

Iitaka Fibrations for Vector Bundles

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A vector bundle on a smooth projective variety, if it is generically generated by global sections, yields a rational map to a Grassmannian, called Kodaira map. We investigate the asymptotic behavior of the Kodaira maps for the symmetric powers of a vector bundle, and we show that these maps stabilize to a map dominating all of them, as it happens for a line bundle via the Iitaka fibration. Through this Iitaka-type construction, applied to the cotangent bundle, we give a new characterization of Abelian varieties.

1 Introduction

The aim of this work is to extend the notions of stable base loci, Kodaira maps and Iitaka fibrations to vector bundles, in the same framework as [1].

This work is based on a key concept in birational geometry, that is that, given an algebraic variety X , the global sections of a line bundle on X naturally induce a (rational) map in some projective space. It is well known, for example, that sections of some powers of an ample line bundle give an embedding in a projective space. For more general line bundles this does not hold anymore. There is, however, a classical and well developed theory of Iitaka fibrations describing all the different possible outcomes, see [9].

In the construction of the Iitaka fibration there are three main steps. The first one is to decide if and where these maps are well defined (as regular morphisms), by studying the stable base locus of the given line bundle. Second, the fibration needs to

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behave nicely in an asymptotic way, hence the maps, that we will call *Kodaira maps*, induced by different powers of the same line bundle have to be related. Finally, the images shall stabilize, be controlled by a morphism and give information about the original variety. All of this is known in the case of line bundles.

The main difference respect to the case of line bundles is that global sections of vector bundles naturally induce maps into Grassmannians rather than projective spaces. Whereas the main technical issue is that the rank of the bundle grows as we take powers of it, we are still able to prove that it is possible to give an Iitaka-type construction in this more general case. The results are similar to the ones achieved for line bundles, up to a finite map. In particular, we do not always obtain a fibration, a surjective morphism with connected fibers.

The article is organized as follows:

In Section 2, we recall the main notions and results on asymptotic base loci for vector bundles introduced in [1], fixing some of the definitions. Furthermore, we correct the original definition of augmented and restricted base loci given in [1], as there were some inaccuracies.

In Section 3, we focus on the definition of semiamplessness for vector bundles pointing out that the two characterizations, often considered equivalent in the folklore of vector bundles and positivity constructions, are actually not. In the same section, we provide an Iitaka construction for *strongly semiample* vector bundles.

In the last section, we give an Iitaka-type statement for *Asymptotically Generically Generated* vector bundles. We conclude the section with a characterization of Abelian varieties via Iitaka-type invariants and base loci of the cotangent bundle.

2 Base Loci and Projectivizations

In [1], the authors established relations between positivity properties of vector bundles on a projective variety X and some corresponding properties of asymptotic base loci in X , in the same flavor of what happens for line bundles.

Because of some extra difficulties arising in the case of vector bundles, the various definitions of positivity (i.e., bigness, semiamplessness, etc.) appearing in the literature do differ according to different authors.

In this section, we review the main properties of asymptotic base loci, their relations to positivity properties, and some of the different definitions appearing in the literature, focusing on what will be needed in the Iitaka construction: semiamplessness and asymptotic generic generation.

Even though we will consider only stable base loci, we will also give the definitions and main properties of augmented and restricted base loci, in order to slightly change some definitions stated in [1] which were not completely correct (cf. Remark 2.6).

Notation 2.1. Let X be a normal projective variety over the complex numbers, and E a vector bundle over X . For a point $x \in X$, $E_x = E \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$ denotes the stalk of E at the point x , and $E(x) = E \otimes_{\mathcal{O}_X} \kappa(x)$ denotes the fiber of E at the point x , where $\kappa(x)$ is the residue field at x . Clearly, $E(x)$ is a vector space of dimension $r = \operatorname{rk} E$. \square

Definition 2.2. We define the *base locus* of E (over X) as the subset

$$\operatorname{Bs}(E) := \{x \in X \mid H^0(X, E) \rightarrow E(x) \text{ is not surjective}\},$$

and the *stable base locus* of E (over X) as

$$\mathbb{B}(E) := \bigcap_{m>0} \operatorname{Bs}(\operatorname{Sym}^m E). \quad \square$$

Remark 2.3. The assertions below follow immediately from the definition:

- (i) E is globally generated, that is generated by its global sections, if and only if $\operatorname{Bs}(E) = \emptyset$.
- (ii) As $V := \operatorname{Im}(\bigwedge^{\operatorname{rk} E} H^0(X, E) \rightarrow H^0(X, \det E))$ is a linear subspace of the space of global sections $H^0(X, \det E)$ of the line bundle $\det E$, then the base locus $\operatorname{Bs}(E)$ coincides with the base locus of the linear system $|V|$. Hence it is a closed subset, and carries a natural scheme structure. Therefore the stable base locus is closed as well. \square

