(X^{an}) = H^\bullet(X^{an}, \mathbb{C}).$$

Remark 1 The de Rham coh. is the hypercohomology of $\Omega^\bullet_X$ and $\Omega^\bullet_{X^{an}}$. From GAGA we know that the coh. for each Ω^i we have equality in the coh. between anal. and alg. The spectral sequence will give the identity at the limit.

Remark 2 Because X^{an} has a Kahler structure (projective and smooth). We have a Hodge filtration on $H^\bullet_{dR}(X) = H^\bullet_{dR}(X^{an})$ which, a priori, is only trascendental....it is not the case...

But such a comparison theorem is even better. even if the sheaf cohomology fails to be equal in the non proper case.
ex. $H^0(\mathbb{A}^1 \setminus \{0\}, \mathcal{O})$ doesn't contain *expx* which is an element of $H^0(\mathbb{C} \setminus \{0\}, \mathcal{O}^{an})$.

But the de Rham coh. is the good one. In fact

Theorem-Grothendieck Publ. Sc. IHES 1966

If $X/\mathbb{C}$ is a smooth algebraic variety then

$$H^\bullet_{dR}(X) = H^\bullet_{dR}(X^{an}) = H^\bullet(X^{an}, \mathbb{C}).$$

Hence the singular coh. can be calculated by algebraic methods.
For the proof: see also Deligne Lect. Not. Math. 163.

How to prove this? 1)We are in ch.0. Hence we have resolution of singularities. we may find and compactification $j : X \to \overline{X}$ where $\overline{X}$ is proper and smooth and the complement $\overline{X} \setminus X$ is a normal crossing divisor. (j^{an} for the analogue map).
It means that locally analytically $\overline{X}^{an} \simeq \mathbb{C}^n$ in coordinates $z_1, \ldots, z_n$ and $\overline{X}^{an} \setminus Xan$ is given by $z_1 \ldots z_k = 0$. This is a corollary of two things: Nagata compactification and resolutions of singularities(we are in ch.0).

2) Because j is affine and J^{an} is stein $Rj_\star \mathcal{F} = j_\star \mathcal{F}$ on each coherent $\mathcal{F}$ as well as $Rj^{an}_\star \mathcal{G} = j^{an}_\star \mathcal{G}$, for any (analytic coherent) $\mathcal{G}$.

We then have

$$\mathbb{H}(X, \Omega^\bullet_X) = \mathbb{H}(\overline{X}, j_\star \Omega^\bullet)$$

and

$$\mathbb{H}(X^{an}, \Omega^\bullet_{X^{an}}) = \mathbb{H}(\overline{X}^{an}, j^{an}_\star \Omega^\bullet_{X^{an}}).$$

and $j_\star \Omega^\bullet$ and $j^{an}_\star \Omega^\bullet_{X^{an}}$ are quasi-coherent.

3) $\overline{X}$ is proper. Hence we have GAGA for it. we apply it to $j_\star\Omega^\bullet$ which is limit (direct) of coherent shaves $j_\star\Omega^\bullet = \lim_\rightarrow \mathcal{F}_i$ We have the commutativity between $\lim_\rightarrow$ and global sections functor

$$\mathbb{H}(\overline{X}, j_\star\Omega^\bullet) = \mathbb{H}(\overline{X}, \lim_{\rightarrow} \mathcal{F}_i) = \lim \mathbb{H}(\overline{X}, \mathcal{F}_i) = \lim_{\rightarrow} \mathbb{H}(\overline{X}^{an}, \mathcal{F}_i^{an})$$

$$= \mathbb{H}(\overline{X}^{an} \lim_{\rightarrow} \mathcal{F}_i^{an}) = \mathbb{H}(\overline{X}^{an}, (j_\star\Omega^\bullet)^{an}).$$

4) Hence in $\overline{X}$ it will be enough to show that the inclusion

$$(j_\star\Omega^\bullet)^{an} \rightarrow j_\star^{an}\Omega^\bullet_{X^{an}}$$

(meromorphic in essential sing.) is a quasi iso.
This will be done by showing that both are quasi isomorphic to a third complex (which is contained in both)

$$\Omega^\bullet_{\overline{X}^{an}} < log >$$

of the log-diffferential along the divisor at ∞ in $\overline{X}^{an}$ (i.e. the complement of X)

5) Hence we will find that

$$\mathbb{H}(X, \Omega^{\bullet}_{X}) = \mathbb{H}(X^{an}, \Omega^{\bullet}_{X^{an}}) = \mathbb{H}(\overline{X}^{an}, \Omega^{\bullet}_{\overline{X}^{an}} < log >)$$

6) Note that $\Omega^{\bullet}_{\overline{X}^{an}} < log >= (\Omega^{\bullet}_{\overline{X}} < log >)^{an}$ and $\Omega^{\bullet}_{\overline{X}} < log >$ is coherent. Hence we may re-write

$$\mathbb{H}(X, \Omega^{\bullet}_{X}) = H(X^{an}, \Omega^{\bullet}_{X^{an}}) = H(\overline{X}, \Omega^{\bullet}_{\overline{X}} < log >)$$

all in the algebraic setting!

