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On linear stability and syzygy stability

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Abstract
In previous works, the authors investigated the relationships between linear stability of a generated linear series $|V|$ on a curve C, and slope stability of the syzygy vector bundle $M_{V,L} := \ker(V \otimes \mathcal{O}_C \to L)$. In particular, the second named author and L. Stoppino conjecture that, for a complete linear system $|L|$, linear (semi)stability is equivalent to slope (semi)stability of M_L. The first and third named authors proved that this conjecture holds in the two opposite cases: hyperelliptic and generic curves. In this work we provide a counterexample to this conjecture on any smooth plane curve of degree 7.

Keywords Vector bundle · Stability · Syzygy bundles · Butler's conjecture · Mistretta-Stoppino conjecture

1 Introduction

Let C be a smooth projective curve over $\mathbb{C}$, and let L be a globally generated line bundle on C, with $\deg L = d$ and $\dim L = h^0(C, L) - 1$, let $V \subseteq H^0(C, L)$ a subspace of dimension $r + 1$ generating L. Then $M_{V,L} := \ker(V \otimes \mathcal{O}_C \to L)$ is a rank r vector bundle, which appears in different ways and has been given different names in the literature (cf. [4, 6, 7, 9, 11–13]).

Slope semistability of $M_{V,L}$ for a generic linear subsystem of a generic generated line bundle on a generic curve was conjectured by Butler in (cf. [7]) and proven in (cf. [4]). An analogue conjecture is still open for higher rank vector bundles.

In [15] the second named author and L. Stoppino investigate the relationships between linear (semi)stability of the linear series $|V| \subseteq |L|$, and slope (semi)stability of $M_{V,L}$. In particular, it is immediate to show that slope (semi)stability of $M_{V,L}$ implies linear (semi)stability of $|V|$ (cf. Lemma 2.3 below), and they prove that the two conditions are equivalent in some cases and give some examples when they aren't. Furthermore they conjecture that for

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complete linear systems, the two conditions are always equivalent. The only evidence on that conjecture lied in the fact that the conjecture seemed likely to hold on general curves.

In fact the first and third named authors showed in [8] that the conjecture holds when C is a hyperelliptic curve or a Brill-Noether-Petri general curve.

The purpose of this work is to show that this conjecture does not hold for smooth plane curves. In particular, we prove it does not hold on any smooth plane curve of degree 7:

Theorem 1.1 *Let C be a smooth plane curve of degree $d = 7$. Then $W^2_{15}(C)$ is non-empty and a general element L in any of its components satisfies*:

i. The complete linear series $|L|$ is linearly stable.
ii. The vector bundle M_L is not semistable.

2 Notations and previous results

Throughout this work, C will denote a smooth projective curve of genus $g \geqslant 2$ over the field of complex numbers. We will denote $\gamma = \gamma(C)$ the gonality of C, *i.e.* γ is the smallest integer such that there exists a degree γ map to $\mathbb{P}^1$. Another invariant that we make use of, closely related to gonality, is the Clifford Index of curve, it defined for a curve C of genus $g \geqslant 4$ as

$$\mathrm{Cliff}(C) := \min\{\deg(L) - 2(h^0(L) - 1) \mid L \in \mathrm{Pic}(C)\ ,\ h^0(L) \geqslant 2\ , h^1(L) \geqslant 2\}\ ,$$

it is defined to be $\mathrm{Cliff}(C) = 0$ when $g = 2$ and $\mathrm{Cliff}(C) = 0$ or 1 for $g = 3$, according to whether C is hyperelliptic or not. The Clifford Index satisfies the following inequalities with respect to the gonality:

$$\gamma(C) - 3 \leqslant \mathrm{Cliff}(C) \leqslant \gamma(C) - 2\ ,$$

the case $\mathrm{Cliff}(C) \leqslant \gamma(C) - 2$ holding for general $\gamma(C)$-gonal curves in the moduli space $\mathcal{M}_g$ of smooth curves of genus g. Furthermore the equality $\mathrm{Cliff}(C) = 0$ holds if and only if C is hyperelliptic.

We will denote $\mathrm{Pic}(C)$ the Picard group of line bundles, and $\mathrm{Pic}^d(C)$ those of degree d. For $L \in \mathrm{Pic}(C)$ we will denote $|L| = \mathbb{P}(H^0(C, L)^*)$ the complete linear series of effective divisors linearly equivalent to L, and $|V|$, with $V \subset H^0(C, L)$, a linear subseries of $|L|$.

To simplify some of the notations we will use divisors (up to linear equivalence) and the additive notation instead of line bundles, writing $|D|$ for $|\mathcal{O}_C(D)|$, $h^0(D)$ for $h^0(C, \mathcal{O}_C(D))$, and so on. We will denote ω_C the canonical line bundle and K_C a canonical divisor.

