Cohomology theories IV

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Road Map

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We are not going to give a complete discussion. What we want to show is the fact that the classical cycles maps from the Bloch-Ogus axioms are going to give (together with their compatibilities) the notion of *regulator* from motivic cohomology to some kind of other cohomologies.

We recall the notion of cycle class. If Z is closed in a smooth X and of cod.c, then we may associate to it a class (we are in Bloch-Ogus..) $\eta_Z \in H_{2n-2c}(X)(n-c)$ ($n = dimX$). We then have a duality

$$H^{2c}_Z(X)(c) \simeq H_{2n-2c}(X)(n-c)$$

hence we end in associating to Z and element η_Z of $H^{2c}(X)(c)$ (this if X is smooth. In general we will end in homology...). Hence we end in a cohomology group but also in a part given by the twist.

In case we are over $\mathbb{C}$ then we will have a Betti(topological class) in $(2\pi i)^c\mathbb{Z}$ and a de Rham class in the F^p of the Hodge filtratiion.. And they are compatible (up to sign. deligne Hodge II).

Remark. We may also try to be brave, and to take a variety over a DVR, $\mathcal{V}$, of mixed characteristic (k, K), $X_{\mathcal{V}}$. We suppose it smooth. We take Z a closed subscheme flat over $\mathcal{V}$ (hence equidimensional). It has two fibers Z_k in ch.p and Z_K in ch.0. We have two classes: a rigid one for Z_k: η_k and a de Rham one η_K.

Then there is a natural map $sp : H^i_{dR}(X_K) \to H^i_{rig}(X_k)$ which is not an iso in general, but it sends one class to the other! they are compatible.

It has been proved by Messing-Gillet (Duke 55(1977)) for proper and smooth: then the rigid was the crystalline one and the crystalline cohomology was iso to de Rham one. Recently by chiar.-Ciccioni-Mazzari in full generalities. Note that even in the smooth case the map between rigid and de Rham is not iso.

Why did we introduce cycles? Because we want to present the Motivic-Cohomology using the Higher Chow groups and then the *regulator* maps would be generalizations of the cycle maps.

We follow in order to construct Higher Chow groups $CH^*(X,*)$ Bloch's construction (see Adv. Math. 61(1986) "Algebraic cycles and higher K-theory").

We define $Z^*(Y)$ as the free abelian group graded by codimension $*$ with generators the irreducible closed subvarieties of Y. Z^* is a covariant for proper morphism and contro for flat....a kind of homology.

For each i contruct a complex $Z^i(X,\bullet)$ whose j-th cohomology would be $CH^i(X,j)$. It is a complex made by cycles in $X \times \Delta_\bullet$ of codimesnion i meeting in the good way the faces. In particular $Z^*(X,0) = Z^*(X)$ the usual cycles...and $CH^*(X,0) = CH^*(X)$. i.e. up to rational equivalence.

Properties.

Functoriality co-variant for proper , controvar. for flat.
we have homotopy $CH^*(X, \bullet) = CH^*(X \times \Delta_n, \bullet)$.....multiplicativity.
It turns out it has enough axioms to have a map from K-theory via higher Chern classes from the Higher K-groups (Quillen) to them.
Actually an iso: I.e. a map

$$\oplus_i gr^i_\gamma G_n(X)_\mathbb{Q} \simeq G_n(X)_\mathbb{Q} \simeq \oplus_i CH^i(X, n)_\mathbb{Q}$$

as it has been proved by Bloch (ibidem) Using Gillet work.

We may indicate $H^{p,q}(X) = CH^q(X.2q - p)$ and then

$$H^{p,q}(X) = Hom_{DM^{eff}_{Nis}(k)}(\mathbb{Z}_{tr}(X), \mathbb{Z}_{tr}(q))[p]$$

Hence the higher Chow groups can be seen as *Hom* (...*Ext*) in an appropriate category. We want to see how to extend the cycles class to the Higher Chow groups hence defining regulators

Consider a n-dimensional algebraic manifold X over $\mathbb{C}$. It admits a good compactification

$$X \to \overline{X}$$

where $\overline{X} \setminus X$ is a NCD. We the take $\Omega^\bullet_{\overline{X}} < log >$ along such a divisor. We know that

$$\mathbb{H}^\bullet(\overline{X}, \Omega^\bullet_{\overline{X}} < log >) = H^\bullet_{dR}(X)$$

We know moreover that the stupid filtrartion $\{\Omega^{\geq p}_{\overline{X}} < log >\}_p$ will induced in coh. the Hodge filtration. We call

$$F^p_{\mathscr{D}} = [0 \cdots \to \Omega^p_{\overline{X}} < log > \to \Omega^{p+1}_{\overline{X}} < log > \to \dots$$

the natural maps

$$\mathbb{H}^\bullet(\overline{X}, F^p_{\mathscr{D}}) \to H^\bullet_{dR}(X)$$

are all injective and they will indude the Hodge filtartion $F^p H^\bullet_{dR}(X)$. We can use Zariski topology to do that.

