STABLE HOMOTOPY THEORY

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CONTENTS

1. SPECTRA AND THE STABLE HOMOTOPY CATEGORY

1.1. **Topological spaces and the pointed homotopy category.** To avoid point-set pathology, the foundations of homotopy theory should be developed using a *good* category of topological spaces (see [May99, Chapter 5] or [Gra75, Chapter 8], for example). Take $\mathscr{T}$ to be the category of compactly-generated, weak Hausdorff spaces. (A space X is weak Hausdorff if the image $g(K)$ if closed for any compact space K and continuous map $g : K \to X$; a weak Hausdorff space X is compactly-generated if $F \subset X$ is closed $\Leftrightarrow F \cap K$ is closed for every compact $K \subset X$. There is a functor k from weak Hausdorff spaces to compactly-generated weak Hausdorff spaces which is right adjoint to the forgetful functor, equipping the space with the induced compactly-generated topology.)

In particular, the product $X \times Y$ of two topological spaces from $\mathscr{T}$ is $k(X \times Y)$ the function space $F(X, Y)$ is $k(\mathrm{Map}(X, Y))$, where the mapping space is given the compact-open topology.

Remark 1.1.

(1) The foundations can be developed by using CW-complexes, which are, in particular, compactly-generated topological spaces. (A CW complex is a particular form of cell complex; a cell e^n of dimension n is a Euclidean ball of dimension n and boundary the sphere S^{n-1} is attached to a space X by means of a attaching map $f : S^{n-1} \to X$ by forming the coproduct $X \cup_f e^n$.)

(2) An alternative approach is to use simplicial sets to give a combinatorial model for topological spaces. There is a topological realization functor which is left adjoint to the singular simplicial set functor:

$$|-| : \Delta^{\mathrm{op}}\mathfrak{Set} \rightleftarrows \mathscr{T} : \mathrm{Sing}$$

and the homotopy theory of topological spaces can be developed using simplicial sets [Qui67, GJ99].

Definition 1.2. Let

(1) $\mathscr{T}_\bullet$ denote the category of pointed (compactly-generated) topological spaces;

(2) $(-)_+ : \mathscr{T} \to \mathscr{T}_\bullet$ be the left adjoint to the functor which forgets the base-point, so that $Y_+ = Y \cup *$, pointed by $*$;

(3) $\vee : \mathscr{T}_\bullet \times \mathscr{T}_\bullet \to \mathscr{T}_\bullet$ denote the *wedge*, defined for pointed spaces X, Y by

$$X \vee Y := X \amalg Y / *_X \sim *_Y,$$

which is the categorical coproduct in $\mathscr{T}_\bullet$;

(4) $\wedge : \mathscr{T}_\bullet \times \mathscr{T}_\bullet \to \mathscr{T}_\bullet$ be the *smash product*:

$$X \wedge Y := (X \times Y)/\{*_X \times Y \cup X \times *_Y\}.$$

(5) $F_\bullet(-,-) : \mathscr{T}_\bullet^{\mathrm{op}} \times \mathscr{T}_\bullet \to \mathscr{T}_\bullet$ be the *pointed function space* of base-point preserving maps, pointed by the constant map.

Example 1.3.

(1) The interval $I = [0, 1]$ gives rise to the pointed interval I_+.

(2) The 0-sphere is $S^0 \cong *_+$ and the circle S^1 is homeomorphic to $I/0 \sim 1$, pointed by the image of the endpoints. The n-sphere is homeomorphic to the n-iterated smash product $(S^1)^{\wedge n}$.

Remark 1.4. To perform homotopy theory, a *cofibrancy* condition on pointed spaces is often required; this corresponds to $(X, *)$ being *well-pointed* or the inclusion $* \hookrightarrow X$ being a *NDR*.

Proposition 1.5. *The category* $(\mathscr{T}_\bullet, \wedge, S^0)$ *is closed symmetric monoidal, with internal function object* $F_\bullet(-,-) : \mathscr{T}_\bullet^{\mathrm{op}} \times \mathscr{T}_\bullet \to \mathscr{T}_\bullet$.

In classical homotopy theory, two morphisms are homotopic if they are related by a continuous deformation, which is parametrized by the interval I. In the pointed situation, all morphisms respect the basepoint, and the space $X \wedge I_+$ plays the rôle of a *cylinder object*. Morphisms $f, g : X \to Y$ of $\mathscr{T}_\bullet$ are (left) homotopic if there exists a *homotopy* $H : X \wedge I_+ \to Y$ which makes the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{i_0} & X \wedge I_+ & \xleftarrow{i_1} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle g} & \\ & & Y, & & \end{array}$$

where i_0, i_1 are induced by the inclusions of the endpoints of I. Write $f \sim g$ (respectively $f \sim_H g$) to indicate that f and g are pointed homotopic (resp. by H).

Remark 1.6. The notion of right homotopy is defined using the path space $F_\bullet(I_+, Y)$.

Homotopy defines an equivalence relation; the set of (pointed) homotopy classes of maps from X to Y is denoted $[X, Y]$. Composition of maps induces composition:

$$[Y, Z] \times [X, Y] \to [X, Z].$$

This leads to the notion of homotopy equivalence:

Definition 1.7. Two pointed spaces X, Y are *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

A morphism $f : X \to Y$ which admits a homotopy inverse (as above) is called a *homotopy equivalence*.

There is a weaker notion of equivalence, defined using homotopy groups.

Definition 1.8. For n a natural number and X a pointed topological space, let $\pi_n(X)$ denote $[S^n, X]$.

The set $\pi_0(X)$ is the set of path components of X; $\pi_1(X)$ is the fundamental group of X, with group structure induced by the composition of paths, and $\pi_n(X)$ has a natural abelian group structure for $n > 1$, which is induced by the pinch map $S^n \to S^n \vee S^n$ which collapses the equator to a point.

Definition 1.9. A morphism $f : X \to Y$ of $\mathscr{T}_\bullet$ is a weak homotopy equivalence if, for every choice of basepoint of X, the morphism $\pi_n(f)$ is a bijection $\forall n$. (If X is path-connected, the basepoint condition can be omitted.) The class of weak equivalences in $\mathscr{T}_\bullet$ is written $\mathscr{W}$.

Remark 1.10. A homotopy equivalence is a weak homotopy equivalence but, without restricting the class of spaces considered, the converse is false.

Definition 1.11. The *homotopy category of pointed topological spaces* is the localized category

$$\mathscr{H}_\bullet := \mathscr{T}_\bullet[\mathscr{W}^{-1}]$$

obtained by inverting the class of weak equivalences.

There is a more concrete description of the category $\mathscr{H}_\bullet$ obtained by using the *ad theorem*: if ${\mathscr{T}_\bullet}^{\mathrm{CW}}$ is the subcategory of pointed CW-complexes and cellular maps, then there is an equivalence of categories

$$\mathscr{H}_\bullet \cong {\mathscr{T}_\bullet}^{\mathrm{CW}} / \sim,$$

the category obtained by passing to homotopy classes of pointed maps. (This justifies the terminology *homotopy category*.)

Remark 1.12. The above result can be understood in the framework of abstract homotopy theory [Qui67, Hov99, DS95] as follows:

(1) the pointed CW-complexes are *cofibrant* in the category $\mathscr{T}_\bullet$ (and all objects are *fibrant*);
(2) in the framework of simplicial sets, all objects are *cofibrant* but the *fibrant* objects are those satisfying the *Kan extension condition* [Qui67, GJ99].

Proposition 1.13. *The smash product passes to the pointed homotopy category and defines a symmetric mondoidal category* $(\mathscr{H}_\bullet, \wedge, S^0)$.

Remark 1.14. The interchange map $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is a homotopy equivalence but is not homotopic to the identity: it is homotopic to the morphism $S^2 \to S^2$ of degree -1. More generally, the interchange of the factors $S^p \wedge S^q \cong S^{p+q}$ is of degree $(-1)^{pq}$. This is the origin of the signs which occur in classical algebraic topology.

More general signs appear when considering *generalized* spheres; for instance:
(1) in G-equivariant homotopy theory, for a finite group G, one considers *representation spheres* of the form S^V, the one-point compaticification of an orthogonal G-representation V;
(2) in motivic homotopy theory, spheres of the form $S_s^{\wedge m} \wedge \mathbb{G}_m^{\wedge n}$ are considered.

1.2. **Stabilization.** The reduced suspension functor $\Sigma : \mathscr{T}_\bullet \to \mathscr{T}_\bullet$ is the functor $\Sigma X := S^1 \wedge X$; observe that $\Sigma S^n \cong S^{n+1}$, for $n \in \mathbb{N}$. The functor Σ has right adjoint $\Omega : \mathscr{T}_\bullet \to \mathscr{T}_\bullet$, $X \mapsto F_\bullet(S^1, X)$, the based loop space. These functors induce an adjunction at the level of the pointed homotopy category

$$\Sigma : \mathscr{H}_\bullet \leftrightarrows \mathscr{H}_\bullet : \Omega.$$

Remark 1.15. In general, when passing to the homotopy category, it is necessary to consider *derived* functors. For instance, if using pointed simplicial sets as the underlying category, to define the *derived* loop space functor, one must first replace a simplicial set by an equivalent Kan simplicial set.