We will denote as usual the Brill-Noether loci by

$$W^r_d(C) = \{P \in \mathrm{Pic}^d(C) \mid h^0(C, P) \geqslant r + 1\}\ .$$

When the expected dimension of $W^r_d(C)$ is greater than or equal to 0, this locus is non-empty and every component of such a locus has dimension greater than or equal to this expected dimension, which is the Brill-Noether number

$$\rho(d, r, g) = g - (r + 1)(g - d + r)\ .$$

Let E be a vector bundle on C, the slope of E is

$$\mu(E) := \frac{\deg E}{\mathrm{rk} E}\ .$$

Definition 2.1 We say that the vector bundle E is *stable* (respectively *semistable*) if any subbundle $0 \neq F \subsetneq E$ satisfies

$$\mu(F) < \mu(E) \quad (\text{respectively } \mu(F) \leqslant \mu(E)).$$

Let L be a line bundle on C, and let $V \subseteq H^0(C, L)$ a linear subspace. We say that the linear series $|V|$ generates the line bundle L', if L' is the image of the map $V \otimes \mathcal{O}_C \to L$. The dimension of a linear series $|V|$ is $\dim |V| = \dim V - 1$.

We say that a linear series $|V|$, with V a linear subspace in $H^0(C, L)$, is a g^r_d if $\deg L = d$ and $\dim V \geqslant r + 1$.

Similarly, we say that a line bundle L is a g^r_d if $\deg L = d$ and $h^0(C, L) \geqslant r + 1$. We say that L is a *complete* g^r_d if $\deg L = d$ and $h^0(C, L) = r + 1$.

We denote $M_{V,L} := \ker(V \otimes \mathcal{O}_C \to L)$ and $M_L := M_{H^0(L),L}$. If V generates L then

$$\mu(M_{V,L}) = -\frac{\deg L}{\dim |V|}$$

Definition 2.2 We say that a pair (L, V), where L is a line bundle and $V \subseteq H^0(C, L)$, is a *generating pair* if the global sections in V generate L. We say the linear series $|V|$, or the generating pair (L, V), is *linearly stable* (respectively *linearly semistable*) if for any subspace $W \subsetneq V$ with $\dim W \geqslant 2$, the line bundle L' generated by W satisfies

$$\frac{\deg L'}{\dim |W|} > \frac{\deg L}{\dim |V|} \quad \left(\text{respectively } \frac{\deg L'}{\dim |W|} \geqslant \frac{\deg L}{\dim |V|}\right).$$

The following lemma is proven in [15]:

Lemma 2.3 *If a generating pair* (L, V) *is such that* $M_{V,L}$ *is a* (*semi*)*stable vector bundle, then* (L, V) *is linearly* (*semi*)*stable.*

Proof This is Remark 3.2 of [15]. The main point is that for any subspace $0 \neq W \subsetneq V$ generating $L' \subseteq L$ we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{W,L'} & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{V,L} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0\ ,
\end{array} \tag{1}$$

where $\mu(M_{V,L}) = -\frac{\deg L}{\dim |V|}$ and $\mu(M_{W,L'}) = -\frac{\deg L'}{\dim |W|}$, therefore (semi)stability of $M_{V,L}$ implies linear (semi)stability of (L, V).

The second named author and L. Stoppino investigate the reverse implication of the above Lemma in [15], in particular they conjecture that equivalence holds in the following cases:

Conjecture 2.4 *If* (L, V) *is a generating pair on* C, *such that*

$$\deg L \leqslant 2\ \dim|V| + \mathrm{Cliff}(C)$$

then linear (semi)stability of (L, V) *is equivalent to (semi)stability of* $M_{V,L}$.

In the same work, the authors prove that the conjecture above holds if $V = H^0(C, L)$ is a complete linear series, and they apply this to prove the stability of some syzygy bundles M_L. They remark that in general the equivalence does not hold, and observing the counterexamples they can construct, they state the stronger conjecture:

Conjecture 2.5 *If* (L, V) *is a generating pair on* C, *such that*

$$\deg L \leqslant \gamma(C) \cdot \dim|V|$$

then linear (semi)stability of (L, V) *is equivalent to (semi)stability of* $M_{V,L}$.

Concerning complete linear series, they conjecture that equivalence always holds in this case.

Conjecture 2.6 *If* $V = H^0(C, L)$ *is a complete base point free linear series, then linear (semi)stability of the linear series* $|L|$ *is equivalent to (semi)stability of* M_L.

The first and third named author proved in [8] that Conjecture 2.6 does hold in the two opposite cases: when C is a hyperelliptic curve and when C is a Brill-Noether-Petri general curve.

