Cohomology theories III

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Road Map

Contents

So far we have introduced good cohomology theories for smooth and projective varieties. Some problems have been not addressed.

-in $ch = p$ a definition of p-adic cohomology. I.e with coefficients in $\mathbb{Q}_p$.

-for open and non smooth varieties?

For the first we would like to introduce first the Crystalline coh. and then its de Rham style generalization: the rigid one. For the second we will introduce the Bloch-Ogus axioms which should rule a good cohomology theory taking care of the case open and non smooth.

Ètale coh. with coeffiencients in $\mathbb{Q}_p$ is not a good cohomology. It can be defined, too. Then one would like to introduce a de Rham style cohomology. To do that directly in $ch = p$ is not possibile. What one tries to do is to take lifting from $ch = p$ to $ch = 0$.

If we start with a finite field k then one can associate a DVR: the ring of Witt vectors $W(k)$. Its maximal ideal is (p) and $W(k)/(p) = k$. If $k = \mathbb{F}_p$ then $W(\mathbb{F}_p) = \mathbb{Z}_p$. We have also intermediate liftings $W_n = W(k)/(p)^n$. The idea is to associate to any X_k a cohomological groups $H(X_k, W_n)$ which are W_n-modules. This will done by crystalline coh. It will be given by a sheaf topology, but of a particular site.

$$Cris(X_k/W_n) = (X_k/W_n)_{crys}$$

The objects are couple are closed immersions $U \to V$ where U is zariski open in X and V is a scheme over W_n. The immersion is nilpotent and the ideal $\mathscr{I} = Ker(\mathcal{O}_V \to \mathcal{O}_U)$ is endowed with a Divided Powers structure i.e. maps

$$\gamma_n \mathscr{I} \to \mathcal{O}_V$$

such that $x \to \gamma_n(x) = ``\frac{x^n}{n}$
In principle we whould write $\{U, V, \gamma\}$ for an object.

Coverings: $\{(U_i, V_i)\}$, is a covering family for (U, V) if $V_i \to V$ are open and $V = \cup V_i$.

We have a notion of sheaf : for any (U, V) a sheaf on V: $\mathcal{F}_V$ and for any map $(U, V) \xrightarrow{f} (U', V')$ a map $f^*\mathcal{F}_V \to \mathcal{F}_{V'}$

Definition If such a morphism is an isomorphism for each f then $\mathcal{F}$ is a crystal.

We have a structural sheaf $\mathcal{O}_{X/W_n}$. To (U, V) it associates $\mathcal{O}_V$. We then have a topology and then we may define the W_n-modules

$$H^i_{crys}(X/W_n) = H^i((X/W_n)_{crys}, \mathcal{O}_{X/W_n})$$

And the W-modules

$$H^i_{crys}(X/W) = \lim_{\leftarrow} H^i((X/W_n)_{crys}, \mathcal{O}_{X/W_n})$$

It is functorial on X and iit gives a good coh. theory for smooth and projective varieties. The twist is given by multiplying the Frobenius by q where q is the number of elements of k. It is a Weil cohomology.

The proof of the Riemann hyp. for the Frobenius action has been proved using a result of Katz-Messing (Inv. Math. 1974): this was based on the proof for étale. A complete indipendent proof from the étale has been given recently by Kedlaya (Compositio 2006), and announced by Mebkhout.

It looks hard to calculate such a crystalline cohomology. But we have the following result by Berthelot and Ogus (see book): Given X as before, suppose to have a closed immersion $X \to Z$ in a smooth W_n-scheme (not nilpotent). Then we may associate to it a W_n-PD envelope (closed immersion..)

$$J : X \to P_Z(X)$$

which is universal among the PD immersions as before. On Z we have $\Omega^\bullet_Z$. Then we may take the complex

$$\Omega^\bullet_Z \otimes \mathcal{O}_{P_Z(X)} = \Omega^\bullet_{P_Z(X)}$$

We then have

Theorem $\mathbb{H}^\bullet(X, Omega^\bullet_{P_Z(X)}) = H^\bullet_{crys}(X/W_n)$.
If $\mathscr{X}$ is a lifting of X in $\mathbb{Z}_p$ and it is smooth and projective (as well as X), then $\mathscr{X}/p^n$ is a lifting as before for W_n.