We have defined $\mathbb{Z}(p) = (2\pi i)^p\mathbb{Z}$. We have a natural map of complexes of sheaves on X (analytic)

$$\mathbb{Z}(p) \to \Omega^\bullet_X$$

We may take injective resolutions and then to have a map of complexes

$$\mathbb{R}j_*\mathbb{Z}(p) \to \mathbb{R}j_*\Omega^\bullet_X \simeq j_*\Omega^\bullet_X$$

because j is Stein. acyclic. . Now we have a natural map

$$\epsilon : \Omega^\bullet_{\overline{X}} < log > \to j_*\Omega^\bullet_X$$

(a quasi-iso) Hence we get a map of complexes $\iota : F^p_{\mathscr{D}} \to \mathbb{R}j_*\Omega^\bullet_X$

If we have a map of complexes $u : A^\bullet \to B^\bullet$ we may define the cone.

Lemma. Suppose to have $u_1 : A_1^\bullet \to B^\bullet$ and $u_2 : A_2^\bullet \to B^\bullet$ maps of complexes in abelian categories. Then one may define $C^\bullet = Cone[A_1^\bullet \oplus A_2^\bullet \xrightarrow{u_1 - u_2} B^\bullet][-1]$. We then have

$$... \to H^q(C^\bullet) \to H^q(A_1^\bullet) \oplus H^q(A_2^\bullet) \to H^q(B^\bullet) \to ...$$

We then define the Beilinson-Deligne complex (A a subring of $\mathbb{R}$..think about $\mathbb{Z}$) as

$$A(p)_{\mathscr{D}} = Cone[\mathbb{R}j_* A(p) \oplus F^p_{\mathscr{D}} \xrightarrow{\epsilon-\iota} \mathbb{R}j_*\Omega^\bullet_X][-1]$$

and then the cohomology

$$H^q_{\mathscr{D}}(X, \mathbb{Z}(p)) = \mathbb{H}^q(X, A(p)_{\mathscr{D}})$$

It fits in a long exact sequence

$$...H^{q-1}(X, \mathbb{C}) \to H^q_{\mathscr{D}}(X, \mathbb{Z}(p)) \to H^q(X, \mathbb{Z}(p)) \oplus F^p H^q_{dR}(X, \mathbb{C}) \to H^q(X, \mathbb{C})..$$

Then if we have a cycle of codimension p in X, we then have a classes $(\eta_{B,Z}, \eta_{dR,Z}) \in H^{2p}_B(X, \mathbb{Z}(p)) \oplus F^p H^{2p}_{dR}(X)$.
which coincide in $H^{2p}(X, \mathbb{C})$. Then they should give a unique element in the Deligne cohomology. ..

As a matter of fact one can prove that Deligne-Beilinson cohomology has plenty of properties: it has Homology version, it has a Poincaré' duality..

see the articles of Esnault-Viehweg and Jannsen in "Beilinson Conjectures on Special Values of L-functions". Moreover D-B coh. has also: homotopy invariants... Then it has a natural map

$$CH^p(X, \mathbb{Z}(q)) \to H^p_{\mathscr{D}}(X, \mathbb{Z}(q))$$

(see Deininger-Scholl "The Beilinson Cohomology", London Math. Soc. Lect. Notes 153(1991)).

Remark It has been proved by Beilinson (Notes on absolute Hodge theory, cont.math.vol.55, partI, 1986) that such a Deligne cohomology can be seen as an absolute cohomology in the derived category of the exact category of mixed hodge structure $D^b(\mathscr{H})$. In fact to X one can associate an element of this category (category which admits shift): $\underline{\mathbb{R}}\Gamma(X, \mathbb{Z})$. Then

$$H^i_{\mathscr{D}}(X, \mathbb{Z}(j)) \simeq \mathbb{R}^i Hom_{D^b(\mathscr{H})}(\mathbb{Z}, \underline{\mathbb{R}}\Gamma(X, \mathbb{Z})(j))$$

where $\mathbb{Z}$ is the "trivial Hodge structure". As we saw Voedvosky gave a similar interpretation of the motivic cohomology (i.e. as an Hom) but in contro-variant terms.