7) Note that if we have started from a smooth variety over $\mathbb{Q}$: $X_{\mathbb{Q}}$ then we could have built its de Rham Coh. as $\mathbb{Q}$-vector spaces $\mathbb{H}(X_{\mathbb{Q}}, \Omega^{\bullet}_{X_{\mathbb{Q}}}) = H_{dR}(X_{\mathbb{Q}})$. And by base change

$$H_{dR}(X_{\mathbb{Q}}) \otimes \mathbb{C} = H_{dR}(X_{\mathbb{C}}) = H_{dR}(X_{\mathbb{C}}^{an})$$

where $X_{\mathbb{C}} = X_{\mathbb{Q}} \times \mathbb{C}$.

Hodge Theory

For a smooth variety $X_{\mathbb{Q}} = X$ we may associate a smooth compactification with Normal Crossings Divisors. We then obtain a complex $\Omega^{\bullet}_{\overline{X}} < log >$ all over $\mathbb{Q}$. We may filter it by two filtrations

1) A decreasing one $F^i \Omega^{\bullet}_{\overline{X}} < log >= \Omega^{\geq i}_{\overline{X}} < log >$ (Hodge)

2) An increasing one $W_i \Omega^{\bullet}_{\overline{X}} < log >$ based on the number of log-poles we admit (weight)

At the limit i.e. in the $\mathbb{Q}$-vector spaces $H_{dR}(X_{\mathbb{Q}})$ they will induce two filtrations. They will form a mixed Hodge structure.....

If $X_{\mathbb{Q}}$ were projective (other than smooth) then we would have only one filtration the Hodge one in $H_{dR}(X_{\mathbb{Q}})$. pause On the other hand the extension $X_{\mathbb{C}}^{an}$ is a projective Kahler analytic manifold. Hence by Hodge theory we have a decomposition of its coh. $\mathbb{C}$-vector spaces $H(X_{\mathbb{C}}^{an}, \mathbb{C})$.

But

$$H_{dR}(X_{\mathbb{Q}}) \otimes \mathbb{C} = H_{dR}(X_{\mathbb{C}}^{an}) = H(X_{\mathbb{C}}^{an}, \mathbb{C})$$

And the filtration in each $H_{dR}^{i}(X_{\mathbb{Q}})$ extended to $\mathbb{C}$ will be the Hodge filtration on $H^{i}(X_{\mathbb{C}}^{an}, \mathbb{C})$ coming from the Harmonic differentials....

The non smooth case

We focused so far only on smooth varieties. We want to deal now with the non smooth-case. If $X_{\mathbb{Q}}$ is a non smooth variety, then to $X_{\mathbb{C}}$ it is always possible to associate a topological space (actually an analytic space): $X_{\mathbb{C}}^{an}$.

To it we can associate $H^{\bullet}(X_{\mathbb{C}}^{an}, \mathbb{C})$. How to calculate it by algebraic methods? Do we have a Hodge structure? This will be done by means of the descent methods.

(Hodge III, Deligne section 5). If Δ is the category whose objects are ordered sets $[n] = \{0, 1, 2, \ldots, n\}$. and maps respecting ordering. A simplicial object in a category $\mathscr{C}$ is a controvariant $C_\bullet : \Delta \to \mathscr{C}$. To $[n]$ we associate C_n.... We will have a simplicial scheme $\mathscr{C} = Sch$..... a

simplicial topological space $\mathscr{C} = Top$

We will indicate a simplicial scheme (top. space) by $X_\bullet$. A sheaf on a simplicial space $X_\bullet$ will be $\mathcal{F}_\bullet$ where each $\mathcal{F}_n$ is a sheaf on X_n and for each $f : [n] \to [m]$ we should have $f^\star \mathcal{F}_n \to \mathcal{F}_m$...a lot of maps....

we define global section functor

$$\Gamma(X_\bullet, \mathcal{F}_\bullet) = ker(\Gamma(X_0, \mathcal{F}_0) \to \Gamma(X_1, \mathcal{F}_1))$$

where the map is the difference of the maps induced from $\partial_0, \partial_1 : [0] \to [1]$. Let $H^i(X_\bullet, \mathcal{F}_\bullet)$ be the $i - th$ derived functors. One takes a Godement resolutions...of $\mathcal{F}_\bullet$....it is automatically compatible with all the maps...
We then have an approximating spectral sequence

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_q) => H^{p+q}(X_\bullet, \mathcal{F}_\bullet)$$