Definition 2.4. Let $r = p/q \in \mathbb{Q}^{>0}$ be a positive rational number, with p and q positive integers (not necessarily coprime, cf. Remark 2.5), and A a line bundle on X . We will use the following notation:

$$\begin{aligned} \mathbb{B}(E + rA) &:= \bigcap_{m>0} \operatorname{Bs}(\operatorname{Sym}^{mq} E \otimes A^{mp}), \quad \text{and} \\ \mathbb{B}(E - rA) &:= \bigcap_{m>0} \operatorname{Bs}(\operatorname{Sym}^{mq} E \otimes A^{-mp}). \end{aligned}$$

Let A be an ample line bundle on X , we define the *augmented base locus* of E as

$$\mathbb{B}_+^A(E) := \bigcap_{r \in \mathbb{Q}^{>0}} \mathbb{B}(E - rA),$$

and the *restricted base locus* of E as

$$\mathbb{B}_-^A(E) := \bigcup_{r \in \mathbb{Q}_{>0}} \mathbb{B}(E + rA). \quad \square$$

Remark 2.5. If $r = p'/q'$ is another representation of r as a fraction, with p' and q' positive integers, then $q'p = p'q$, hence

$$\mathrm{Sym}^{q'q}E \otimes A^{q'p} \simeq \mathrm{Sym}^{q'q}E \otimes A^{p'q},$$

therefore, $\mathrm{Bs}(\mathrm{Sym}^{mq'q}E \otimes A^{mq'p}) = \mathrm{Bs}(\mathrm{Sym}^{mqq'}E \otimes A^{mqp'})$ and hence $\mathbb{B}(E + rA)$ is well defined. A similar argument shows that $\mathbb{B}(E - rA)$ is well defined as well. \square

Remark 2.6. In [1], the loci $\mathbb{B}(E + rA)$ and $\mathbb{B}(E - rA)$ were defined by $\mathbb{B}(E + rA) = \mathbb{B}(\mathrm{Sym}^qE \otimes A^p)$ and $\mathbb{B}(E - rA) = \mathbb{B}(\mathrm{Sym}^qE \otimes A^{-p})$. However, as $\mathrm{Sym}^m(\mathrm{Sym}^qE \otimes A^p) \neq \mathrm{Sym}^{mq}E \otimes A^{mp}$, the loci defined here better suite our purposes. \square

Remark 2.7. The definitions above yield the following properties:

- (i) The loci $\mathbb{B}_+^A(E)$ and $\mathbb{B}_-^A(E)$ do not depend on the choice of the ample line bundle A , so we can write $\mathbb{B}_+(E)$ and $\mathbb{B}_-(E)$ for the augmented and restricted base locus of E , respectively.
- (ii) For any $r_1 > r_2 > 0$ we have $\mathbb{B}(E + r_1A) \subseteq \mathbb{B}(E + r_2A)$ and $\mathbb{B}(E - r_2A) \subseteq \mathbb{B}(E - r_1A)$.
- (iii) In particular, for any $\epsilon \in \mathbb{Q}_{>0}$ we have $\mathbb{B}(E + \epsilon A) \subseteq \mathbb{B}(E) \subseteq \mathbb{B}(E - \epsilon A)$.
- (iv) Therefore we have that

$$\mathbb{B}_+(E) := \bigcap_{q \in \mathbb{N}} \mathbb{B}(E - (1/q)A) \quad \text{and} \quad \mathbb{B}_-(E) := \bigcup_{q \in \mathbb{N}} \mathbb{B}(E + (1/q)A).$$

- (v) It follows that $\mathbb{B}_+(E)$ is closed but, even for line bundles, the locus $\mathbb{B}_-(E)$ is not closed in general: Lesieutre [10] proved that this locus can be a proper dense subset of X , or a proper dense subset of a divisor of X . \square

Remark 2.8. In the case of a line bundle over the variety X these loci correspond to the well-known stable base locus $\mathbb{B}(L)$, augmented base locus $\mathbb{B}_+(L)$, and restricted base locus $\mathbb{B}_-(L)$. Positivity properties of line bundles are related to asymptotic base loci as summarized in the following table:

	$\mathbb{B}_-(L)$	$\mathbb{B}(L)$	$\mathbb{B}_+(L)$
$= \emptyset$	nef	Semiample	Ample
$\neq X$	Pseudo-effective	Effective	Big

□

In the case of higher rank, some of the positivity properties can be generalized in various non-equivalent ways: one of the most natural way follows.