The adjunction $\Sigma \dashv \Omega$ has unit $X \to \Omega\Sigma X$, which induces the natural suspension morphism

$$\sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X).$$

This is not an isomorphism in general, but is one in dimensions small with respect to the *connectivity* of X. For instance, there is the following special case of the *Freudenthal suspension theorem*.

Theorem 1.16. *The suspension morphism*

$$\pi_r(S^n) \to \pi_{r+1}(S^{n+1})$$

is an isomorphism for $r < 2n - 1$ *and a surjection if* $r = 2n - 1$.

Stable homotopy theory is the study of properties of $\mathscr{H}_\bullet$ which are *stable under suspension*. The Freudenthal suspension theorem motivates studying the *stable homotopy groups*, defined as follows:

Definition 1.17. The stable homotopy group $\pi_n^S(X)$ of a pointed topological space X, for $n \in \mathbb{Z}$, is

$$\pi_n^S(X) := \lim_{\to} \pi_{n+k}(\Sigma^k X).$$

Remark 1.18. The *compact* objects of $\mathscr{H}_\bullet$ are the objects represented by finite pointed CW-complexes (ie having a finite number of cells). The compact objects form a full subcategory of $\mathscr{H}_\bullet$ and it is possible to define the stabilization of this category by the *Spanier-Whitehead* construction. This category has objects the finite pointed CW-complexes and morphisms

$$\{X, Y\} := \lim_{\to} [\Sigma^k X, \Sigma^k Y].$$

This is not sufficient for studying cohomology theories, since the representing objects for interesting theories are seldom compact.

1.3. **Cohomology theories.** The pointed homotopy category $\mathscr{H}_\bullet$ is not an additive category; it does, however, have additional structure reminiscent of that of a triangulated category, with Σ playing the rôle of suspension. (Note that Σ is *not* an equivalence of categories.)

Definition 1.19. For $f : X \to Y$ a map of $\mathscr{T}_\bullet$, let $Y \to Cf$ denote the *mapping cone* or *homotopy cofibre* of f:

$$Cf := Y \cup_f CX,$$

where $CX := X \wedge I_+ / X \wedge 1_+$ is the reduced cone and the gluing is defined with respect to $Y \stackrel{f}{\leftarrow} X \hookrightarrow CX$, equipped with the canonical inclusion $Y \hookrightarrow CX$.

This is an example of a *homotopy colimit*; in particular, if $f \sim g$, then $Cf \sim Cg$. Moreover

(1) the composite $X \stackrel{f}{\to} Y \to Cf$ is *null-homotopic* (homotopic to the constant map);
(2) a pointed map $g : Cf \to Z$ is equivalent to a map $Y \to Z$ together with a null-homotopy $H : X \wedge I_+ \to Z$ for the composite $X \stackrel{f}{\to} Y \hookrightarrow Cf \stackrel{g}{\to} Z$.

The homotopy cofibre of $Y \hookrightarrow Cf$ is homotopy equivalent to ΣX and this gives rise to a sequence in $\mathscr{H}_\bullet$:

$$X \stackrel{f}{\to} Y \to Cf \to \Sigma X \stackrel{-\Sigma f}{\to} \Sigma Y \to \Sigma Cf \to \ldots.$$

Definition 1.20. A (reduced) cohomology theory is a functor $E^* : \mathscr{H}_\bullet^{\mathrm{op}} \to \mathfrak{Ab}^{\mathrm{gr}}$, with values in graded abelian groups, which satisfies the following axioms:

(1) (exactness) for any $f : X \to Y$, the sequence

$$E^*(Cf) \to E^*(Y) \stackrel{E^*(f)}{\to} E^*(X)$$

is exact;
(2) (stability) there is a natural suspension isomorphism $\sigma : E^*(-) \stackrel{\cong}{\to} E^{*+1}(\Sigma -)$;
(3) (additivity) for any set $\{X_i | i \in \mathscr{I}\}$ of pointed topological spaces, the natural morphism

$$E^*(\bigvee_{\mathscr{I}} X_i) \to \prod_{\mathscr{I}} E^*(X_i)$$

is an isomorphism.

Remark 1.21. The usual *homotopy* axiom is subsumed in the hypothesis that E^* is a functor defined on the pointed homotopy category.

Example 1.22.

(1) For $A \in \mathfrak{Ab}$ an abelian group, (reduced) singular cohomology $X \mapsto H^*(X; A)$ is a cohomology theory. It is an *ordinary* cohomology theory in the sense that it factors across a functor to the derived category of abelian groups

$$\mathscr{H}_\bullet \to \mathscr{D}\mathfrak{Ab}$$

induced by the singular chains functor.
(2) Complex K-theory $X \mapsto KU^*(X)$ is a cohomology theory. In degree zero, for Y a finite complex, $KU^0(Y_+)$ is the group completion of the monoid $\mathrm{Vect}(Y)$ of $\mathbb{C}$-vector bundles on Y, with sum induced by Whitney sum of vector bundles.

The fact that KU^0 extends to a cohomology theory is a consequence of the Bott periodicity theorem. Namely, for X a pointed space, there is a natural isomorphism $KU^0(X) \cong KU^0(\Sigma^2 X)$. The corresponding cohomology theory is 2-periodic.

(3) Orthogonal (or real) K-theory KO^* is defined by using $\mathbb{R}$-vector bundles; in this case the associated cohomology theory is 8-periodic, by the orthogonal version of the Bott periodicity theorem.
(4) Cobordism theories, such as complex cobordism MU^* (see later).

Remark 1.23. If E^* is a cohomology theory, then so is the functor $E^*[n]$, defined by $E^*[n](-) := E^{*+n}(-)$, for any integer n.

Definition 1.24. A stable cohomology operation of degree n from E^* to F^* is a natural transformation $E^* \to F^*[n]$ of functors $\mathscr{H}_\bullet^{\mathrm{op}} \to \mathfrak{Ab}^{\mathrm{gr}}$ which is compatible with the suspension isomorphisms.

Example 1.25.
(1) The Bockstein defines a cohomology operation $\beta : H^*(-;\mathbb{F}_p) \to H^*(-;\mathbb{F}_p)$.
(2) There is a stable operation $KO^* \to KU^*$, which is induced by complexification.

Proposition 1.26. *Cohomology theories and stable cohomology operations form a (graded) additive category, equipped with an equivalence of categories* [1].

The *Milnor short exact sequence* gives a way of calculating the cohomology of a direct limit of topological spaces; for simplicity, this is stated for CW-complexes.

Proposition 1.27. *Let E^* be a cohomology theory and let $\hookrightarrow X_i \hookrightarrow X_{i+1} \hookrightarrow \ldots$ be a direct system of inclusions of CW-complexes with $X := \bigcup X_i$. Then there is a short exact sequence:*

$$0 \to \lim_{\leftarrow,i}{}^1 E^{n-1}(X_i) \to E^n(X) \to \lim_{\leftarrow,i} E^n(X_i) \to 0.$$

1.4. **Towards the stable homotopy category.** Stable homotopy theory provides a richer version of the category of Proposition 1.26. For example, this allows the construction of new theories from old ones: given a natural transformation of cohomology theories $\alpha : E^* \to F^*$, the notion of cofibre or fibre of α has a sense.

Remark 1.28. The stable homotopy category, $\mathscr{SH}$, should be
(1) an additive (in fact, *triangulated*) category, in particular equipped with a suspension functor [1] (which will also be denoted by Σ) which is an auto-equivalence;
(2) equipped with a functor $\Sigma^\infty : \mathscr{H}_\bullet \to \mathscr{SH}$ which is compatible with the suspension functors and which sends the sequence $X \xrightarrow{f} Y \to Cf \to$ to a distinguished triangle in $\mathscr{SH}$.
(3) The category $\mathscr{SH}$ should be *generated* as a triangulated category by the object S^0 (ie the smallest full triangulated subcategory of $\mathscr{SH}$ containing S^0 and stable under arbitrary coproducts is $\mathscr{SH}$.)
(4) Finally and fundamentally, cohomology theories should be *representable* in the category $\mathscr{SH}$.

Remark 1.29. Suppose that $\mathscr{SH}$ is a triangulated category as above. If E is an object of $\mathscr{SH}$, then the usual cohomological functor

$$E^n(-) := [-, E[n]]_{\mathscr{SH}}$$

restricts to a cohomology theory, via $\Sigma^\infty : \mathscr{H}_\bullet \to \mathscr{SH}$.

A further desired property of $\mathscr{SH}$ is that the smash product structure $(\mathscr{H}_\bullet, \wedge, S^0)$ for the homotopy category of pointed spaces should extend to a symmetric monoidal structure $(\mathscr{SH}, \wedge, S)$, where $S = \Sigma^\infty S^0$, which is compatible with the triangulated structure.

Definition 1.30. The homology theory associated to an object $E \in \mathrm{Ob}\, \mathscr{SH}$ is the functor $\mathscr{SH} \to \mathfrak{Ab}^{\mathrm{gr}}$ defined by

$$E_n(Z) := [S[n], Z \wedge E]_{\mathscr{SH}}.$$

Example 1.31. Taking $S = \Sigma^\infty S^0$ gives a homology theory $Z \mapsto [S[n], Z]$; this should coincide with the *stable homotopy* functor.