The aim of this work is to show that Conjecture 2.6 does not hold in general: we can give counterexamples on any smooth plane curve of degree 7 (so genus 15 and gonality 6). In particular we will show that on such a curve C a generic line bundle in any component of the (non-empty) Brill-Noether locus $W^2_{15}(C)$ is globally generated, has $h^0(C, L) = 3$, is linearly stable, but M_L is a rank 2 vector bundle which is not semistable, hence not stable.

These constructions do not contradict Conjecture 2.5 however, as they provide line bundles L with $\deg L = 15 > \gamma(C) \cdot \dim|V| = 12$.

3 Dimension 2 linear series on higher gonality curves

In this section we construct dimension 2 complete linear series on curves with high gonality, and show that they are linearly stable. Most of the results we will make use of are stated in Voisin's work [18].

The main results are obtained as consequences of the following lemmas, to be found in [17] and [10] (cf. [1], Chapter IV, for more details):

Lemma 3.1 (Mumford) *Let* C *be a non-hyperelliptic curve of genus* $g \geqslant 4$. *Let* d, r *be two integers such that* $2 \leqslant d \leqslant g-2$ *and* $0 < 2r \leqslant d$. *If* $\dim W^r_d(C) \geqslant d - 2r - 1$ *then* C *is trigonal or bielliptic or a smooth plane quintic. In particular* $\gamma(C) \leqslant 4$.

Lemma 3.2 (Keem) *Let C be a curve of genus $g \geqslant 11$. Let d, r be two integers such that $4 \leqslant d \leqslant g+r-4$ and $r>0$. If* $\dim W^r_d(C) \geqslant d-2r-2 \geqslant 0$ *then* $\gamma(C) \leqslant 4$.

We give the complete proof of the following theorem and of the constructions provided in [18], we think it is worthwhile to go over all details in the proof, for clarity of the exposition, for better understanding of the proofs involved later, and for the the interest in itself of the proof.

Theorem 3.3 (Voisin) *Let C be a curve of genus $g \geqslant 11$ and gonality $\gamma \geqslant 5$. Then a general element D in any component of $W^1_{g-2}(C)$ satisfies*:

i. $|D|$ *is base point free and* $h^0(D)=2$.
ii. $|K_C - D|$ *is base point free and* $h^0(K_C-D)=3$.

Proof The expected dimension of $W^1_{g-2}(C)$ is $\rho(g-2,1,g)=g-6$, and according to Mumford's Lemma 3.1 above, as the gonality of the curve is $\gamma \geqslant 5$, no component has dimension greater than or equal to $g-5$.

Applying Keem's Lemma 3.2 above, $\dim W^1_{g-3}(C) \leqslant g-8$. Therefore if we denote

$$W^1_{g-3}+(C) := \{P \in \mathrm{Pic}^{g-2}(C) \mid P = Q \otimes \mathcal{O}_C(p)\,,\ Q \in W^1_{g-3}(C)\,,\ p \in C\}$$

then $W^1_{g-3}+(C)$ has empty interior in $W^1_{g-2}(C)$ by a dimensional count.

Now an element $L \in W^1_{g-2}(C)$ lies in $W^1_{g-3}+(C)$ if either it has $h^0(C,L) \geqslant 3$ or it has a base point. Therefore a general element of a component of $W^1_{g-2}(C)$ is base point free and has $h^0(C,L)=2$, and this proves the first point.

In order to prove the second point, we prove the following

Claim every component of $W^2_{g-1}(C)$ has dimension at most $g-8$.

If the Claim holds, then we can proceed as above: set

$$W^2_{g-1}+(C) := \{P \in \mathrm{Pic}^{g}(C) \mid P = Q \otimes \mathcal{O}_C(p)\,,\ Q \in W^2_{g-1}(C)\,,\ p \in C\}\,,$$

then, as every component of $W^2_g(C)$ has dimension at least $\rho(g,2,g)=g-6$, $W^2_{g-1}+(C)$ has empty interior in $W^2_g(C)$ as it has smaller dimension. Therefore a generic point F in a component of $W^2_g(C)$ does not lie in $W^2_{g-1}+(C)$ so it is a base point free 2 dimensional linear series.

As the map $P \mapsto P^* \otimes \omega_C$ is an isomorphism $\mathrm{Pic}^g(C) \to \mathrm{Pic}^{g-2}(C)$ which restricts to an isomorphism $W^2_g(C) \to W^1_{g-2}(C)$, then a generic point in a component of $W^1_{g-2}(C)$ correspond to a generic point in a component of $W^2_g(C)$ and the second point is proven.

Proof of the Claim. By contradiction, suppose there is a component X of $W^2_{g-1}(C)$ of dimension greater than or equal to $g-7$. Then a general element of such a component does not lie in $W^2_{g-2}+(C)$, as by Keem's Lemma (Lemma 3.2 above) $\dim W^2_{g-2} \leqslant g-9$. Therefore a general $L \in X$ is base point free and has $h^0(C,L)=3$. By same argument, as $L^* \otimes \omega_C$ varies in a component of W^2_{g-1} of dimension at least $g-7$ as well, then for a general such L we have that $L^* \otimes \omega_C$ is base point free with $h^0(C, L^* \otimes \omega_C)=3$.