A good example in order to understand it would be $Spec\, k \to spec W_n[t]$ as a smooth embedding.

Theorem Suppose that $X_{\mathcal{V}}$ a proper and smooth scheme over $\mathcal{V}$ a DVR with residue field k and fraction field K. Then

$$H^i_{dR}(X_K) = H^i_{crys}(X_k/W(k)) \otimes K$$

(without tensoring: if the ramification of K is $\leq p-1$ if the residue field has ch.p.)

Crystalline co. is a good cohomology for smooth and projective varieties. It is a Weil one. Twist : if $k = \mathbb{F}_q$, $q = p^r$, then the twist given by $Frac(W(k))(1)$ where the inverse of the r-th iterated of the Froebnius acts by multiplication by q^{-1}.

But it doesnt' give a good results for open and non-smooth varieties. This gives to you the opportunity of discussion how to define a good coh. in ch.p but also to come back to ch0.

M-W cohomology

M-W cohomology is just a particular case of the Rigid one. But it contains all the ingredients. It is a de Rham type cohomology. And this

will give us also a hint of how to work out the case non smooth for the de Rham cohomology (i.e. Hartshorne "de Rham Cohomology", IHES 1975).

We want to justify the construction of M-W with an example.

1.) Consider $\mathbb{A}^1_{\mathbb{F}_p} \setminus \{0\} = Spec\mathbb{F}_p[x, x^{-1}]$. If we take directly de Rham in ch$p$. Then a lot of constants: $d(x^p) = 0$.

2. Then one try to make a lift on $\mathbb{Z}_p$. It is not unique

$$Spec\frac{\mathbb{Z}_p[x, y]}{(xy - 1)} \quad Spec\frac{\mathbb{Z}_p[x, y, z]}{(xy - 1, xz - (1 + p))}$$

which one? and moreover x^{p-1} is not integrable...

3.) A way of getting rid of the two previous problems will be to tensor by $\mathbb{Q}_p$ and more interersting to take the p-adic completion of the polynomials rings. So we will indicate by $\mathbb{Z}_p < x_1, \ldots, x_s >$ the p-adic completion of the polynomial ring $\mathbb{Z}_p[x_1, \ldots, x_s]$. I.e. the set of power series in X_i such that

$$\sum_{\alpha \in \mathbb{N}^s} a_\alpha x^\alpha \quad , \lim |a_\alpha|_p = 0 \quad as\, |\alpha| \to +\infty$$

Then

$$\frac{\mathbb{Q}_p < x, y >}{(xy - 1)} \simeq \frac{\mathbb{Q}_p < x, y, z >}{(xy - 1, xz - (1 + p))}$$

But again if we take $\mathbb{A}^1_{\mathbb{F}_p}$ then the lift will be $\mathbb{Q}_p < x >$ and $\sum p^{i-1} x^{p^i - 1}$ cannot be integrated..... H^1 would be too large.

The idea is to leave the radious free.not just one.

Definition Consider $A_{\mathbb{F}_p} = \mathbb{F}_p[x_1, \dots x_s]/A$ such that its *spec*, X is a smooth affine variety. Then by Elkik we may find $\mathcal{A}_{\mathbb{Z}_p} = \mathbb{Z}_p[x_1, \dots x_s]/\mathscr{A}$ a smooth $\mathbb{Z}_p$-variety. Call it $X_{\mathbb{Z}_p}$ is affine and smooth. The de Rham complex of $\mathcal{A}_{\mathbb{Z}_p}$ will be $\Omega^\bullet_{\mathcal{A}_{\mathbb{Z}_p}}$. Then we take the weak completion of $\mathcal{A}$, $\mathcal{A}^\dagger_{\mathbb{Q}_p}$ as

$$\mathbb{Z}_p\{x_1, \dots x_s\}^\dagger/\mathscr{A}$$

where $\mathbb{Z}_p\{x_1, \dots x_s\}^\dagger$ is the set of all $\sum_{\alpha \in \mathbb{N}^s} a_\alpha x^\alpha$ $a_\alpha \in \mathbb{Q}_p$, such that there exists $\epsilon > 0$

$$\lim |a_\alpha|_p(1+\epsilon)^{|\alpha|} = 0 \quad as\, |\alpha| \to +\infty$$

We may then take

$$\Omega^\bullet_{\mathcal{A}_{\mathbb{Z}_p}} \otimes \mathcal{A}^\dagger_{\mathbb{Q}_p}$$