Let E be a vector bundle on a normal projective variety X , $\pi: \mathbb{P}(E) \rightarrow X$ the projective bundle of rank one quotients of E , and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient line bundle of π^*E on $\mathbb{P}(E)$. The most common way to relate positivity properties of line bundles to positivity properties of vector bundles is applying the definition for line bundles to $\mathcal{O}_{\mathbb{P}(E)}(1)$:

Definition 2.9. We say that E is a *nef* (respectively, *semiample*, *ample*, *big*) vector bundle on X if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef (respectively, semiample, ample, big) line bundle on $\mathbb{P}(E)$. □

Remark 2.10. We immediately have

$$\pi(\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathbb{B}(E).$$

More precisely, we have $\pi(\text{Bs}(\mathcal{O}_{\mathbb{P}(E)}(1))) = \text{Bs}(E)$ (cf. [1]), and we will show in the following section that the inclusion $\pi(\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathbb{B}(E)$ of stable base loci is strict in general.

However in [1], some useful connections are established relying on properties of augmented and restricted base loci, which exhibit a more predictable behavior with respect to the map π . □

In [1], we established the following:

Proposition 2.11. Let E be a vector bundle on a normal projective variety X , $\pi: \mathbb{P}(E) \rightarrow X$ the projective bundle of one-dimensional quotients of E , and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient of π^*E on $\mathbb{P}(E)$. Then

- (i) $\pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) = \mathbb{B}_-(E)$;
- (ii) $\pi(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))) = \mathbb{B}_+(E)$.

□

We recall some definitions appearing in the literature which can be related to the asymptotic base loci defined above:

Definition 2.12. Let E be a vector bundle on the variety X .

- (i) We say that E is *pseudo-effective* if $\mathbb{B}_-(E) \neq X$ (cf. [3], where for a line bundle L the locus $L_{\text{non-nef}}$ coincides with $\mathbb{B}_-(L)$).
- (ii) We say that E is *weakly semipositive over the (non-empty) open subset U* if $\mathbb{B}_-(E) \subseteq X \setminus U$ (cf. [13]).
- (iii) We say that E is *weakly semipositive* if it is weakly semipositive over some non-empty open subset in X (cf. [13]).
- (iv) We say that the vector bundle E is *strongly semiample* if $\text{Sym}^m E$ is globally generated for $m \gg 0$. (cf. [5]).
- (v) We say that E is *asymptotically generically generated (AGG)* if there exists a non-empty open subset $U \subseteq X$ and a positive integer $m > 0$ such that $\text{Sym}^m E$ is generated by global sections over the points in U .
- (vi) We say that E is *strongly big* if there exists an ample line bundle A , and a positive integer $c > 0$, such that $\text{Sym}^c E \otimes A^{-1}$ is weakly positive (cf. [8]). \square

It was proven in [1] that the following table holds for the above definitions on vector bundles:

	$\mathbb{B}_-(E)$	$\overline{\mathbb{B}_-(E)}$	$\mathbb{B}(E)$	$\mathbb{B}_+(E)$
$= \emptyset$	nef	nef	Strongly semiample	Ample
$\neq X$	Pseudo-effective	Weakly positive	AGG	Strongly big

In the following sections, we further investigate the semiampleness and the stable base loci of vector bundles, and we provide some constructions which generalize Iitaka fibrations to vector bundles, when they are strongly semiample or AGG.

3 Semiample Vector Bundles

In this section, we describe and compare the two different notions of semi-amplitude for vector bundles existing in the literature and provide an Iitaka construction for strongly semiample vector bundles.

3.1 Strongly semiample vector bundles

In the literature two different definitions for semiampleness can be found:

- either a vector bundle is called semiample when $\mathcal{O}_{\mathbb{P}(E)}(1)$ is semiample (one such reference can be found in [12], *Open Problems*);
- or it is called semiample when $\mathrm{Sym}^m E$ is globally generated for some $m > 0$ (cf. [5]).

In order to distinguish the two definitions, we call the latter *strong semiampleness*, according to definition 2.12 above.

Remark 3.1. Clearly, a vector bundle E is strongly semiample if and only if $\mathbb{B}(E) = \emptyset$; and E is semiample if and only if $\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1)) = \emptyset$. \square

It is often believed that the two definitions coincide (cf. [5]), and it is easy indeed to prove that strong semiampleness implies semiampleness that is, $\pi(\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathbb{B}(E)$. However, it is not hard to construct examples where semiampleness holds while strong semiampleness does not, showing therefore that the inclusion $\pi(\mathbb{B}(\mathcal{O}_{\mathbb{P}(E)}(1))) \subseteq \mathbb{B}(E)$ can be strict:

Example 3.2. Let C be a smooth projective curve of genus $g \geq 1$. And let $E = \mathcal{O}_C \oplus L$ with L being a non-trivial torsion line bundle satisfying $L^{\otimes 2} = \mathcal{O}_C$. Clearly L is not globally generated as it has no non-trivial global sections. As in $\mathrm{Sym}^c E$ there is a direct factor isomorphic to L , then for any $c > 0$ the vector bundle $\mathrm{Sym}^c E$ cannot be generated by global sections in any point.