Now, let us consider the morphism $\varphi_L : C \to \mathbb{P}^2$ induced by L. It cannot be an immersion, otherwise its image would be a plane curve of degree $g-1$ and genus g, which is impossible for $g \geqslant 11$. Therefore there is a couple $(p,q) \in C^2$ such that $H^0(C, L(-p-q)) = H^0(C, L(-p)) = H^0(C, L(-q)) \cong \mathbb{C}^2$. In the following, we choose divisor notation. Let us choose a divisor $\Delta \in |L|$ and let us denote $\Delta' := \Delta - p - q$, then $\mathcal{O}_C(\Delta') = L(-p-q) \in W^1_{g-3}$.

As both $|\Delta|$ and $|K_C - \Delta|$ are base point free, we can see that $|K_C - \Delta'| = |K_C - \Delta + p + q|$ is base point free as well: in fact the only base points could be $p, q \in C$, however, as $h^0(C, L(-p-q)) = h^0(C, L(-p)) = h^0(C, L(-q)) = 2$, applying Riemann-Roch we have that $h^0(K_C - \Delta + p) = h^0(K_C - \Delta + q) = 3$, and $h^0(K_C - \Delta + p + q) = 4$.

Therefore we have a base point free linear series $K_C - \Delta'$, let us consider the induced map:

$$\varphi_{K_C - \Delta'} : C \to \mathbb{P}^3 ,$$

then either this map is birational, and therefore there is a finite number of couples $(x, y) \in C^2$ such that

$$h^0(K_C - \Delta' - x - y) = h^0(K_C - \Delta' - x) = 3$$

and so

$$h^0(\Delta' + x + y) = 3 = h^0(\Delta') + 1 ;$$

or the map is a degree m morphism

$$\varphi_{K_C - \Delta'} : C \to \overline{C} \subset \mathbb{P}^3 ,$$

and in this case there are infinite couples (x, y) satisfying $h^0(\Delta' + x + y) = 3 = h^0(\Delta') + 1$, these are all the couples contained in the fibers of $\varphi_{K_C - \Delta'}$. However, as $|K_C - \Delta' - p - q| = |K_C - \Delta|$ is base point free, we see that such fibers cannot contain more than two points (counted with multiplicity), so the degree of $\varphi_{K_C - \Delta'}$ must be $m = 2$ if it is not a birational map, and the set of such couples (x, y) has dimension 1.

Now, let us consider the following scheme:

$$\begin{aligned} Y := \{ (\Delta', x, y) \mid & |\Delta'| \text{ is a complete } g^1_{g-3} , \ |\Delta' + x + y| \text{ and } |K_C - \Delta' - x - y| \\ & \text{are complete and base point free } g^2_{g-1} \} , \end{aligned}$$

and the two maps:

$$pr_1 : Y \to W^1_{g-3}(C) \text{ and } \pi : Y \to W^2_{g-1}(C), (\Delta', x, y) \mapsto \Delta' + x + y .$$

According to the argument above, there must be a component Y_0 dominating $X \subset W^2_{g-1}(C)$ by the morphism π, and the fibers of $pr_1 : Y_0 \to W^1_{g-3}$ have dimension at most 1. Now as $\dim X \geqslant g - 7$ by hypothesis, and $\dim W^1_{g-3} \leqslant g - 8$ by Lemma 3.2, then we deduce that $pr_1(Y_0)$ must contain an open dense subset U_0 of a $(g-8)$-dimensional component $W \subset W^1_{g-3}$, and the general fibers of pr_1 must be 1-dimensional.

According to the description above, for $\Delta' \in U_0$ the residual linear series $K_C - \Delta'$ is a complete and base point free g^3_{g+1} on C, inducing a degree 2 map

$$\varphi_{K_C - \Delta'} : C \to \overline{C} \subset \mathbb{P}^3 .$$

Let us consider the normalization $\nu : \widetilde{C} \to \overline{C}$, and the map $\widetilde{\varphi} : C \to \widetilde{C}$. Then $\widetilde{C}$ has a complete linear series of dimension 3 and degree $\frac{g+1}{2}$ therefore it is not a rational curve. Then when Δ' varies in U_0 the curve $\widetilde{C}$ and the map $\widetilde{\varphi}$ are fixed.