The restriction of the homology theory $E_*(-)$ to $\mathscr{H}_\bullet$ is a *reduced homology theory*. (The reader can supply the appropriate axioms.)

1.5. **Naïve spectra.** The aim of this section is to give an indication of the construction of a model for the category $\mathscr{SH}$. The approach outlined is the classical approach due to Boardman, which is explained in [Ada95, Part III] and in [Swi02, Chapter 8]. This has the advantage of being explicit, relatively elementary and of yielding the correct homotopy category $\mathscr{SH}$. There are more highly structured models which have better formal properties.

The definition of the category of *spectra* is motivated by a version of the *Brown representability theorem* for functors $\mathscr{H}_\bullet^{\mathrm{op}} \to \mathfrak{Ab}$. This implies that, if $F : \mathscr{H}_\bullet^{\mathrm{op}} \to \mathfrak{Ab}$ is a functor which satisfies the exactness and additivity axioms from Definition 1.20, then F is represented by a pointed CW-complex. In particular, if E^* is a cohomology theory, then there exists a sequence of pointed spaces $\{E_n | n \in \mathbb{Z}\}$ such that the functor $X \mapsto E^n(X)$ is naturally equivalent to the functor $X \mapsto [X, E_n]$, $\forall n$.

The Yoneda lemma implies that the suspension axiom is equivalent to the existence of natural transformations $\forall n$:

$$\tilde{\sigma}_n : E_n \overset{\simeq}{\to} \Omega E_{n+1},$$

which are weak equivalences (ie isomorphisms in $\mathscr{H}_\bullet$). By adjunction, it is equivalent to specify structure morphisms $\sigma_n : \Sigma E_n \to E_{n+1}$. The weak equivalence condition is clearly best expressed in terms of $\tilde{\sigma}_n$.

Conversely, a sequence $(E_n, \tilde{\sigma}_n)$ as above represents a cohomology theory.

Example 1.32.

(1) For singular cohomology theory $H^*(-; A)$ (for $A \in \mathrm{Ob}\, \mathfrak{Ab}$), it is classical that the nth Eilenberg-MacLane space $K(A, n)$ represents $H^n(-; A)$. For $n \geq 0$, the homotopy type of a pointed CW complex $K(A, n)$ is determined by the condition that $\pi_*(K(A, n))$ is concentrated in degree n, where it is isomorphic to A; for $n < 0$, the space is a point. Hence, it is clear that $\Omega K(A, n+1) \simeq K(A, n)$.

(2) For complex K-theory, KU and $n \in \mathbb{Z}$, the space representing $KU^{2n}(-)$ is $\mathbb{Z} \times BU$, where BU is the classifying space of the infinite unitary group; the space representing $KU^{2n+1}(-)$ is U. Bott periodicity provides the homotopy equivalence $\Omega SU \simeq BU$, where SU is the special unitary group, which implies that $\Omega U \simeq \mathbb{Z} \times BU$ (since $U \simeq S^1 \times SU$).

Definition 1.33. The category of spectra, $\mathscr{Sp}$, is the category with

- objects: $(E_n, \sigma_n | n \in \mathbb{N})$ such that $E_n \in \mathrm{Ob}\, \mathscr{T}_\bullet$ and $\sigma_n : \Sigma E_n \to E_{n+1}$;
- morphisms: $f : (E_n, \sigma_n) \to (F_n, \sigma_n)$, a sequence of morphisms $f_n : E_n \to F_n$ of $\mathscr{T}_\bullet$ which are compatible with the structure morphisms, ie

$$\begin{array}{ccc} \Sigma E_n & \longrightarrow & E_{n+1} \\ {\scriptstyle \Sigma f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_n & \longrightarrow & F_{n+1} \end{array}$$

commutes, $\forall n$.

A spectrum (E_n, σ_n) for which each E_n has the homotopy type of a pointed CW-complex and the adjoint structure morphisms $\tilde{\sigma}_n : E_n \to \Omega E_{n+1}$ are all weak equivalences is called an Ω-spectrum.

The *suspension spectrum* functor $\Sigma^\infty : \mathscr{T}_\bullet \to \mathscr{S}p$ is defined by $(\Sigma^\infty X)_n := \Sigma^n X$, with $\sigma_n : \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ the natural homeomorphism; the behaviour on morphisms is analogous.

Remark 1.34.

(1) One can also use $\mathbb{Z}$-indexed spectra.
(2) The diagrams are commutative in the category $\mathscr{T}_\bullet$. There is a technique for making homotopy commutative diagrams strictly commutative; this involves replacing the domain by a *homotopy equivalent* spectrum, via the *telescope* construction.
(3) The simplicity of the definition of the suspension spectrum functor Σ^∞ is one justification for using σ_n rather than $\tilde{\sigma}_n$ in the definition of a spectrum.
(4) A cohomology theory E^* gives rise to a weak Ω-spectrum, as outlined above.

Definition 1.35. Define functors

(1) $\wedge : \mathscr{S}p \times \mathscr{T}_\bullet \to \mathscr{S}p$ by $(E_*, \sigma_*) \wedge X := (E_* \wedge X, \sigma_* \wedge X)$;
(2) $[k] : \mathscr{S}p \to \mathscr{S}p$, for $k \in \mathbb{Z}$, by

$$((E_*, \sigma_*)[k])_n := \begin{cases} E_{n+k} & n+k \geq 0 \\ * & n+k < 0; \end{cases}$$

(3) $\bigvee : \mathscr{S}p \times \mathscr{S}p \to \mathscr{S}p$ by $(E_*, \sigma_*) \bigvee (F_*, \sigma_*) := (E_* \vee F_*, \sigma^E_* \vee \sigma^F_*)$.

The structure of $\mathscr{S}p$ already allows for negative-dimensional spheres:

Definition 1.36. For $n \in \mathbb{Z}$, define the n-dimensional sphere spectrum S^n in $\mathscr{S}p$ by

$$S^n := \begin{cases} \Sigma^\infty S^n & n \geq 0 \\ (\Sigma^\infty S^0)[n] & n < 0. \end{cases}$$

Remark 1.37. The functor $[k]$ is close to being an equivalence of categories: there is a natural transformation $[-k] \circ [k] \to 1_{\mathscr{S}p}$, such that $(E[k][-k])_n \to E_n$ is the identity for $n \geq -k$.

Remark 1.38. There are two natural ways of defining the suspension of a spectrum:

(1) $- \wedge S^1 : \mathscr{S}p \to \mathscr{S}p$;
(2) $[1] : \mathscr{S}p \to \mathscr{S}p$.

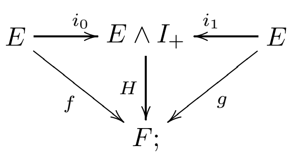
From the point of view of compatibility with Σ^∞, the first is the natural construction; however, it is easier to show that $[1]$ induces an equivalence of the associated homotopy category. The key point is to compare the two; it is tempting to assert that the structure morphisms induced a natural morphism

$$(E_*, \sigma_*) \wedge S^1 \dashrightarrow (E_*, \sigma_*)[1].$$

This is only true after allowing for signs associated to the interchange of factors $S^1 \wedge S^1$ and a process of rigidification of homotopy commutative diagrams.

The inclusion $\{0, 1\} \hookrightarrow I$ leads to a *cylinder object* $E \wedge I_+$ for the spectrum E and hence to the evident notion of homotopy.

Definition 1.39. Two morphisms of spectra $f, g : E \to F$ are homotopic by a homotopy $H : E \wedge I_+ \to F$ if the following diagram commutes



write $f \sim g$ (or $f \sim_H g$).

Remark 1.40. The category $\mathscr{SH}$ cannot simply be defined as $\mathscr{S}p/\sim$, for two reasons:

(1) it is necessary to restrict to the appropriate *cofibrant* objects - there is a natural (and elementary) definition of CW-spectrum which plays this rôle;
(2) more seriously, CW-spectra are not *fibrant*. The appropriate fibrant objects turn out to be CW-spectra which are Ω-spectra.

The second difficulty is illustrated by the comparison of $\Sigma^\infty X$ and $\Sigma^\infty X[-1][1]$, for X a pointed CW-complex. The canonical morphism $\Sigma^\infty X[-1][1] \to \Sigma^\infty X$ should be an equivalence in the category $\mathscr{SH}$. However, if $X \neq *$, there is no non-trivial morphism from $\Sigma^\infty X$ to $\Sigma^\infty X[-1][1]$, hence these objects cannot possibly be homotopy equivalent.

Boardman's approach [Ada95, Swi02] is to get around this problem by adding more morphisms; from a modern point of view, it is more natural to use the methods of *homotopical algebra* [Qui67].

Definition 1.41. The stable homotopy functor $\pi_* : \mathscr{S}p \to \mathfrak{Ab}^{\mathrm{gr}}$ is defined by

$$\pi_t(E) := \lim_{k,\to} \pi_{t+k}(E_k),$$

for $t \in \mathbb{Z}$, where the morphisms of the direct system are the composites

$$\pi_{t+k}(E_k) \to \pi_{t+k+1}(\Sigma E_k) \stackrel{\pi_{t+k+1}(\sigma_k)}{\to} \pi_{t+k+1}(E_{k+1}).$$

A morphism $f : E \to F$ of spectra is a *weak stable homotopy equivalence* if $\pi_*(f)$ is an isomorphism of graded abelian groups. The class of weak stable homotopy equivalences is denoted $\mathscr{W}_{st}$.