The M-W cohomology of X is

$$H^i_{MW}(X/\mathbb{Q}_p) = H^i(\Omega^\bullet_{\mathcal{A}_{\mathbb{Z}_p}} \otimes \mathcal{A}^\dagger_{\mathbb{Q}_p}).$$

The finiteness and the indipencence upon the lifting of such a coh. theorty have been proved in the framework of the Rigid cohomology. Rigid coh. which coincides with the crystalline on the proper and smooth case and with M-W when smooth and affine, but it can be defined for all varieties.

The idea at the basis of M-W cohomlogy is to embed the variety in a compactification which is smooth and then to think about the behaviour at the complement.

For an introduction to Rigid cohomology see book Le Stum "Rigid cohomology" (cambridge univ. press 2007) The finiteness for such a coh. was proved in the late 97 (inv.math vol.128) by Berthelot using de Jong alteration.
Rigid coh. theory works for non proper and non smooth varieties and satisfies a series of axioms which characterize the Bloch-Ogus formalism. On the other hand the techniques we have introduced for the ch*p* case , will help us to define some Bloch-Ogus coh. also in ch.0.

Bloch-Ogus axioms

As for the Weil coh. theories we are going to discuss the various axioms which forms a Bloch-Ogus formalism. They will cover the Poincaré duality for also for non smooth varieties...and they will introduce homology..

Definition Bloch-Ogus Let $\mathscr{V}$ a category of algebraic schemes. $\mathscr{V}^*$ the category of couple (X, Y) with a closed immersion $X \to Y$, and morphisms: cartesian squares.

1) A twisted cohomological theory with support. I.e. a sequence of controvariant functors

$$\mathscr{V}^* \to \text{"}\textit{graded abelian groups}\text{"}$$

to $X \to Y$ one associates $\oplus_{i \in \mathbb{Z}} H^i_X(Y, n)$, $n \in \mathbb{N}$. Such that if we have a series of closed immersions $Z \to X \to P$ then we have a long exact sequence

$$\cdots \to H^i_Z(Y, n) \to H^i_X(Y, n) \to H^i_{X \setminus Z}(Y \setminus Z, n) \to H^{i+1} \ldots$$

If $X \to Y$ in $\mathscr{V}^*$ and $U \subset Y$ open containing the immage of X then $H^i_X(Y, n) = H^i_X(U, n)$.

2) A twisted homology theory on $\mathscr{V}_*$, where $ob\mathscr{V} = ob\mathscr{V}_*$. but the maps are only proper. We then have a **co**variant functor

$$\mathscr{V}_* \to \bigoplus_{i \in \mathbb{Z}} H_i(X, n)$$

(again on graded abalian groups). **Contro**variant for étale maps. If we have a cartesian

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha} & Y \end{array}$$

where f, g proper and α, β étale. Then we have an associated diagram for the homology groups.

Moreover if $(X, Y) \in \mathscr{V}^*$ and $\alpha : Y \setminus X \to Y$ the relative open immersion then

$$\cdots \to H_i(X, n) \to H_i(X, n) \to H_i(Y \setminus X, n) \to H_{i-1} ...$$

We denote such a long exact sequence by $H_\bullet(X, Y, Y \setminus X)$.
Start now with $f : Y' \to Y$ proper , X' closed in Y', then $X = f(X')$ and $\alpha : Y' \setminus f^{-1}(X) \to Y' \setminus X'$
then we have a commutitive diagramm from $H_\bullet(X', Y', Y' \setminus X')$ to $H_\bullet(X, Y, Y \setminus X)$ (make it.)