Now for a general point $K_C - \Delta$ in $K_C - X$, we have $K_C - \Delta = K_C - (\Delta' + x + y)$, where x, y are points in a fiber of $\widetilde{\varphi}$. Therefore $K_C - \Delta$ is the pull back through $\widetilde{\varphi}$ of a $g^2_{(g-1)/2}$ on $\widetilde{C}$. So we have

$$\dim W^2_{(g-1)/2}(\widetilde{C}) \geqslant \dim X \geqslant g - 7 .$$

By the Riemann-Hurwitz formula, the genus $g' = g(\widetilde{C})$ satisfies $g' \leqslant (g+1)/2$. So we have the following inequalities:

$$g - 7 \leqslant g' \leqslant (g+1)/2$$

and therefore we have that g is odd and satisfies $11 \leqslant g \leqslant 15$ and $\widetilde{C}$ is a curve of genus g' such that $g - 7 \leqslant g' \leqslant (g+1)/2$ and that $\dim W^2_{(g-1)/2}(\widetilde{C}) \geqslant g - 7$.

We can show that these inequalities cannot hold, and so we have proven the claim. In fact, according to the inequalities above, we have the following cases:

i. if $g = 15$ then $g' = 8$ and $\dim W^2_7(\widetilde{C}) \geqslant 8$ which is impossible, as it would imply that $W^2_7(\widetilde{C}) = \mathrm{Pic}^7(\widetilde{C})$;
ii. if $g = 13$ then $g' = 7$ or $g' = 6$ and $\dim W^2_6(\widetilde{C}) \geqslant 6$ which is impossible, as the case $g' = 6$ would imply that $W^2_6(\widetilde{C}) = \mathrm{Pic}^6(\widetilde{C})$, and the case $g' = 7$ would imply that $W^2_6(\widetilde{C}) \subsetneq \mathrm{Pic}^6(\widetilde{C})$ has codimension at most 1 and is therefore equal to the theta divisor;
iii. if $g = 11$ then $g' = 6$ or $g' = 5$ or $g' = 4$ and $\dim W^2_5(\widetilde{C}) \geqslant 4$ which is impossible for similar arguments.

This completes the proof of the Claim and therefore of the Theorem.

We need the following Lemma proven in [15, Proposition 8.1]:

Lemma 3.4 *Let $V \subseteq H^0(C, L)$ be a dimension 2 linear series on C, generating L, such that*

$$\varphi_{|V|} : C \to \overline{C} \subset \mathbb{P}^2$$

is a birational morphism.

Then $|V|$ is linearly (semi)stable if and only if all the points $p \in \overline{C}$ have multiplicity $m_p(\overline{C}) < \deg L/2$ (or $m_p(\overline{C}) \leqslant \deg L/2$ for semistability).

Now we can prove the following:

Theorem 3.5 *Let C be a curve of genus $g \geqslant 11$ and gonality $\gamma \geqslant 5$. Then a general element L in any component of $W^2_g(C)$ satisfies:*

i. The complete linear series $|L|$ is base point free and $h^0(C, L) = 3$.
ii. The complete linear series $|L|$ induces a birational morphism $\varphi_L : C \to \overline{C} \subset \mathbb{P}^2$, where $\overline{C} \subset \mathbb{P}^2$ is a singular curve of degree g.
iii. The complete linear series $|L|$ is linearly stable.

Proof The first point follows from Theorem 3.3 above. We have to show that such a general element induces a birational map to its image and is linearly stable.

Let us prove that for a general divisor $D \in W^2_g$ in an irreducible component of W^2_g, the linear series $|D|$ induces a birational morphism $\varphi_D : C \to \overline{C} \subset \mathbb{P}^2$. Let us observe that by the first point the linear series $|D|$ and $|K_C - D|$ are complete and base point free g^2_g and g^1_{g-2} .

As φ_D cannot be an embedding, then there exist $p, q \in C$ such that $H^0(D-p) = H^0(D-q) = H^0(D-p-q) \cong \mathbb{C}^2$. So the divisor $D' := D - p - q$ satisfies:

i. $|D'|$ is a complete g^1_{g-2};
ii. $|K_C - D'|$ is a complete base point free g^2_g;
iii. $|D' + p + q|$ is a complete base point free g^2_g;
iv. $|K_C - D' - p - q|$ is a complete base point free g^1_{g-2}.

Furthermore, two points $p, q \in C$ satisfy $\varphi_D(p) = \varphi_D(q)$ if and only if the divisor $D' = D - p - q$ satisfies the conditions above.

Now let us consider the following scheme:

$$\begin{aligned} Y := \{(D', x, y) \mid\ & |D'| \text{ is a complete } g^1_{g-2}\ , \\ & |D' + x + y| \text{ is a complete and base point free } g^2_g\ , \\ & |K_C - D' - x - y| \text{ is a complete and base point free } g^1_{g-2}\}\ , \end{aligned}$$

and the two maps:

$$pr_1 : Y \to W^1_{g-2}(C) \text{ and } \pi : Y \to W^2_g(C), (D', x, y) \mapsto D' + x + y\ .$$

According to the description above, the morphism π is dominant on every component of $W^2_g(C)$, and the fiber of the morphism π, over a general divisor D in a component of $W^2_g(C)$, is the set of all triples $(D - x - y, x, y)$ such that $\varphi_D(x) = \varphi_D(y)$. Remark that the fiber over a divisor $D' \in \mathrm{Im}(pr_1)$ of $pr_1 : Y \to W^2_g(C)$ is the set of all triples (D', x, y) such that $\varphi_{K_C - D'}(x) = \varphi_{K_C - D'}(y)$.