Remark 1.42. Stable homotopy π_* takes values in abelian groups.

Lemma 1.43. *For $k \in \mathbb{Z}$, the natural transformation $[-k] \circ [k] \to 1_{\mathscr{S}p}$ is a natural weak stable homotopy equivalence.*

Remark 1.44.

(1) If $X \in \mathrm{Ob}\, \mathscr{T}_\bullet$, then there is a natural isomorphism $\pi_t^S(X) \cong \pi_t(\Sigma^\infty X)$.
(2) The stable homotopy groups $\pi_i(S^0)$ are trivial for $i < 0$ and $\pi_0(S^0) \cong \mathbb{Z}$. This *connectivity property* is not a formal consequence of the definitions.

The following definition can be formalized and made more explicit using the theory of model categories.

Definition 1.45. The stable homotopy category $\mathscr{SH}$ is the localization $\mathscr{S}p[\mathscr{W}_{st}^{-1}]$.

Theorem 1.46. *The category $\mathscr{SH}$ is triangulated and the functor $\Sigma^\infty : \mathscr{T}_\bullet \to \mathscr{S}p$ induces a functor $\Sigma^\infty : \mathscr{H}_\bullet \to \mathscr{SH}$ which is compatible with the suspension functors and which sends the sequence $X \to Y \to Cf \to$ to a distinguished triangle in $\mathscr{SH}$.*

Remark 1.47. The first construction of $\mathscr{SH}$ using abstract homotopy theory was by Bousfield and Friedlander, who constructed a model category structure on a suitable category of spectra in pointed simplicial sets; a version of their construction is given in [GJ99]. A general approach to stabilizing model categories has been given by Hovey [Hov01]; this clarifies the rôle of the fibrant objects (which are analogues of Ω-spectra).

Example 1.48. By the isomorphism $[S,S]_{\mathscr{SH}} \cong \pi_0(S) \cong \mathbb{Z}$, for any natural number $n > 0$, there is a morphism $n : S \to S$ which induces multiplication by n in homotopy (or homology). The cofibre of this morphism S/n has the property

$$H\mathbb{Z}_*(S/n) \cong \begin{cases} \mathbb{Z}/n & * = 0 \\ 0 & \text{Otherwise} \end{cases}$$

This is an example of a *Moore spectrum*.

The construction shows that the stable homotopy functor $\pi_n(-)$ is naturally isomorphic to the functor $[S^n, -]_{\mathscr{SH}}$.

Theorem 1.49. *The sphere spectra $\{S^n | n \in \mathbb{Z}\}$ form a set of compact generators of the category $\mathscr{SH}$. Moreover, the category $\mathscr{SH}$ has a canonical t-structure $(\mathscr{SH}; \mathscr{SH}_{\geq 0}, \mathscr{SH}_{<0})$ such that*

$$X \in \mathscr{SH}_{\geq 0} \quad \Leftrightarrow \quad \pi_i(X) = 0, \ \forall i < 0$$
$$X \in \mathscr{SH}_{<0} \quad \Leftrightarrow \quad \pi_i(X) = 0, \forall i \geq 0.$$

The heart of the t-structure $\mathscr{SH}_{[0,0]}$ is the category $\mathfrak{A}b$ and the functor

$$\mathfrak{A}b \cong \mathscr{SH}_{[0,0]} \to \mathscr{SH}$$

is the Eilenberg-MacLane functor $H : \mathfrak{A}b \to \mathscr{SH}$.

Remark 1.50. This result is essentially the classical theory of *Postnikov systems* for $\mathscr{SH}$. For a spectrum X, there is a distinguished triangle:

$$X_{\geq 0} \to X \to X_{<0} \to$$

where $X_{\geq 0} \in \mathscr{SH}_{\geq 0}$ and $X_{<0} \in \mathscr{SH}_{<0}$.

The spectrum $X_{<0}$ is constructed by attaching cells to kill the non-negative homotopy groups; by the connectivity of $\mathscr{SH}$, this can be done by induction on the dimension of the cells, starting with cells of dimension zero.

The theorem implies that the objects of the heart $\mathscr{SH}_{[0,0]}$ are those spectra with homotopy groups concentrated in degree zero. This is precisely the definition of the Eilenberg-MacLane spectra. The Eilenberg-Steenrod axioms for ordinary cohomology show that the associated cohomology theory for HA is $H^*(-; A)$.

Example 1.51. The Eilenberg-MacLane spectrum $H\mathbb{Z}$ is the spectrum $S_{\leq 0}$, constructed from S by killing all homotopy groups in degrees > 0.

Theorem 1.52. *(Brown representability.) Let E^* be a cohomology theory on $\mathscr{H}_\bullet$; then E^* is represented by a spectrum E, considered as an object of $\mathscr{SH}$.*

Proof. The result is proved by using standard techniques for triangulated categories. □

Remark 1.53. The functor $\Sigma^\infty : \mathscr{H}_\bullet \to \mathscr{SH}$ has a right adjoint $\Omega^\infty : \mathscr{SH} \to \mathscr{H}_\bullet$. This is a *derived* version of the evaluation functor $\mathscr{S}p \to \mathscr{H}_\bullet$, $(E_*, \sigma_*) \mapsto E_0$. In particular,

$$\Omega^\infty E \simeq \lim_{\to} \Omega^n E_n.$$

If E is an Ω-spectrum, then $\Omega^\infty E \simeq E_0$.

For example, the space $\Omega^\infty \Sigma^\infty X$, for X a pointed CW-complex, is the associated infinite loop space

$$QX := \lim_{\to} \Omega^n \Sigma^n X.$$

The functor $Q : \mathscr{H}_\bullet \to \mathscr{H}_\bullet$ is highly non-trivial.

The Milnor short exact sequence (Proposition 1.27) leads to a method for calculating the group of stable operations between two cohomology theories. (To simplify the hypotheses, the result is stated for CW-spectra; this corresponds to a *cofibrany* condition, which can be ignored here.)

Proposition 1.54. *Let E^*, F^* be cohomology theories with representing (CW) spectra E, F respectively and write $[E, F]$ for the group of stable cohomology operations from E^* to F^* of degree zero. There is a short exact sequence*

$$0 \to \lim_{\leftarrow, n}{}^1 F^{n-1}(E_n) \to [E, F] \to \lim_{\leftarrow, n} F^n(E_n) \to 0.$$

Proof. $[E, F]$ is isomorphic to $F^0(E)$, by Yoneda's Lemma. The result follows by writing E as a suitable direct limit, which is equivalent to a homotopy colimit in the triangulated category $\mathscr{SH}$, and deriving the Milnor exact sequence in the standard way. □

Example 1.55. The mod p Steenrod algebra, for p a prime, is the graded algebra $[H\mathbb{F}_p, \Sigma^* H\mathbb{F}_p]$. This can be calculated as a graded $\mathbb{F}_p$-vector space by

$$[H\mathbb{F}_p, \Sigma^* H\mathbb{F}_p] \cong \lim_{\leftarrow, n} H^{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p),$$

since the $\lim^1$-term vanishes (exercise: why?).

2. MULTIPLICATIVE STRUCTURES

2.1. **The homotopy smash product.** Recall that $(\mathscr{H}_\bullet, \wedge, S^0)$ is symmetric monodial.

Theorem 2.1. *The category $\mathscr{SH}$ has a symmetric monoidal structure $(\mathscr{SH}, \wedge, S)$ such that $\Sigma^\infty : \mathscr{H}_\bullet \to \mathscr{SH}$ is strictly symmetric monoidal.*

There is an elementary definition of a naïve smash product at the level of $\mathscr{S}p$: for spectra $E = (E_*, \sigma^E_*)$ and $F = (F_*, \sigma^F_*)$, the spectrum $E \wedge F$ is defined by

$$(E \wedge F)_k := \begin{cases} E_n \wedge F_n & k = 2n \\ \Sigma(E_n \wedge F_n) & k = 2n+1. \end{cases}$$

The structure map $\Sigma(E \wedge F)_{2n+1} \to (E \wedge F)_{2n+2}$ is induced by $\sigma^E_n \wedge \sigma^F_n$.

Remark 2.2. This definition does *not* define a symmetric monoidal structure on the category $\mathscr{S}p$: this naïve smash product is certainly not associative. (It is possible to refine the definition to obtain an associative product, but not a commutative one.) However, the naïve smash product construction *does* induce a functor $\mathscr{SH} \times \mathscr{SH} \xrightarrow{\wedge} \mathscr{SH}$ which is symmetric monodial.

For considering the properties of multiplicative cohomology theories, the structure $(\mathscr{SH}, \wedge, S)$ is sufficient; however it is *not* sufficient for the purposes of *brave new algebra*. For example, the construction of the theory of *topological modular forms* requires a more rigid theory.

2.2. **Spanier-Whitehead duality.** The compact objects of $\mathscr{SH}$ are easy to describe:

Proposition 2.3. *A spectrum E of $\mathscr{SH}$ is compact if and only if there exists a finite pointed CW-complex X and an integer n such that $E \cong \Sigma^\infty X[n]$.*

Remark 2.4. The compact objects form a full subcategory $\mathscr{SH}_{\mathrm{Cpct}}$ of the stable homotopy category $\mathscr{SH}$; this is equivalent to the Spanier-Whitehead category.