However it can be proven that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is semiample: it is easily proven by hand that $\mathcal{O}_{\mathbb{P}(E)}(2)$ is globally generated on $\mathbb{P}(E)$, but we remark that we can apply Catanese–Dettweiler criterion for semiampleness (cf. [4]), as E is a unitary flat (poly-stable of degree 0 over a curve) vector bundle on the curve C , and clearly the image of its monodromy representation is a finite group of order 2. \square

The example above also shows that strong semiampleness is not stable with respect to finite morphisms, contrarily to what happens to semiampleness.

3.2 Semiample “fibration”

Let E be a rank r strongly semiample vector bundle on a normal complex projective variety X . Set $\mathbf{M}(E) := \{m \in \mathbb{N} \mid S^m E \text{ is globally generated}\}$. Denote

$$\sigma_m(r) := \mathrm{rk} S^m E = \binom{m+r-1}{m}$$

then for any $m \in \mathbf{M}(E)$ evaluation on global sections of $S^m E$ yields a morphism:

$$\begin{aligned}\varphi_m: X &\rightarrow \mathbb{G}(H^0(X, S^m E), \sigma_m(r)) \\ x &\mapsto [H^0(X, S^m E) \rightarrow S^m E(x)],\end{aligned}$$

where $\mathbb{G}(H^0(S^m E), \sigma_m(r))$ is the Grassmannian of $\sigma_m(r)$ -dimensional quotients. We call these morphisms *Kodaira maps* (cf. [2]).

Ampleness and some other positivity properties cannot be detected through the geometry of Kodaira maps (cf. [9], 6.1.6), however it is interesting to observe the asymptotical behavior of these Kodaira maps φ_m : in the case of (semiample) line bundles, these maps factor through a tower of finite maps, so that their images all have the same dimensions, and for large m one gets a given fibration, which is the Stein factorization of any of the maps φ_m .

Something similar happens for higher rank vector bundles, but we do not get a fibration in general. The purpose of this section is to investigate this asymptotic construction.

Lemma 3.3. Let E be a strongly semiample vector bundle on a normal projective variety X , let φ_m be the Kodaira maps described above, for any integer $m \in \mathbf{M}(E)$, and call Y_m the normalization of the image of the map φ_m . Then for any positive integer k , the integer km lies in $\mathbf{M}(E)$ and the map φ_m factors through φ_{km} and a finite morphism $Y_{km} \rightarrow Y_m$. \square

In particular, this shows that all the images Y_m have the same dimension.

Proof. Obviously if $S^m E$ is globally generated so is $S^{km} E$.

We want to compare the maps $\varphi_m: X \rightarrow Y_m \rightarrow \mathbb{G}(H^0(X, S^m E), \sigma_m(r))$ and $\varphi_{km}: X \rightarrow Y_{km} \rightarrow \mathbb{G}(H^0(X, S^{km} E), \sigma_{km}(r))$, by factoring the first one through a finite map $Y_{km} \rightarrow Y_m$.

Let us call $\mathbb{F}_{k,m}$ the flag variety of quotients $S^k H^0(X, S^m E) \twoheadrightarrow Q_1 \twoheadrightarrow Q_2$, with $\dim Q_1 = \sigma_k(\sigma_m(r))$ and $\dim Q_2 = \sigma_{km}(r)$, and $\psi_{k,m}: X \rightarrow \mathbb{F}_{k,m}$ the map $x \mapsto [S^k H^0(X, S^m E) \twoheadrightarrow S^k(S^m E(x)) \twoheadrightarrow S^{km} E(x)]$. Let us call $\widehat{Y}_{k,m}$ the normalization of the image of X in $\mathbb{F}_{k,m}$, π_1 and π_2 the natural projections from the flag variety to the two Grassmannians.