3) All the previous data will satisfy
i) A pairing with support. For $(X, Y) \in \mathscr{V}^*$ then

$$\cap : H_i(X, m) \otimes H^j_X(Y, n) \to H_{i-j}(Y, m-n)$$

Such that if
-$(X, Y) \in \mathscr{V}^*$ and $(\beta, \alpha) : (X', Y') \to (X, Y)$ is an etale map.Then for $a \in H_X(Y, n)$ e $z \in H_i(X, m)$ then

$$\alpha^*(a) \cap \alpha^*(z) = \beta^*(a \cap z)$$

-If f is proper in $\mathscr{V}^*$, $f : (X_1, Y_1) \to (X_2, Y_2)$. $\forall\, a \in H^j_{X_2}(Y_2, n)$ and $z \in H_i(Y_1, m)$ then

$$H_i(f)(z) \cap a = H_i(f_X)(z \cap H^j(f)(a))$$

where? (projection formula)

-*fundamental class* If $X \in ob\mathscr{V}$ irreducible of dimension d, there exists $\eta_X \in H_{2d}(X, d)$.
If $\alpha : X' \to X$ étale, then $\alpha^* \eta_X = \eta_{X'}$.

-*Poincaré duality* If $(X, P) \in \mathscr{V}^*$ with P smooth of $dim\, P = d$. Then the cap product gives an iso

$$\eta_P \cap - : H_X^{2d-i}(d-n) \to H_i(Y, n)$$

Remark Nothing has been asked about to have an ordinary cohomology. Note that we ask only the existence of a homology theory. Nothing about the compact support cohomology.

Examples. We have several kind of examples. The étale cohomology we start with varieties defined over a field k not algebraically closed (bloch-Ogus Annales S. Ec. Normale Sup. 1974, §2). The for l prime to ch. of k, $\nu \in \mathbb{N}$. We may consider η the étale sheaf of l^ν -th roots of unity on $Spec\, k$ and let $\eta^n = \eta \otimes \dots \eta$, $\eta^{-n} = Hom(\eta^n, \mathbb{Z}/l^\nu\mathbb{Z})$. For X a variety, we define π_X the structural map to $Spec\, k$. We then define

$$H^i(X, n) = H^i_Y(X, \pi_X^* \eta^n) \quad H_i(X, n) = H^{-i}(X, \pi_X^! \eta^{-n})$$

Note that we don't ask k algebraically closed. In case it were algebraically closed we would have $H^i(X, n) = H^i_Y(X, \pi_X^* \eta^n) = H^i(X, \mathbb{Z}/l^\nu\mathbb{Z}) \otimes \mathbb{Z}/l^\nu\mathbb{Z}(n)$ (not a shift by degree...?!!??) Twist in Bloch-Ogus it is not a iso! it changes the space not only its strurture letting an iso without structure.

What is the meaning of $^!$? Compact support in étale? If we define for a sheaf as the section having support over all complete subvarieties. In the Zaariski setting this is a problem. In fact if U is affine over an alg. closed field then the only complete subvarieties are the finite ones. And then $H_Z(U, \mathcal{F}) = \oplus \mathcal{H}_\ddagger(\mathcal{U}, \mathcal{F})$ and $z \in Z$(finite). Hence the limit is even not finite...if we take value in a constant sheaf. If $dimU = d$, we have only $H_z^{2d}(X, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ and we have to take the sum all over the point...it is not finite.
We are left with another definitiion: we take for U a compactification(by Nagata it exists) X and then we take for any torsion sheaf $j_!$= extension by 0. Then

$$H_c^i(U, \mathcal{F}) = H^i(X, j_! \mathcal{F})$$

it is not the derived functor of $j_!$ (pag.165 milne's book).

Examples Rigid cohomology for variety in $ch = p$. we have compact support cohomology and the homology is defined as dual and the twist is just the shift of the Frobenius (see Le Stum's book).

Example $k = \mathbb{C}$, then algebraic varieties of finite type over $\mathbb{C}$. Then we may take the classical sheaf cohomology on the constant sheaf $\mathbb{Z}$ for the analytic topology. And then the homology is the Borel-Moore homology i.e. a sort of dual of the compact support cohomology (now we have the functor compact support...). No shift. see Iversen's book and Verdier duality for paracompacts spaces. $\mathbb{Z}$ can be replaced by any ring.

Example Le last example I want to deal with is $ch.k = 0$ and we take teh category $\mathscr{V}$ of all schemes which are embeddable in a smooth scheme over k. Then Hartshorne (IHES 1974) defined a cohomology "de Rham" as $X \to P$, P smooth, then one takes $\Omega^\bullet_{P\hat{|}X}$ the de Rham complex of P completed along X. A kind of "tube" around X.And then

$$H_{dR}(X) = \mathbb{H}(X, \Omega^\bullet_{P\hat{|}X})$$

-$Speck = X \to Speck[t]$....