Theorem 2.5. *There is a duality functor $D : \mathscr{SH}^{\mathrm{op}}_{\mathrm{Cpct}} \to \mathscr{SH}_{\mathrm{Cpct}}$ which defines a strong duality, induced by the canonical morphisms*

$$\begin{aligned} S &\to DX \wedge X \\ DX \wedge X &\to S, \end{aligned}$$

for $X \in \mathrm{Ob}\, \mathscr{SH}_{\mathrm{Cpct}}$. In particular, for $A, B \in \mathrm{Ob}\, \mathscr{SH}$, there is an isomorphism

$$[A \wedge X, B]_{\mathscr{SH}} \cong [A, DX \wedge B]_{\mathscr{SH}}.$$

The functor D is strictly monodial and DD is naturally equivalent to the identity functor.

Remark 2.6. The duality functor D has a geometric interpretation (cf. Alexander duality).

Duality provides a relation between homology and cohomology.

Corollary 2.7. *Let $X \in \mathrm{Ob}\, \mathscr{SH}_{\mathrm{Cpct}}$ be a compact object and $E \in \mathrm{Ob}\, \mathscr{SH}$ be a spectrum with associated cohomology theory E^* and homology theory E_*. Then there are natural isomorphisms:*

$$E_n(X) \cong E^{-n}(DX).$$

Proof. A formal consequence of the duality and the definitions of the associated (co)homology theories. □

Remark 2.8. Taking $X = S$, this gives the isomorphism $E_n(S) \cong E^{-n}(S)$, which corresponds to the usual identification between homological and cohomological grading.

2.3. **Multiplicative structures.**

Definition 2.9.

(1) The category of (commutative) ring spectra is the category of (commutative) monoids in $(\mathscr{SH}, \wedge, S)$: a ring spectrum is a triple (R, μ, η), where $R \in \mathrm{Ob}\,\mathscr{SH}$, $\mu : R \wedge R \to R$ and $\eta : S \to R$ such that the following diagrams commute in $\mathscr{SH}$:

$$\begin{CD} R\wedge R\wedge R @>{\mu\wedge R}>> R\wedge R \\ @V{R\wedge\mu}VV @VV{\mu}V \\ R\wedge R @>>{\mu}> R, \end{CD}$$

$$\begin{array}{ccccc} R\cong S\wedge R & \xrightarrow{\eta\wedge R} & R\wedge R & \xleftarrow{R\wedge\eta} & R\wedge S\cong R \\ & {\scriptstyle R}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle R} & \\ & & R. & & \end{array}$$

The ring spectrum R is commutative if $\mu \circ \tau = \mu$, where $\tau : R \wedge R \to R \wedge R$ switches the factors. The definition of a morphism of ring spectra is the evident one.

(2) If R is a ring spectrum in $\mathscr{SH}$, the category of (left) R-modules is the category of left modules over the monoid R. Thus, an R-module is a spectrum M equipped with a structure map $\varphi : R \wedge M \to M$ which satisfies the usual associativity and unit conditions. The definition of a morphism of R-modules is again standard. (There is an analogous category of right R-modules.)

Example 2.10.

(1) Let A be an associative ring; then the Eilenberg-MacLane spectrum HA is a ring spectrum, which is commutative if and only if A is commutative.

The integral Eilenberg-MacLane spectrum $H\mathbb{Z}$ is a commutative ring spectrum; the unit $S \to H\mathbb{Z}$ is an isomorphism on π_0 and induces the Hurewicz homomorphism

$$\pi_*(X) \cong [S[*], X] \to [S[*], H\mathbb{Z} \wedge X] \cong H_*(X; \mathbb{Z}).$$

(2) The K-theory spectra KU and KO are both commutative ring spectra; the multiplication is induced by the tensor product of vector bundles.

(3) Let p be a prime number; then S/p has the structure of a commutative ring spectrum if p is odd but not for $p = 2$.

Proposition 2.11. *The category of ring spectra has a symmetric monoidal structure induced by $\wedge$, with unit S. This resticts to a symmetric monoidal structure on commutative ring spectra.*

Remark 2.12. The category of R-modules in $\mathscr{SH}$ over a ring spectrum R does not, *a priori*, have a triangulated structure. This makes it difficult to carry out constructions of new R-modules within this framework; for example, in this framework, it is necessary to check that the cofibre of a morphism of R-modules $f : M \to N$ has a natural R-module structure. In a good derived category of R-modules, this should be formal; however, this requires more structure on R.

The smash product induces an evident exterior multiplication for spectra E, F:

$$\pi_*(E) \otimes \pi_*(F) \to \pi_*(E \wedge F).$$

Definition 2.13. The coefficient ring of a ring spectrum R is the graded abelian group $R_* := \pi_*(R)$ equipped with multiplication induced by μ and the exterior multiplication.

Remark 2.14. If R is commutative, then R_* is a graded commutative ring.

Lemma 2.15. *Let R be a ring spectrum and M be a left R-module. Then $M_* := \pi_*(M)$ has the structure of a graded left R_*-module.*

Example 2.16. There are a number of products which are induced by an R-module structure on M, for instance, the following exterior products, natural in X and Y:

$$\begin{array}{rcl} R^*(X) \otimes_{R^*} M^*(Y) & \rightarrow & M^*(X \wedge Y) \\ (X \xrightarrow{f} R[s]) \otimes (Y \xrightarrow{g} M[t]) & \mapsto & X \wedge Y \xrightarrow{f \wedge g} R \wedge M[s+t] \xrightarrow{\varphi} M[s+t]; \end{array}$$

$$\begin{array}{rcl} R_*(X) \otimes_{R_*} M_*(Y) & \rightarrow & M_*(X \wedge Y) \\ (S^a \xrightarrow{\alpha} R \wedge X) \otimes (S^b \xrightarrow{\beta} M \wedge Y) & \mapsto & S^{a+b} \xrightarrow{\alpha \wedge \beta} (R \wedge X) \wedge (M \wedge Y) \xrightarrow{\varphi \circ \tau_{X,M}} M \wedge X \wedge Y. \end{array}$$

These satisfy obvious associativity properties.

The exterior products can give rise to internal products in the presence of the appropriate structure. For instance, the diagonal map $U \rightarrow U \times U$ for $U \in \mathrm{Ob}\,\mathscr{T}$ induces $U_+ \rightarrow (U \times U)_+ \cong U_+ \wedge U_+$ in $\mathscr{T}_\bullet$ with counit $U_+ \rightarrow S^0$ induced by $U \rightarrow *$.

Proposition 2.17. *Let R be a commutative ring spectrum. Then the functor $U \mapsto R^*(U_+)$ takes values in the category of graded commutative R^*-algebras.*

Remark 2.18. This result admits a converse, a version of Brown representability. In practice, the natural graded commutative ring structure usually comes from an explicit product at the level of representing objects, so this is not always useful.

2.4. **Complex cobordism.** For $n \in \mathbb{N}$, let γ_n be the universal n-dimensional $\mathbb{C}$-vector bundle over the classifying space $BU(n)$. The pullback of the bundle γ_{n+1} via $BU(n) \rightarrow BU(n+1)$ (induced by $U(n) \hookrightarrow U(n+1)$) is isomorphic to $\gamma_n \oplus \Theta^1$, where Θ^1 is the trivial $\mathbb{C}$-bundle of rank one.

Remark 2.19. If ξ is a $\mathbb{C}$-vector bundle with a Hermitian metric, then there is an associated *disk bundle* $D(\xi)$, given by the vectors of norm at most one, and sphere bundle $S(\xi)$ of vectors with norm one. The Thom space of ξ is, by definition, the quotient space:

$$\mathrm{Thom}(\xi) := D(\xi)/S(\xi),$$

which is canonically pointed. If the base space is compact, $\mathrm{Thom}(\xi)$ is homeomorphic to the one point compatification of $D(\xi)$.

If ξ and ζ are complex vector bundles over X, Y respectively, the exterior sum is a vector bundle $\xi \boxplus \zeta$ over $X \times Y$ and there is a homeomorphism:

$$\mathrm{Thom}(\xi \boxplus \zeta) \cong \mathrm{Thom}(\xi) \wedge \mathrm{Thom}(\zeta).$$

In particular, if Θ^1 is the trivial $\mathbb{C}$-vector bundle over $*$, then $\xi \boxplus \Theta^1$ is the Whitney sum of ξ with the trivial vector bundle Θ^1 over X and, hence,

$$\mathrm{Thom}(\xi \oplus \Theta^1) \cong \Sigma^2 \mathrm{Thom}(\xi),$$

since the Thom space of Θ^1 over $*$ is S^2. Hence, Thom spaces can be considered as *twisted suspensions*.