Therefore, in order to prove point (ii) in the statement of the theorem, let us suppose by contradiction that for a general divisor D in a component X of $W^2_g(C)$ the morphism $\varphi_D : C \to \mathbb{P}^2$ is not birational to its image. Then the fibers of π have positive dimension, so there is a component $Y_0 \subseteq Y$ such that $\dim Y_0 > \dim X = g - 6$. That component Y_0 must be dominant through pr_1 onto a component W_0 of $W^1_{g-2}(C)$ as well, otherwise the generic fiber of pr_1 would have dimension 2 which is impossible. Then we deduce that for a general element $D' \in W_0$, the morphism $\varphi_{K_C - D'} : C \to \overline{C} \subset \mathbb{P}^2$ is not birational, and has degree $m > 1$. With the same argument as in the proof of Theorem 3.3, as we know that $|K_C - D' - x - y|$ is a complete base point free g^1_{g-2} for some $x, y \in C$, we see that in fact it must be of degree 2 in this case.

Then we proceed as in the proof of Theorem 3.3, considering the normalization $\nu : \widetilde{C} \to \overline{C}$, and the map $\widetilde{\varphi} : C \to \widetilde{C}$. Then $\widetilde{C}$ has a complete linear series of dimension 2 and degree $\frac{g}{2}$ therefore it is not a rational curve. And so when D' varies in W_0 the curve $\widetilde{C}$ and the map $\widetilde{\varphi}$ are fixed. Let us call $g' = g(\widetilde{C})$ its genus.

The divisor $K_C - D'$ is the pull back through $\widetilde{\varphi}$ of a $g^2_{g/2}$ on $\widetilde{C}$. So we have

$$(g+1)/2 \geqslant g' \geqslant \dim W^2_{g/2}(\widetilde{C}) \geqslant \dim W_0 = g - 6\ ,$$

the first inequality following from Riemann-Hurwitz formula. As $g \geqslant 11$ by hypothesis, and even, then we must have $g = 12$, $g' = g/2 = 6$, and $\dim W^2_{g/2}(\widetilde{C}) = g - 6 = g'$ which is impossible. So this completes the proof of point (ii).

So we have proven that for a general divisor D in a component of W^2_g the linear series $|D|$ and $|K_C - D|$ are base point free, and the map $\varphi_D : C \to \overline{C} \subset \mathbb{P}^2$ is birational. Let us prove that the multiplicity of any point $p \in \overline{C}$ is at most 2.

Consider the scheme defined above:

$$\begin{aligned} Y := \{(D', x, y) \mid\ & |D'| \text{ is a complete } g^1_{g-2}\ , \\ & |D' + x + y| \text{ is a complete and base point free } g^2_g\ , \\ & |K_C - D' - x - y| \text{ is a complete and base point free } g^1_{g-2}\}\ , \end{aligned}$$

and the two maps:

$$pr_1 : Y \to W^1_{g-2}(C) \text{ and } \pi : Y \to W^2_g(C), (D', x, y) \mapsto D' + x + y\ .$$

We claim that every component of Y dominating a component of W^2_g through π, dominates through pr_1 a component of W^1_{g-2} as well.

If the claim holds, we can show that the multiplicity $m_p(\overline{C})$, of any point of $p \in \overline{C} = \varphi_D(C) \subset \mathbb{P}^2$, is at most 2. In fact in that case, for a general D in a component of W^2_g, we have that for any couple $(x, y) \in C^2$ such that $H^0(D - x) = H^0(D - y) = H^0(D - x - y)$, the divisor $D' = D - x - y$ is general in a component of W^1_{g-2} so it is a base point free divisor, therefore the image $\varphi_D(x) = \varphi_D(y)$ cannot have multiplicity higher than 2.

Let us prove now that the claim above holds: we have to show that any component Y_0 of Y that dominates through π a component of W^2_g dominates a component of W^1_{g-2} through p_1.

Recall that by Mumford's Lemma 3.1 all components of W^1_{g-2} have dimension $g - 6$ (hence all components of W^2_g as well).

Suppose by contradiction that there is a component Y_0 of Y, dominating a component of W^2_g, which does not dominate any component in W^1_{g-2}. Then the component Y_0 has dimension $g - 6$, as the fiber $\pi^{-1}(D)$ of a generic divisor $D \in W^2_g$ is finite. The image $pr_1(Y_0)$ is then a locus strictly contained in a component of W^1_{g-2}, and therefore of dimension $g - 7$, as the fibers cannot have dimension greater than 1.