Then $H_{dr}(X) \otimes \mathbb{C} = H(X^{an}_{\mathbb{C}}, \mathbb{C})$. Then Hartshorne was able to define homology without defining compact support

$$H^{dR}_q(X) = H^{2n-q}_X(P, \Omega^\bullet_{P\hat{|}X})$$

The twist is trivial. In particular $H^{dR}_i(X) \otimes \mathbb{C} = H^{BM}_i(X^{an}_{\mathbb{C}}, \mathbb{C})$. by the way: the stupid filtration in $\Omega^\bullet_{P\hat{|}X}$ doesn't give any good filtration

In case we have a Weil Cohomology (coefficients in a field of ch.0) which extends to a Bloch-Ogus one with compact support cohomology then for such a cohomology retaining the twist from bloch-ogus we would have

$$H_Y^{2d-i}(X) \otimes H_c^i(Y) \to H^{2d}(X) \to K(-d)$$

Then $H_Y^{2d-i}(X) = Hom(H_c^i(Y), K(-d))$. Hence if we define $H_i(Y, n) = (H_c^i(Y, n))^\vee$ we conclude to have $H_Y^{2d-i}(X)(d-n) = H_Y^{2d-i}(X, d-n) = H_i(Y, n)$

In the article Gillet "Riemann-Roch theorems....", Adv.Math. 40, 203-289(1981), it is presented a version of Bloch-Ogus axioms by associating to a category of schemes $\mathscr{V}$ for any n, a complex of sheaves: $\underline{\Gamma}^*(n)$ which is a sheaf on $\mathscr{V}_{Zar}$.Such that for X, we will have $H^i(X,n) = \mathbb{H}(X, \underline{\Gamma}^*(n))$. In particular, we will speak about a graded coh. theory.

$$\oplus_{i\in\mathbb{Z}}\underline{\Gamma}^*(i)$$

Among the examples the Higher Chow groups.....

Let's put together all the compatibilities we have found so far. We will try to explain what is a system of realizations and which kind of comparisons we expect.
We won't expect to be complete. We refer to Deligne's article "le group fondamental de la droite projective moins .." in the book about the Galois's group of $\overline{\mathbb{Q}}/\mathbb{Q}$.
What we should have in mind is a variety over $\mathbb{Z}$. Then it is clear that we may take its complexification which we may think to be smooth. On the other hand sometimes its reduction *mod p* is not good...even if the complexification was good.

What do we mean for a system of realizations?

1) a $\mathbb{Q}$ vector space, M_B the *betti realization*

2)M_{dR} over $\mathbb{Q}$ the de Rham realization

3) for all l prime a $\mathbb{Q}_l$-module the l-adic (étale) realization; M_l

4)for almost all p a $\mathbb{Q}_p$ vector space $M_{crys,p}$ (crystalline realization=rigid)

5) we have comparison $M_B \otimes \mathbb{C} \simeq M_{dR} \otimes \mathbb{C}$, $M_B \otimes \mathbb{Q}_l \simeq M_l$, $M_{crys,p} \simeq M_{dR} \otimes \mathbb{Q}_p$.

6) $M_B, M_{dR}, M_l, M_{crys}$ have increasing filtartions called weights. And the comparison should respect them.

7) in $M_{dR} \otimes C$ we have a Hodge filtration decreasing $F^{\bullet}$

8) In M_l we have an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ which respects $W_{\bullet}$.

9) M_{crys} has a Frobenius, respecting weights.. ....

-) $(M_B, W_{\bullet}, F^{\bullet})$ is a Mixed Hodge Structure.
.... -) For $l \neq p$ the eigenvalues of the Frobenius at p on $Gr_n^W(M_l)$ and theose of the Frobenius in $Gr_n^W(M_{crys})$ are all algebraic numbers and they concide... i.e. we are talking about indipendence on l. This is not known even in the case where we start with a open variety over $\mathbb{F}_p$ for the l-adic for different l....... (see Katz's discussion on the volume Motives conference in seattle 1991).

What about the p with bad reduction? or a family of varieties...Limit Hodge Structure..