Definition 2.20. The complex cobordism spectrum MU is defined by

$$MU_n := \begin{cases} \mathrm{Thom}(\gamma_d) & n = 2d \\ \Sigma \mathrm{Thom}(\gamma_d) & n = 2d+1, \end{cases}$$

with structure morphism $\Sigma MU_{2d} \rightarrow MU_{2d+1}$ the identity and

$$\Sigma MU_{2d+1} \cong \Sigma^2 \mathrm{Thom}(\gamma_d) \cong \mathrm{Thom}(\gamma_d \oplus \Theta^1) \rightarrow MU_{2d+2} \cong \mathrm{Thom}(\gamma_{d+1})$$

the map between Thom spaces induced by the morphism of vector bundles $\gamma_d \oplus \Theta^1 \to \gamma_{d+1}$.

Remark 2.21. The space MU_2 is, by definition, $\mathrm{Thom}(\gamma_1)$, the Thom space of the canonical line bundle over $\mathbb{C}P^\infty$; this is homeomorphic to $\mathbb{C}P^\infty$ (similarly, the Thom space of the canonical bundle η_n over $\mathbb{C}P^n$ is homeomorphic to $\mathbb{C}P^{n+1}$ - see [KT06, Lemma 3.8] for example).

This defines a canonical cohomology class

$$x_{MU} \in MU^2(\mathbb{C}P^\infty).$$

Proposition 2.22. *The spectrum MU is a commutative ring spectrum with product induced by the Whitney sum of vector bundles, classified by maps*

$$BU(m) \times BU(n) \to BU(m+n)$$

corresponding to the usual inclusion $U(m) \times U(n) \to U(m+n)$ by block sum of matrices.

Remark 2.23. The homology theory represented by MU has a geometric interpretation in terms of cobordism classes of almost complex manifolds; see Quillen's paper [Qui71] for consequences of this interpretation.

2.5. **A glimpse of highly-structured spectra.** For many purposes, a category of highly-structured spectra is useful and, sometimes, essential. Such a category $\mathscr{S}$ should have the following properties:

(1) be equipped with a (stable) model structure [Qui67, Hov99] such that the associated homotopy category is equivalent, as a triangulated category, to $\mathscr{SH}$;
(2) admit a symmetric monoidal structure $(\mathscr{S}, \wedge, \mathbb{S})$ which is compatible with the model structure (see [SS00]), hence passes to the homotopy category, where it should coincide with $(\mathscr{SH}, \wedge, S)$.

Remark 2.24. There are a number of models for highly-structured spectra; the paper [MMSS01] provides a unified approach via the study of diagram spectra which allows the study of relations between them. Amongst popular models are:

(1) symmetric spectra [HSS00];
(2) S-modules [EKMM97];
(3) orthogonal spectra [MMSS01].

Each model has its advantages.

The construction of the smash product follows the following recipe: first define an external smash product to a category of bi-spectra and then use *Kan extension* (sometimes known in this setting as the *Day convolution product*) to get back to spectra.

Example 2.25. Orthogonal spectra are indexed by finite-dimensional real inner product spaces, rather than by the natural numbers $\mathbb{N}$; in particular, the automorphisms of inner product spaces have to be taken into account.

Namely, let $\mathscr{I}$ be the topological category of finite-dimensional real inner product spaces and linear isometries, so that $\mathscr{I}(V, W)$ is empty if $\dim V \neq \dim W$ and is homeomorphic to the orthogonal group $O(V) \cong O(W)$, if $\dim V = \dim W$. There is a natural continuous functor

$$S^- : \mathscr{I} \to \mathscr{T}_\bullet$$

defined by one point compactification: $V \mapsto S^V$.

The category $\mathscr{I}$ has a symmetric monodial structure induced by orthogonal sum $\perp$, with unit the zero space. Observe that, if $W \subset V$ is a sub inner product space, then there is an isomorphism $W \cong V \perp V^\perp$ and $S^W \cong S^V \wedge S^{V^\perp}$.

An orthogonal spectrum is a continuous functor $\mathscr{I} \to \mathscr{T}_\bullet$, together with a structure as a module over the functor S^-. This is determined by structure maps

$$X_V \wedge S^W \to X_{V \perp W}$$

which are $O(V) \times O(W)$-equivariant and satisfy associativity and unit conditions.

There is a natural notion of external smash product from $\mathscr{I}$-diagram spaces to $\mathscr{I} \times \mathscr{I}$-diagram spaces. This induces an internal smash product, by the general process of topological Kan extension. This smash product then induces a smash product on the category of orthogonal spectra. This can be given explicitly:

$$(X \wedge Y)_V = \bigvee_{i=0}^{\dim V} O(V)_+ \wedge_{O(V_i) \times O(V_i^\perp)} X_{V_i} \wedge Y_{V_i^\perp},$$

where $V_i \subset V$ such that $\dim V_i = i$.

There is a model category structure on orthogonal spectra which induces the structure $(\mathscr{SH}, \wedge, S)$ on the associated homotopy category.

Example 2.26. Symmetric spectra are indexed by the category Σ of finite sets and their permutations. The spectra have underlying based spaces $X_{\mathbf{n}}$, indexed by natural numbers n, equipped with an action of the symmetric group. The smash product of two such sequences of spaces is given by

$$(X \wedge Y)_{\mathbf{n}} \cong \bigvee_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} X_{\mathbf{i}} \wedge Y_{\mathbf{j}}.$$

The *sphere spectrum* is given by $\mathbf{n} \mapsto S^n$, equipped with the action of the symmetric group permuting smash product factors in $(S^1)^{\wedge n}$ and a *symmetric spectrum* has structure maps

$$S^- \wedge X \to X$$

satisfying the obvious associativity condition. It is the introduction of the action of the symmetric group on S^- which allows the definition of a symmetric monoidal structure at the level of spectra.

2.6. **Brave new algebra.** Given a highly-structured category of spectra $(\mathscr{S}, \wedge, \mathbb{S})$, it is natural to study

1. monoids in $\mathscr{S}$ (these are a modern formulation of A_∞-spectra);
2. commutative monoids in $\mathscr{S}$ (a modern formulation of E_∞-spectra).

It is clear that a (commutative) monoid in $\mathscr{S}$ defines a (commutative) ring spectrum in $\mathscr{SH}$. This begs the following fundamental question:

Question 2.27. *Given a commutative ring spectrum R, is it represented by a monoid in $\mathscr{S}$ (resp. a commutative monoid)? If so, is this structure unique?*

Similar questions can be asked for morphisms of ring spectra.

This question is highly-non-trivial (see [GH04], for example); their approach is to study the *moduli space* of all commutative monoids in $\mathscr{S}$ which realize the commutative ring spectrum R. This moduli space is non-empty if and only if R can be realized. The homotopy type of the path components of the moduli space R give important information on the realizations.

There are significant gains in having a multiplicative structure at the level of spectra. For instance:

1. commutative monoids in $\mathscr{S}$ have a *model structure*, hence this facilitates the construction of new ring spectra;
2. if E is a monoid in $\mathscr{S}$, the category of E-modules in $\mathscr{S}$ has a model category structure; the associated homotopy category gives a *derived category of E-modules*.

Example 2.28. The spectra $H\mathbb{Z}$ and MU are fundamental examples of commutative monoids in $\mathscr{S}$.

Example 2.29. The construction of the Hopkins-Miller *topological modular forms* (see [Hop95] for an early survey using only A_∞-structures) requires highly-structured ring spectra.

Remark 2.30. The spectrum KO can be constructed from KU as follows; complex conjugation induces an action of $\mathbb{Z}/2$ on KU and there is an associated *homotopy fixed point spectrum* $KU^{h\mathbb{Z}/2}$ which is equivalent to KO. This observation has deep generalizations (the Hopkins-Miller theorem): to construct the appropriate homotopy fixed point spectrum, one needs a sufficiently rigid group action on a spectrum; this uses highly-structured ring spectra [GH04].

3. Complex oriented theories and Quillen's theorem

3.1. **Complex orientations.** Complex projective space $\mathbb{C}P^n$ has a natural cellular structure, associated to the decomposition

$$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n \hookleftarrow \mathbb{A}^n$$

where $\mathbb{C}P^{n-1}$ is the hyperplane at infinity. The complex affine space $\mathbb{A}^n$ corresponds to a cell of dimension $2n$ and the homotopy cofibre of $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ is equivalent to S^{2n} ($\mathbb{C}P^1$ is homeomorphic to the sphere S^2).

Infinite complex projective space $\mathbb{C}P^\infty$ is the direct limit of the natural inclusions $\ldots \hookrightarrow \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1} \hookrightarrow \ldots$ and hence has a CW-structure with one cell in each even dimension.

Remark 3.1. The space $\mathbb{C}P^\infty \in \mathscr{H}_\bullet$ is important in homotopy theory; it can be considered as the classifying space BS^1 or as an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. In particular, for X a CW-complex, $[X_+, \mathbb{C}P^\infty] \cong H^2(X_+; \mathbb{Z}) \cong \mathrm{Pic}(X)$, the topological Picard group of complex line bundles over X, with group structure induced by $\otimes$.

The group structure is induced by the *commutative H-space* structure of $\mathbb{C}P^\infty$. Namely, there is a product

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

which classifies the tensor product of line bundles; this makes $\mathbb{C}P^\infty_+$ an abelian group object in the category $\mathscr{H}_\bullet$.

Definition 3.2. Let E be a commutative ring spectrum.