Finally, after calling W the image of the natural linear map

$$S^k H^0(X, S^m E) \rightarrow H^0(X, S^{km} E),$$

we can construct the following diagram:

$$\begin{array}{ccccc}
 & & Y_{km} & \longrightarrow & \mathbb{G}(H^0(X, S^{km}E), \sigma_{km}(r)) \\
 & \nearrow \varphi_{km} & \downarrow \nu & & \downarrow \downarrow \\
 & & \widetilde{Y}_2 & \longrightarrow & \mathbb{G}(W, \sigma_{km}(r)) \\
 & & \downarrow \wr & & \downarrow \wr \\
 & & \widetilde{Y}_2 & \longrightarrow & \mathbb{G}(S^k H^0(X, S^m E), \sigma_{km}(r)) \\
 & \nearrow \sim & & \nearrow \pi_2 & \\
 X & \xrightarrow{\psi_{k,m}} & \widehat{Y}_{k,m} & \longrightarrow & \mathbb{F}_{k,m} \\
 & \searrow \varphi_m & \downarrow \mu & \searrow \pi_1 & \\
 & & \widetilde{Y}_1 & \longrightarrow & \mathbb{G}(S^k H^0(X, S^m E), \sigma_k(\sigma_m(r))) \\
 & & \uparrow \wr & & \uparrow \wr \\
 & & Y_m & \longrightarrow & \mathbb{G}(H^0(X, S^m E), \sigma_m(r))
 \end{array}$$

where the maps between Grassmannians are constructed as follows:

- the first map above is a (rational) projection induced on the Grassmannians (of quotients) by the inclusion $W \subseteq H^0(X, S^{km}E)$;
- the second map is induced by the quotient $S^k H^0(X, S^m E) \twoheadrightarrow W$;
- the last map below is induced by considering the k -th symmetric product of a quotient $H^0(X, S^m E) \twoheadrightarrow Q$.

As evaluating global sections on a point $x \in X$ commutes with the various symmetric products considered, it is easy to see that the maps above are well defined and make all diagrams commute. As an example the map $\pi_2 \circ \psi_{k,m}$ is obtained on x by evaluating $S^k H^0(X, S^m E) \twoheadrightarrow S^{km}E(x)$, as this maps factors through $S^k H^0(X, S^m E) \twoheadrightarrow W \hookrightarrow H^0(X, S^{km}E)$ then we see that $\pi_2 \circ \psi_{k,m} = \nu \circ \varphi_{km}$.

In order to complete the proof of the lemma we have to show that:

- the map $\widehat{Y}_{k,m} \rightarrow \widetilde{Y}_2$ is an isomorphism;
- the map $\nu: Y_{km} \rightarrow \widetilde{Y}_2$ is finite;
- the map $\mu: \widehat{Y}_{k,m} \rightarrow \widetilde{Y}_1$ is finite.

The first point follows observing that if the evaluation map

$$S^k H^0(X, S^m E) \twoheadrightarrow S^{km}E(x)$$

coincide on two points $x_1, x_2 \in X$ (i.e., their kernels are equal) then also the images of x_1 and x_2 in $\mathbb{F}_{k,m}$ coincide: suppose by contradiction that the evaluation maps $S^k H^0(X, S^m E) \twoheadrightarrow S^{km}E(x)$ coincide on x_1 and x_2 , but the maps $S^k H^0(X, S^m E) \twoheadrightarrow S^k(S^m E(x))$ do not coincide on those points. Then there exists a section $s \in H^0(X, S^m E)$ such that $s(x_1) = 0 \in S^m(E(x_1))$ and $s(x_2) \neq 0 \in S^m(E(x_2))$, so we have $s(x_1)^k = 0 \in S^{km}(E(x_1))$ and $s(x_2)^k \neq 0 \in S^{km}(E(x_2))$ as well, in contradiction with the assumption.

Then the projection π_2 from the flag variety $\mathbb{F}_{k,m}$ to the Grassmannian $\mathbb{G}(S^k H^0(X, S^m E), \sigma_{km}(r))$ induces a bijection on the images from X , and therefore an isomorphism on their normalizations.

The second point follows observing that ν is a regular map on Y_{km} which is the restriction of an affine map (the projection) to a proper variety, hence is finite.

The last point can be proven by contradiction: suppose that an irreducible curve $C \subseteq \widehat{Y}_{k,m}$ is contracted by μ , then there exists an irreducible curve \tilde{C} in X dominating C which is contracted by φ_m . Therefore $S^m E|_{\tilde{C}} \cong \mathcal{O}_{\tilde{C}}^{\oplus \sigma_m(r)}$ is trivial and so are $S^k(S^m E)|_{\tilde{C}}$ and $S^{km} E|_{\tilde{C}}$, so that evaluation on global sections is constant on points of \tilde{C} , that is the map $\psi_{k,m}$ contracts \tilde{C} , which contradicts our hypothesis. ■

The above lemma implies the following

Theorem 3.4 (Iitaka for strongly semiample vector bundles). Let X be a normal projective variety, E a vector bundle as above, φ_m the Kodaira maps and Y_m the normalizations of their images. Then there exists a diagram

$$\begin{array}{ccc} & & Y_\infty \\ & \nearrow \varphi_{\det E} & \downarrow \pi \\ X & & Y_{\mathbb{G}} \\ & \searrow \varphi_{\mathbb{G}} & \end{array}$$

where $\varphi_{\det E}$ is the semiample fibration (Iitaka) induced by the determinant line bundle of E and π is a finite map, such that for any $m \in \mathbf{M}(E)$ and for $k \gg 0$ we have $Y_{km} = Y_{\mathbb{G}}$ and $\varphi_{km} = \varphi_{\mathbb{G}}$. In particular, any Kodaira map φ_m , for $m \in \mathbf{M}(E)$, factors through $\varphi_{\mathbb{G}}$ and a finite map.