Now, for a given $D' \in pr_1(Y_0)$, the divisor $K_C - D'$ is base point free and induces a map $\varphi_{K_C - D'} : C \to \overline{C} \subset \mathbb{P}^2$. As the fiber $pr_1^{-1}(D')$ is positive dimensional, the morphism $\varphi_{K_C - D'}$ is not birational, and as $|K_C - D' - x - y|$ is complete and base point free, then $\varphi_{K_C - D'}$ is a degree 2 morphism.

Proceeding as in the previous proofs, we see that the normalization $\widetilde{C}$ of $\overline{C}$ and the morphism $\widetilde{\varphi} : C \to \widetilde{C}$ do not vary when D' varies in $pr_1(Y_0)$, and that the divisor $K_C - D'$ is the pull back $\widetilde{\varphi}^* E$ of a divisor $E \in W^2_{g/2}(\widetilde{C})$.

Let us call $g' = g(\widetilde{C})$ the genus of $\widetilde{C}$, then we have the following inequalities:

$$(g + 1)/2 \geqslant g' \geqslant \dim W^2_{g/2}(\widetilde{C}) \geqslant g - 7\ .$$

Then g must be even and we have the following cases:

i. $g = 12$ and $5 \leqslant g' \leqslant 6$;
ii. $g = 14$ and $g' = 7$.

The first case would have either $g' = 5$ and $\dim W^2_6(\widetilde{C}) = 5$, which is impossible; or $g' = 6$ and $\dim W^2_6(\widetilde{C}) = 5$ which is impossible as well.

The second case satisfies $g = 14$, $g' = 7$, and $\dim W^2_6(\widetilde{C}) = 7$, which is impossible again.

Therefore we have shown the claim that every component of Y dominating a component of W^2_g through π, dominates through pr_1 a component of W^1_{g-2} as well, and we have seen that this implies that for a generic $D \in W^2_g(C)$ the morphism $\varphi_D : C \to \overline{C}$ is birational and its image $\overline{C}$ has points of multiplicity at most 2. Now to complete the proof of point (iii) in the theorem we just have to apply Lemma 3.4.

Remark 3.6 Theorem 3.5 is a consequence of Voisin's work in [18]. Most of the statements we present here are not proven in that work, (they make use of similar techniques as in previous results in the same article *i.e.* as in the proof of Theorem 3.3). We think it is worthwhile giving a full presentation of those results and of the proof of Theorem 3.3 here, remarking that the originality of the argument is due to Voisin.

We remark that linear stability was introduced by Mumford as it implies Chow stability of the corresponding point in the Hilbert scheme (cf. [2, 6]). We will not use Chow stability in this work, however we notice that we have the following:

Corollary 3.7 *Let C be a curve of genus $g \geqslant 11$ and gonality $\gamma \geqslant 5$, and let L be a general element in any component of $W^2_g(C)$. Then the curve C is Chow stable with respect to L.*

4 Counterexamples on plane curves

In this section we show that any smooth plane curve of degree 7 admits counterexamples to Conjecture 2.6.

Theorem 4.1 *Let C be a smooth plane curve of degree $d = 7$. Then a general element L in any component of $W^2_{15}(C)$ satisfies:*

i. The complete linear series $|L|$ is base point free and linearly stable.
ii. The vector bundle M_L is not semistable.

Proof The first point is given by Theorem 3.5 above, as the curve has genus $g = 15$ and gonality $\gamma = 6$. We have to exhibit a destabilization of M_L in this case.

Let us consider the line bundle $B = \mathcal{O}_{\mathbb{P}^2}(1)_{|C}$, it is a line bundle of degree 7 with $h^0(C, B) = 3$. Using the exact sequence

$$0 \to M_L \otimes B \to H^0(C, L) \otimes B \to L \otimes B \to 0$$

and passing to cohomology, we have:

$$0 \to H^0(C, M_L \otimes B) \to H^0(C, L) \otimes H^0(C, B) \to H^0(C, L \otimes B)\,.$$

Now, let us call $W = W^0_6(C) \subset \mathrm{Pic}^6(C)$ the locus of effective divisors of degree 6, then clearly $\dim W = 6$, so the locus

$$(\omega_C \otimes B^*) - W := \{L \in W^2_{15}(C) \mid L = \omega_C \otimes B^* \otimes F^* \ , \ F \in W\}$$

has dimension 6 as well. As every component of $W^2_{15}(C)$ has dimension at least $g - 6 = 9$, then a general line bundle L in such a component is not contained in $(\omega_C \otimes B^*) - W$. Therefore for a general element L of a component of $W^2_{15}(C)$ we have:

$$H^0(C, \omega_C \otimes B^* \otimes L^*) \cong H^1(C, B \otimes L)^* = 0 \ ,$$

so by Riemann-Roch we have

$$h^0(C, B \otimes L) = \deg(B \otimes L) + 1 - g = 8 < \dim H^0(C, L) \otimes H^0(C, B) = 9 \ .$$

Therefore $H^0(C, M_L \otimes B) \neq 0$ and we have an injection:

$$B^* \hookrightarrow M_L$$

which provides a destabilization as $\mu(B^*) = -7 > \mu(M_L) = -15/2$.