(1) E is *complex oriented* if there is a class $x_E \in E^2(\mathbb{C}P^\infty)$ such that the restriction of x_E along $S^2 \cong \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ is the unit $1 \in E^2(S^2) \cong E^0$.
(2) A class x_E satisfying the above condition is a *complex orientation* of E.

Example 3.3. The spectra $H\mathbb{Z}$, $H\mathbb{F}_p$, KU are complex oriented ring spectra. However, the spectrum KO is a commutative ring spectrum which is *not* complex orientable. There is no class in $KO^2(\mathbb{C}P^2)$ which restricts to the unit, since KO *detects* the Hopf map $S^3 \stackrel{\eta}{\to} S^2$, which is the attaching map for the top cell of $\mathbb{C}P^2$.

Lemma 3.4. *Let E, F be commutative ring spectra.*

(1) *If $f : E \to F$ is a morphism of ring spectra and x_E is a complex orientation of E, then $x_F := f_*(x_E)$ is a complex orientation of F.*
(2) *If E, F are both complex oriented, the morphisms of ring spectra $E \to E \wedge F \leftarrow F$ induce two complex orientations on $E \wedge F$.*

3.2. **Algebraic structure associated to a complex orientation.**

Proposition 3.5. *Let E be a complex oriented ring spectrum, with orientation x_E. Then, there are isomorphisms of (topological) rings*

(1) $E^*[[x_E]] \stackrel{\cong}{\to} E^*(\mathbb{C}P^\infty_+)$
(2) $E^*[[x_E^1, x_E^2]] \stackrel{\cong}{\to} E^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+)$.

Proof. (Indications.) Consider the first case; the second is similar. It is straightforward to show that there is a natural isomorphism

$$E^*[x_E]/(x_E^{n+1}) \stackrel{\cong}{\to} E^*(\mathbb{C}P^n).$$

The passage to $\mathbb{C}P^\infty$ follows from the Milnor exact sequence, Proposition 1.27. □

Remark 3.6. The complex orientation x_E is not unique in general; x_E can be replaced by $g(x_E)$, where $g(x_E) \in (E^*[[x_E]])^2$ is a formal power series with $g(0) = 0$ and $g'(0) = 1$. There is a natural morphism of graded commutative rings $E^* \hookrightarrow$

$E^*(\mathbb{C}P^\infty_+)$ (induced by $\mathbb{C}P^\infty \to *$) and an augmentation $E^*(\mathbb{C}P^\infty_+) \to E^*$ (induced by any choice of a basepoint of $\mathbb{C}P^\infty$).

If E is complex oriented, then the *formal scheme* associated to $E^*(\mathbb{C}P^\infty_+)$ is isomorphic to the formal affine line $\hat{\mathbb{A}}^1$, equipped with a canonical section. The complex orientation corresponds to a *choice* of isomorphism, or a *choice of coordinate*.

The only formal groups which are considered here are commutative of dimension one.

Definition 3.7. A formal group law over a commutative ring R is a formal power series $F(x,y) \in R[[x,y]]$ such that

(1) $F(x,0) = x = F(0,x)$;
(2) (commutativity) $F(x,y) = F(y,x)$;
(3) (associativity) $F(x, F(y,z)) = F(F(x,y),z)$.

If F, G are formal group laws over R, a morphism of formal group laws $F \xrightarrow{f} G$ is a formal power series $f(x) \in R[[x]]$ such that $f(0) = 0$ and $f(F(x,y)) = G(f(x), f(y))$.

Composition of morphisms corresponds to composition of formal power series; a morphism $f(x)$ is an *isomorphism* if $f'(0)$ is invertible in R and is a *strict isomorphism* if $f'(0) = 1$.

Remark 3.8. If F is a formal group law over R, there exists a unique formal power series $i(x) \in R(x)$ such that $i(0) = 0$ and $F(x, i(x)) = x$. More generally, for $n \in \mathbb{Z}$, there is a unique formal power series $[n]_F(x)$ (the n-series), which is an endomorphism of F, such that $[1]_F x = x$ and $F([m]_F(x), [n]_F(x)) = [m+n]_F(x)$, so $[-1]_F x = i(x)$.

Proposition 3.9. *Formal group laws and morphisms over R form a category* $\mathrm{FGL}(R)$, *which contains the groupoid* $\mathrm{FGL}^{\mathrm{SI}}(R)$ *of formal group laws and strict isomorphisms.*

Moreover, a morphism of commutative rings $\varphi : R \to S$ induces functors

$$\begin{array}{ccc} \mathrm{FGL}^{\mathrm{SI}}(R) & \xrightarrow{\varphi_*} & \mathrm{FGL}^{\mathrm{SI}}(S) \\ \big\downarrow & & \big\downarrow \\ \mathrm{FGL}(R) & \xrightarrow[\varphi_*]{} & \mathrm{FGL}(S). \end{array}$$

Theorem 3.10. *(Lazard's theorem.) There exists a universal formal group law F_U defined over a commutative ring L and L is isomorphic to a polynomial algebra $\mathbb{Z}[a_i | 0 < i \in \mathbb{N}]$; moreover, L can be taken to be (homologically) graded with $|a_i| = 2i$, so that $F_U(x,y)$ is homogeneous of degree -2, where $|x| = |y| = -2$.*

Remark 3.11. The set of ring morphisms $\mathrm{Hom}(L, R)$ is in bijection with the set of formal group laws over R, via $\varphi \mapsto \varphi_* F_U$.

More generally, the functor $R \mapsto \mathrm{FGL}^{\mathrm{SI}}(R)$, with values in small groupoids, is represented by an *affine groupoid scheme*; this structure is usually referred to as a *Hopf algebroid* in algebraic topology.

It is useful to consider the associated algebraic stack of formal groups (defined with respect to the *fpqc* topology).

The relevance of formal group laws to algebraic topology is explained by the following result:

Theorem 3.12. *Let E be a complex oriented commutative ring spectrum.*

(1) *A complex orientation x_E induces a formal group law $F_{x_E} \in \mathrm{Ob}\,\mathrm{FGL}(E^*)$.*
(2) *If $x'_E = g(x_E)$ is a second complex orientation, then g defines a strict isomorphism $F_{x_E} \xrightarrow{g} F_{x'_E}$.*

Proof. (Indications.) The product $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces a morphism of E^*-algebras:

$$E^*(\mathbb{C}P^\infty_+) \cong E^*[[x_E]] \to E^*((\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+) \cong E^*[[x^1_E, x^2_E]],$$

which is determined by $F_{x_E} \in E^*[[x^1_E, x^2_E]]$, the image of x_E. The associativity and the commutativity of the product in $\mathscr{H}_\bullet$ imply that this is a formal group law.

The naturality statement is straightforward. □

Remark 3.13. In topology, the rings considered are *graded commutative*. The orientation x has cohomological degree 2 and the formal power series F_{x_E} is homogeneous of cohomological degree two. In particular, the coefficients lie in the *commutative* subring E^{2*}.

Remark 3.14. Whereas the formal group law associated to a complex oriented commutative ring spectrum depends on the choice of coordinate, the corresponding 1-dimensional commutative formal group $\mathbb{G}_E$ is canonical.

Example 3.15.

(1) If $E = H\mathbb{Z}$, the coefficient ring is $\mathbb{Z}$, concentrated in degree zero. In this case, there is a *canonical* choice of coordinate, and the associated formal group is the additive formal group $\mathbb{G}_a$, with formal group law

$$F_a(x, y) = x + y.$$

(2) If $E = KU$, the coefficient ring is $\mathbb{Z}[u^{\pm 1}]$, with $u \in KU^{-2}$ the Bott element; the standard choice of coordinate induces the *multiplicative* formal group law:

$$F_m(x, y) = x + y - uxy.$$

(3) The spectrum MU is equipped with a *canonical* complex orientation $x_{MU} \in MU^2(\mathbb{C}P^\infty)$. (For the identification of the coefficent ring MU_*, see Theorem 3.25).

Remark 3.16. *Elliptic cohomology theories* are (roughly-speaking) oriented cohomology theories such that the associated formal group is the formal completion of an elliptic curve. The theory of *topological modular forms* is not complex oriented, but is constructed as the (homotopy) global sections of a sheaf of structured commutative ring spectra on a moduli stack of generalized elliptic curves, which is constructed from elliptic cohomology theories.

An important consequence of the existence of a complex orientation is the Thom isomorphism theorem.

Theorem 3.17. *Let ξ be a rank n $\mathbb{C}$-vector bundle over X and E be a $\mathbb{C}$-oriented commutative ring spectrum. Then there exists a Thom class $u_\xi \in E^{2n}(\mathrm{Thom}(\xi))$ such that $E^*(\mathrm{Thom}(\xi))$ is a free $E^*(X_+)$-module on u_ξ, where the module structure is induced by the Thom diagonal:*

$$\mathrm{Thom}(\xi) \to X_+ \wedge \mathrm{Thom}(\xi).$$

This leads to the construction of *Gysin* or *umkehr* maps, by using *tubular neighbourhoods*.

3.3. **Homology calculations.** Let E be a commutative ring spectrum; there is a *Kronecker pairing*

$$\langle -, - \rangle : E^*(Z) \otimes_{E^*} E_*(Z) \to E_*.$$

This sends $(Z \xrightarrow{f} E[n]) \otimes (S^a \xrightarrow{\alpha} E \wedge Z)$ to the composite:

$$S^a \xrightarrow{\alpha} E \wedge Z \xrightarrow{E \wedge f} E \wedge E[n] \xrightarrow{\mu} E[n],$$

which gives an element of E_{a-n}.