Furthermore, for all $m \in \mathbf{M}(E)$ there exist vector bundles Q_m on $Y_{\mathbb{G}}$ such that $\varphi^* Q_m = S^m E$. In particular, if we restrict the vector bundle E to a fiber of the map $\varphi_{\mathbb{G}}$, we obtain a vector bundle with trivial Kodaira maps. □

Proof. Applying Lemma 3.3 above we see that all Y_m are dominated by Y_{km} and that the corresponding Kodaira maps factor through finite maps, hence there must be an inverse limit of all such maps, corresponding to Y_{km} for $k \gg 0$, call such limit $Y_{\mathbb{G}}$.

Now we have $Y_{\mathbb{G}} \subseteq \mathbb{G}(H^0(X, S^n E), \sigma_n(r)) \subseteq \mathbb{P}(\bigwedge^{\sigma_n(r)} H^0(X, S^n E))$ for some $n \in \mathbf{M}(E)$, where the latter is Plücker embedding. So the map $\varphi_{\mathbb{G}}$ is determined by the linear subseries $W \subseteq H^0(X, (\det E)^{\otimes N})$, where $(\det E)^{\otimes N} = \det(S^n E)$, and $W = \text{Im}(\bigwedge^{\sigma_n(r)} H^0(X, S^n E) \rightarrow H^0(X, (\det E)^{\otimes N}))$.

Then $\varphi_{\mathbb{G}}$ factors through $\varphi_{\det E} : X \rightarrow \mathbb{P}(H^0(X, (\det E)^{\otimes N}))$ and a projection to $\mathbb{P}(W)$, the first map being exactly the Iitaka fibration for the line bundle $\det E$, and the projection being a finite map for the same reason as the map ν in the proof of the above lemma.

The last property stated in the theorem follows pulling back to $Y_{\mathbb{G}}$, for each $m \in \mathbf{M}(E)$, the canonical quotients of the Grassmannians through the maps

$$Y_{\mathbb{G}} \rightarrow Y_m \rightarrow \mathbb{G}(H^0(X, S^m E), \sigma_m(r)). \quad \blacksquare$$

Remark 3.5. The finite maps $Y_{km} \rightarrow Y_m$ induce finite extensions of the fields of rational functions $k(Y_m) \subseteq k(Y_{km})$. These extensions are all included in the field of rational functions of X , in particular they are included in the algebraic closure of $k(Y_m)$ in $k(X)$. For $m \gg 0$ they stabilize to $k(Y_{\mathbb{G}})$. As $\varphi_{\det E}$ is a fibration, then $k(Y_{\infty})$ is algebraically closed in $k(X)$, and it is in fact the algebraic closure of $k(Y_m)$ in $k(X)$. However, the field $k(Y_{\mathbb{G}})$ needs not to be algebraically closed in $k(X)$. This corresponds to the fact that in general the map π appearing in the theorem is not an isomorphism but just a finite map, as it is shown in the example below. \square

Remark 3.6. One can wonder whether the map induced by global sections of the symmetric power of a vector bundle is actually nothing more but the Iitaka fibration of the determinant bundle, seen after Plücker embedding of the Grassmannian. Actually this is not always the case, so that the finite map π appearing in the theorem is not always an isomorphism, and its degree is therefore an invariant of the vector bundle E . An example where the map π is not an isomorphism is given below. \square

Example 3.7. Fix a vector space V of dimension 3, and let $\pi : X \rightarrow \mathbb{P}(V) = \mathbb{P}^2$ be a double cover, ramified over a smooth conic of \mathbb{P}^2 , that is $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Consider the Euler exact sequence on \mathbb{P}^2 :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{Q} \rightarrow 0$$

call $E := \pi^* \mathcal{Q}$, then we can prove that for all powers $S^m E$ the Kodaira maps are finite of degree 2. In fact, $\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, and by projection formula we have:

$$H^0(X, S^m E) = H^0(\mathbb{P}^2, \pi_* S^m E) = H^0(\mathbb{P}^2, S^m \mathcal{Q}) \oplus H^0(\mathbb{P}^2, S^m \mathcal{Q}(-1)),$$

and since for all $m > 0$ we have that $H^0(\mathbb{P}^2, S^m \mathcal{Q}(-1)) = 0$, then all global sections of $S^m E$ over X are pull back of sections of $S^m \mathcal{Q}$ over \mathbb{P}^2 , so the Kodaira maps factor through the degree 2 map π . \square

4 Iitaka-type Properties for Vector Bundles

We now study the Iitaka construction in case of non-strongly semiample vector bundles. In order to have (rational) maps to Grassmannian varieties we need some generation property, which is a rather strong positivity property, that corresponds to effectivity for line bundles.