Remark 4.2 For a line bundle L as above we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^* & \longrightarrow & H^0(C,B)^* \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow \cong & & \big\downarrow & & \\ 0 & \longrightarrow & M_{V,L} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \ , \end{array} \qquad (2)$$

where $F := \ker(H^0(C, B) \otimes \mathcal{O}_C \twoheadrightarrow B)^*$ is a rank 2 vector bundle. However we cannot have a diagram as in (2) which would provide a linear destabilization; in particular, the line bundle L, being generic, does not admit an injection $B \hookrightarrow L$.

5 Applications, further questions, and remarks

We have proven that all smooth plane curves of degree 7 admit counterexamples to Conjecture 2.6.

Remark 5.1 As smooth plane curves of degree 7 are not generic in the moduli space $\mathcal{M}_{15}$ of smooth genus 15 curves, and as all of them do not satisfy Conjecture 2.6, it would be interesting to describe the locus in $\mathcal{M}_{15}$ of curves not satisfying Conjecture 2.6 and its numerical properties, and in general, it would be interesting to investigate the locus in $\mathcal{M}_g$ of curves not satisfying Conjecture 2.6.

Question 5.2 We remark the techniques in Sect. 3 provide linearly stable complete base point free complete linear series of dimension 2 on all curves with genus $g \geqslant 11$ and gonality $\gamma \geqslant 5$. However the very same techniques cannot be applied to find other counterexamples on plane curves with different degrees, in fact their numerical properties do not allow us to construct destabilizations in a similar way. Therefore we have counterexamples to Conjecture 2.6 only on smooth plane curves of degree 7. We would have to use other techniques to investigate on syzygy stability for plane curves in other degrees. It would be interesting to find other counterexamples in higher degrees.

A consequence of Theorem 4.1 is the following

Corollary 5.3 *Let C be a smooth plane curve of degree 7, let $B(n, d, r)$ be the moduli space of semistable vector bundles E on C with rank n, degree d, and such that $h^0(C, E) \geqslant r$. Then all components of $B(2, 15, 3)$ consist of non generated bundles.*

Proof Suppose by contradiction that there exists a globally generated vector bundle $E \in B(2, 15, 3)$, then there is a vector space $V \subset H^0(C, E)$ of dimension 3 that generates E. Then we have the following exact sequence

$$0 \to E^* \to V^* \otimes \mathcal{O}_C \to L \to 0$$

with $L = \det E$ which lies in a component of $W^2_{15}(C)$. As having a semistable syzygy bundle is an open property, we would have that a general line bundle in that component is base point free and with semistable syzygy bundle, which is a contradiction to Theorem 4.1.

Remark 5.4 It would be interesting to describe the space $B(2, 15, 3)$ for a smooth plane curve C of degree 7, for example its dimension, the number of components etc., knowing that it is nonempty by [5], and it consists of non generated bundles by the corollary above.

Remark 5.5 In a similar way, we know that there is a stable globally generated vector bundle in $B(2, 7, 3)$. Namely it is easy to see that the vector bundle M_B^*, where M_B is the syzygy bundle fitting in the exact sequence

$$0 \to M_B \to H^0(C, B)^* \otimes \mathcal{O}_C \to B \to 0$$

with $B = \mathcal{O}_{\mathbb{P}^2}(1)_{|C}$ is the degree 7 divisor on C, is a stable bundle in $B(2, 7, 3)$: in fact a destabilization would be a line bundle $S \subset M_B$ such that $\deg S \geqslant \mu(M_B) = -7/2$; then S^* would be a base point free line bundle with $\deg S^* \leqslant 3$, which is impossible as the curve C has gonality 6.

Furthermore M_B^* is the unique globally generated semistable vector bundle in $B(2, 7, 3)$. It would be interesting to describe for $d \geqslant 8$ the spaces $B(2, d, 3)$ and the locus of globally generated vector bundles inside any of those spaces, knowing that they are nonempty for $d \geqslant 15$ by [5].

Remark 5.6 In the recent works [3, 14, 16] the second named author considers stable base loci, augmented and restricted base loci for vector bundles. It would be interesting to compute explicitly the base loci in these cases for the unstable bundles M_L^* constructed above. In fact these are globally generated vector bundles, therefore they are nef vector bundles, however they are not semistable, and need not be ample.

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