For $Z = \mathbb{C}P^\infty_+$, and E a complex oriented commutative ring spectrum, the Kronecker pairing is perfect. If x_E is a complex orientation of E, let β_i^E denote the class which is dual to x_E^i.

Proposition 3.18. *Let E be a complex oriented commutative ring spectrum, with orientation x_E. There is an isomorphism of E_*-modules*

$$E_*(\mathbb{C}P^\infty_+) \cong E_*\langle \beta_i^E | i \geq 0 \rangle,$$

where the right hand side denotes the free module generated by the classes β_i^E.

Remark 3.19.

(1) The result generalizes to consider $E_*((\mathbb{C}P^\infty)_+^{\wedge n})$. Indeed, $E_*(\mathbb{C}P^\infty_+)$ is a flat E_*-module, so there is a *Künneth isomorphism*:

$$E_*(X \wedge \mathbb{C}P^\infty_+) \cong E_*(X) \otimes_{E_*} E_*(\mathbb{C}P^\infty_+).$$

(2) The multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ gives $E_*(\mathbb{C}P^\infty_+)$ the structure of a *Hopf algebra* over E_*.

Notation 3.20. Let E be a complex oriented commutative ring spectrum, with orientation x_E. Write $b_i^E \in E_{2i}MU$, ($i \geq 1$), for the image of the class $\beta_{i+1}^E \in E_{2(i+1)}(\mathbb{C}P^\infty_+)$ under the morphism in homology

$$E_*(\mathbb{C}P^\infty_+) \to E_{*-2}MU$$

induced by $\mathbb{C}P^\infty \cong MU_2$. (Exercise: give this morphism explicitly.)

Theorem 3.21. *Let E be a commutative ring spectrum with complex orientation x_E. Then E_*MU is a commutative graded ring and there is a natural isomorphism of rings*

$$E_*[b_i^E | i \geq 1] \xrightarrow{\cong} E_*MU.$$

Proof. (Indications.) The complex orientation of E gives a *Thom isomorphism theorem* for E_*. Thus, the structure of $E_*(MU)$ as an E_*-module can be calculated from $E_*BU(n)$, which can be calculated by a generalization of the classical calculation of $H_*(BU(n); \mathbb{Z})$. □

Example 3.22. Taking $E = MU$, there is an isomorphism of rings $MU_*MU \cong MU_*[b_i | i \in \mathbb{Z}]$, where $|b_i| = 2i$.

3.4. Complex orientations and morphisms of ring spectra from MU.

By Lemma 3.4, a morphism of commutative ring spectra $MU \to E$ induces a complex orientation on E. The converse holds:

Theorem 3.23. *Let E be a complex oriented commutative ring spectrum. There is a bijection between the set of $\mathbb{C}$-orientations of E and*

$$\mathrm{Ring}_{\mathscr{SH}}(MU, E)$$

the set of morphisms of ring spectra from MU to E.

Proof. (Indications.) There is an isomorphism

$$[MU, E]_{\mathscr{SH}} \cong \mathrm{Hom}_{E_*}(E_*MU, E_*),$$

a form of *universal coefficients* isomorphism, where the right hand side is the group of E_*-module morphisms. Moreover, the set $\mathrm{Ring}_{\mathscr{SH}}(MU, E)$ identifies with the set of morphisms of E_*-algebras $\mathrm{Hom}_{E_*-\mathrm{alg}}(E_*MU, E_*)$.

The proof is completed by showing that the latter is in bijection with the set of complex orientations of E, using $E_*MU \cong E_*[b_i^E]$ as E_*-algebras, where the generators b_i^E come from $E_*(\mathbb{C}P^\infty)$. The result follows from the corresponding universal coefficients isomorphism

$$E^2(\mathbb{C}P^\infty) \cong \mathrm{Hom}_{E_*}(E_{*+2}(\mathbb{C}P^\infty), E_*).$$

□

Example 3.24. The integral Eilenberg-MacLane spectrum $H\mathbb{Z}$ admits a canonical complex orientation. This corresponds to the *Thom orientation*:

$$MU \to H\mathbb{Z}$$

which is a morphism of ring spectra, which induces an isomorphism on homotopy groups π_n, for $n \leq 0$.

3.5. **Quillen's theorem.** The spectrum MU has a canonical complex orientation, hence there is an induced formal group law F_{MU} which is classified by a morphism of rings $L \to MU_*$, where L is the Lazard ring.

Theorem 3.25. *(Quillen's theorem.) The morphism $L \to MU_*$ is an isomorphism of rings.*

Proof. (Indications.) There are two aspects of this theorem; the statement that the coefficient ring MU_* is a polynomial ring isomorphic to the Lazard ring (as graded rings) and the assertion that the formal group law F_{MU} is universal.

The classical approach to the calculation of MU_* uses the *Adams spectral sequence*; this establishes, in particular, that the ring MU_* has no additive torsion, which implies that the Hurewicz morphism

$$MU_* \to H\mathbb{Z}_* MU$$

is a monomorphism of rings.

The homology $H\mathbb{Z}_*MU$ is the coefficient ring of the spectrum $H\mathbb{Z} \wedge MU$, which has two orientations, by Lemma 3.4. The ring $H\mathbb{Z}_*MU$ represents the strict isomorphisms between the additive formal group law and a second formal group law.

The proof of the theorem depends on an analysis of the Hurewicz morphism on the modules of indecomposables. □

Remark 3.26. Quillen [Qui71] gives a direct proof of this theorem, which exploits a *geometric* interpretation of the (co)homology theory induced by MU.

Corollary 3.27. *Let R be a graded commutative ring equipped with a morphism of graded rings $MU_* \to R$ (equivalently, a graded formal group law G over R). Then there is a bijection between*

$$\mathrm{Hom}_{MU_*-\mathrm{alg}}(MU_*MU, R)$$

and the set of strict isomorphisms of formal group laws of the form $G \to G'$.

Proof. (Indications.) Combine the algebraic statement used at the end of the proof of Theorem 3.23 with Quillen's theorem, Theorem 3.25. □

Remark 3.28. The previous result can be made more precise: (MU_*, MU_*MU) is naturally equipped with the structure of an affine groupoid scheme, which represents the functor $R \mapsto \mathrm{FGL}^{\mathrm{SI}}(R)$.

This is a case of a general construction. A commutative ring spectrum E is said to be flat if E_*E is a flat left E_*-module: if E is flat, then there is a natural isomorphism

$$E_*E \otimes_{E_*} E_*X \stackrel{\cong}{\to} E_*(E \wedge X),$$

for any spectrum X, in particular for $X = E$ and $X = E \wedge E$.

The structure morphisms of the affine groupoid scheme are induced by the following morphisms:

(1) $E \cong S \wedge E \stackrel{\eta \wedge E}{\to} E \wedge E \stackrel{E \wedge \eta}{\leftarrow} E \wedge S \cong E$, inducing the left and right units $\eta_L, \eta_R : E_* \leftrightarrows E_*E$;
(2) $\mu : E \wedge E \to E$, inducing the augmentation $\epsilon : E_*E \to E_*$;
(3) $\tau : E \wedge E \to E \wedge E$, inducing the conjugation $\chi : E_*E \to E_*E$;

(4) $E \wedge \eta \wedge E : E \wedge E \cong E \wedge S \wedge E \to E \wedge E \wedge E$, inducing the diagonal $E_*E \to E_*E \otimes_{E_*} E_*E$.

The structure (E_*, E_*E) is referred to as the *homology cooperations* for the flat commutative ring spectrum E.

Example 3.29. If $E = H\mathbb{Z}/p$, for a prime p, then $H\mathbb{Z}/p_* \cong \mathbb{F}_p$, concentrated in degree zero, and the associated structure is a *Hopf algebra* $H\mathbb{Z}/p_*H\mathbb{Z}/p$, which is the dual of the Steenrod algebra $\mathscr{A} \cong [H\mathbb{Z}/p, H\mathbb{Z}/p]^*$ of graded stable cohomology operations. For $p = 2$, $H\mathbb{Z}/2_*H\mathbb{Z}/2$ is simply the automorphisms of the additive formal group over $\mathbb{F}_2$-algebras (a similar interpretation is available for p odd, taking into account graded commutativity).

3.6. **Consequences of Quillen's theorem.** As observed by Morava [Mor85], Quillen's theorem implies that the geometry of the stack of one dimensional commutative formal groups has profound implications in topology. In particular, working p-locally, the stratification of the moduli stack by height leads to the *chromatic filtration* of stable homotopy theory.

Example 3.30. For n a positive integer and p a prime, there exists a complex oriented commutative ring spectrum $K(n)$ with coefficient ring $\mathbb{F}_p[v_n^{\pm 1}]$, where $|v_n| = 2(p^n - 1)$ and the associated formal group is the Honda group law with p-series $[p](x) = v_n x^{p^n}$.

Remark 3.31. Since MU is a highly structured commutative ring spectrum, it is straightforward to construct $K(n)$ as an MU-module, by killing the appropriate elements of the coefficient ring of MU and localizing.

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