Definition 4.1. We recall the definition of *asymptotic generic generation* given above in Section 2.

- (i) Let X be a normal projective complex variety. Let E be a vector bundle on X , with $\mathrm{rk}(E) = r$, then E is said to be *asymptotically generically generated* (AGG), if global sections of some symmetric power $S^m E$ generate $S^m E$ over an open dense subset of X .
- (ii) Let E be an AGG vector bundle over a variety X . Then denote $\mathbf{N}(E) := \{m \in \mathbb{N} \mid \mathrm{Bs}(S^m E) \neq X\}$ □

Remark 4.2. A vector bundle E is asymptotically generically generated iff for some $m > 0$ the evaluation map $H^0(X, S^m E) \otimes \mathcal{O}_X \rightarrow S^m E$ is surjective over an open subset of X , iff $\mathbb{B}(E) \neq X$. □

For asymptotically generically generated vector bundles, it makes sense to consider the Kodaira maps, which in this case are rational maps

$$\varphi_m: X \dashrightarrow \mathbb{G}(H^0(X, S^m E), \sigma_m(r))$$

and study their asymptotic behavior. For $m \gg 0$ in $\mathbf{N}(E)$ these maps are regular out of the stable base locus $\mathbb{B}(E)$, call Y_m the images. We will construct a diagram similar to the one appearing in Lemma 3.3. We can consider the images of Kodaira maps without normalizing, as in this case we are only interested in the birational behavior.

$$\begin{array}{ccccc}
 & & Y_{km} & \hookrightarrow & \mathbb{G}(H^0(X, S^{km} E), \sigma_{km}(r)) \\
 & \nearrow \varphi_{km} & \downarrow \nu & & \downarrow \vdots \\
 & & \widetilde{Y}_2 & \hookrightarrow & \mathbb{G}(W, \sigma_{km}(r)) \\
 & & \downarrow \wr & & \downarrow \wr \\
 & & \widetilde{Y}_2 & \hookrightarrow & \mathbb{G}(S^k H^0(X, S^m E), \sigma_{km}(r)) \\
 & \nearrow \psi_{k,m} & \nearrow \mu & \nearrow \pi_2 & \nearrow \pi_1 \\
 X \setminus \mathbb{B}(E) & \xrightarrow{\psi_{k,m}} & \widehat{Y}_{k,m} & \longrightarrow & \mathbb{F}_{k,m} \\
 & \searrow \varphi_m & \downarrow \wr & & \downarrow \wr \\
 & & \widetilde{Y}_1 & \hookrightarrow & \mathbb{G}(S^k H^0(X, S^m E), \sigma_k(\sigma_m(r))) \\
 & & \uparrow \wr & & \uparrow \wr \\
 & & Y_m & \hookrightarrow & \mathbb{G}(H^0(X, S^m E), \sigma_m(r))
 \end{array}$$

Remark 4.3. In the latter diagram the maps μ and ν are not necessarily finite maps. \square

This construction yields the following

Theorem 4.4 (Iitaka for AGG vector bundles). Let X be a projective variety and E be an asymptotically generically generated bundle over X . Call $\varphi_m: X \dashrightarrow Y_m$ the evaluation rational maps defined as above. Then there exist projective varieties $X_{\mathbb{G}}$ and $Y_{\mathbb{G}}$ together with regular surjective morphisms $u_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow X$ and $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow Y_{\mathbb{G}}$ such that for every $m \in \mathbf{N}(E)$ and for $k \gg 0$ the regular map $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow Y_{\mathbb{G}}$ is a birational model for the rational map φ_{km} , that is we have the following commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{u_{\mathbb{G}}} & X_{\mathbb{G}} \\ \varphi_{km} \downarrow & & \downarrow \varphi_{\mathbb{G}} \\ Y_{km} & \xleftarrow{\nu_{km}} & Y_{\mathbb{G}} \end{array}$$

where the horizontal maps are birational. \square

Proof. Applying the construction above we have for each m

$$k(X) \supseteq k(Y_{km}) \supseteq k(Y_m)$$

so that all the Y_m 's are dominated by a common limit which is birational to Y_{km} for k big enough. Fixing big enough m and k , we can choose a compactification of Y_{km} to be $Y_{\mathbb{G}}$ and resolving the indeterminacies we have a model $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow Y_{\mathbb{G}}$ which is birational to all the other ones. \blacksquare