Let's start with $a : X_\bullet \to S$ an augmented simplicial scheme...simplicial and a map $X_0 \to S$.
This gives a map $a_n : X_n \to S \; \forall n$. And a functor from sheaves from S to sheaves in $X_\bullet$: $a^\star$. For a sheaf $\mathcal{G}$ in S, $(a^\star \mathcal{G})_n = a_n^\star \mathcal{G}$.
It admits a left adjoint $a_\star$ which can be derived to give a functor

$$\mathbb{R}a_\star : D^+(X_\bullet) \to D^+(S)$$

(from bounded above complexes of sheaves).
We have an adjonction functor

$$\varphi : Id \to \mathbb{R}a_\star a^\star$$

Then a is called of *cohomological descent* if φ is an isomorphism. This means in particular that $(\Gamma(X_\bullet, \mathcal{F}_\bullet) = \Gamma(S, \mathbb{R}a_\star \mathcal{F}_\bullet))$

$$H^i(S, \mathcal{F}) \simeq H^i(S, \mathbb{R}a_\star a^\star \mathcal{F}) \simeq H^i(X_\bullet, a^\star \mathcal{F})$$

hence we may use the spectral sequence

$$E_1^{p,q} = H^q(X_p, (a^\star \mathcal{F})_q) => H^{p+q}(S, \mathcal{F})$$

This applies to sheaf defined over S. We start from $S = X_\mathbb{Q}$ a non smooth, then if we can find a $X_\bullet$ simplicial scheme $a : X_\bullet \to X_\mathbb{Q}$ such that its analytification $a^{an} : X_\bullet^{an} \to X_\mathbb{C}^{an}$ it is of coh. descent

then we may calculate $H^(X_\mathbb{C}^{an}, \mathbb{C})$ by using the sheaf $\mathbb{C}_X$ and then the $H(X_p^{an}, \mathbb{C})$...

Which simplicial schemes/topological spaces do have coh. descent? This is the subject of SGAIV, Vbis. This is the case for etale and proper hypercoverings.
I don't want to go too far in the discussione with introducing skeleton and coskeleta...I want only to say what we mean to say that $X_\bullet \to S$ is a proper hypercovering of S at least at level 0,1. $a_0 : X_0 \to S$ should be

a proper covering (i.e surjectve) we have $\partial_0 : X_1 \to X_0$ and

$\partial_1 : X_1 \to X_0$ which are comptible with a_0.

Hence we get a map $X_1 \to X_0 \times_S X_0$ which we ask to be a proper covering again....
and so on if we ask for an etale cov. or a Zariski...

here it is the main Deligne's result (Hodge III ch.0)+De Jong IHES 83 (1996) ch.p perfect...

Theorem

Let S be a variety defined over a perfect field k. Then there exists a simplicial scheme $\overline{X}_\bullet$ projective and smooth (i.e. each $\overline{X}_n$ is so) over k; a strict normal crossing divisor $D_\bullet$ in $\overline{X}_\bullet$ with open complement $X_\bullet = \overline{X}_\bullet \setminus D_\bullet$ and an augmentation $a : X\bullet \to S$ which is a proper hypercovering.

As a corollary if $k = \mathbb{Q}$ then $H^i(S^{an}_{\mathbb{C}}, \mathbb{C}) = H^i(X^{an}_{\mathbb{C},\bullet}, \mathbb{C}_{X_\bullet})$ we have a spectral sequence

$$E_1^{pq} = H^p(X^{an}_{\mathbb{C},q}, \mathbb{C}_{X^{an}_q}) => H^i(X^{an}_{\mathbb{C},\bullet}, \mathbb{C}_{X^{an}_\bullet})$$

and moreover each X_q is smooth! Hence
$H^p((X^{an}_{\mathbb{C},q}, \mathbb{C}_{X^{an}_q}) = H^p_{dR}(X_q) \otimes \mathbb{C}$!!
Note that $\Omega^{\bullet}_{\overline{X}_{\bullet}} < log >$ is a simplicial sheaf in $\overline{X}_{\bullet}$ By the algebraic spectral sequence (defined over $\mathbb{Q}$)

$$E_1^{pq} = H^p(\overline{X}_q, \Omega^{\bullet}_{\overline{X}_q} < log >) => H^i(X_{\bullet}, \Omega^{\bullet}_{\overline{X}_{\bullet}} < log >)$$

we get an isomorphism between the analogous for the analytic setting. hence

$$H^i(X_{\bullet}, \Omega^{\bullet}_{\overline{X}_{\bullet}} < log >) \times \mathbb{C} = H^i(X^{an}_{\mathbb{C},\bullet}, \mathbb{C}_{X^{an}_{\bullet}}) = H(S^{an}_{\mathbb{C}}, \mathbb{C})$$

again we can calculate in an algebraic way the cohomology of an algebraic singular variety.

1) It can be proved that $H^i(X_\bullet, \Omega^\bullet_{\overline{X}_\bullet} < log >)$ is independent upon the hypercovering...not only after tensoring by $\mathbb{C}$.

2) the interpretation using *log*-differential allows to put a mixed hodge structure on $H^i(X_\bullet, \Omega^\bullet_{\overline{X}_\bullet} < log >) = H_{dR}(S)$.

3) we didn't discuss about coefficients...i.e. connections....