Remark 4.5. We already noticed that, contrary to the line bundle case, the map $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow Y_{\mathbb{G}}$ is not a fibration in general (cf. Remarks 3.5 and 3.6), however one can consider the Stein factorization $X_{\mathbb{G}} \rightarrow Y_{\infty} \rightarrow Y_{\mathbb{G}}$ for this map. In the case of a strongly semiample vector bundle E over X we have that $X_{\mathbb{G}} = X$ and observed that the map $X_{\mathbb{G}} \rightarrow Y_{\infty}$ is exactly the Iitaka fibration for the line bundle $\det E$ over X , therefore in the strongly semiample case the dimension of $Y_{\mathbb{G}}$ is the Iitaka dimension of the determinant of E .

This needs not be the case for non-semiample vector bundles, so it makes sense to ask the following questions. \square

In the following assume E to be AGG:

Question 4.6. Is it always $\dim Y_{\mathbb{G}} = k(X, \det E)$? \square

Question 4.7. Suppose E to be strongly big but not strongly semiample, is $\dim Y_{\mathbb{G}} = \dim X$? \square

Question 4.8. Suppose E is big (and AGG), what can we say about $\dim Y_{\mathbb{G}}$? \square

Question 4.9. Can we relate the complement of the augmented base loci to the Kodaira maps as it happens for line bundles (cf. [2])? \square

Remark 4.10. The last question has a negative answer if we look for a direct generalization of the theorem in [2]: among other results the authors prove that the augmented base locus of a line bundle is the complementary of the biggest open subset where the Iitaka fibration is an isomorphism. However the example 3.7 shows that this cannot be generalized to vector bundles, even if we were to change the word *isomorphism* by *finite regular map*. In fact if we consider the tautological quotient \mathcal{Q} of rank 2 over $\mathbb{P}^2 = \mathbb{P}(V)$, then its Kodaira map φ_1 is the isomorphism $\mathbb{P}(V) \cong \mathrm{Gr}(V^*, 2)$, so all the Kodaira maps φ_m are isomorphisms, however $\mathbb{B}_+(\mathcal{Q}) = \mathbb{P}^2$. \square

Remark 4.11. In the statement of Theorems 3.4 and 4.4 the integer k depends on the chosen m appearing in the symmetric power $S^m E$. In fact the rational model $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow Y_{\mathbb{G}}$ can be fixed once for all, but our methods do not allow us to fix an integer m_0 such that all Kodaira maps stabilize from that integer on, so it would be interesting to prove that such an integer m_0 always exists. \square

In any case we can define some *asymptotic invariants* for an AGG vector bundle:

Definition 4.12. Let E be an AGG vector bundle, using the notations above we call

- (i) *Iitaka index* of E the integer $FI(E) = \deg(Y_{\infty} \rightarrow Y_{\mathbb{G}})$;
- (ii) *Iitaka dimension* of E the integer $k(X, E) = \dim Y_{\mathbb{G}}$; \square

Related to question 4.6 is the following

Question 4.13. Consider an AGG vector bundle E with asymptotic Kodaira map $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \rightarrow Y_{\mathbb{G}}$. Consider a very general fiber of $\varphi_{\mathbb{G}}$, and its image F in X . Is $k(F, E|_F) = 0$? This is the case for line bundles, and for strongly semiample vector bundles. \square

Finally, we remark that in [6] the author gives a characterization for varieties with semiample cotangent bundles and Kodaira dimension 0 or 1. We can prove

the following theorem about varieties with strongly semiample cotangent bundle and Kodaira dimension 0:

Theorem 4.14. Let X be a smooth projective variety, and Ω_X^1 its cotangent bundle. The following conditions are equivalent:

- (i) X is an abelian variety;
- (ii) the cotangent bundle Ω_X is strongly semiample and the Iitaka dimension of Ω_X vanishes: $\mathbb{B}(\Omega_X^1) = \emptyset$ and $k(X, \Omega_X) = 0$;
- (iii) the cotangent bundle Ω_X is strongly semiample and the Kodaira dimension of X vanishes: $\mathbb{B}(\Omega_X^1) = \emptyset$ and $k(X, K_X) = 0$;
- (iv) $\mathrm{Sym}^m \Omega_X^1$ is trivial for some $m > 0$. □

Proof. $\mathbb{B}(\Omega_X^1) = \emptyset$ if and only if Ω_X^1 is strongly semiample, and in this case by Theorem 3.4 $k(X, \Omega_X^1) = k(X, K_X)$, then (ii) and (iii) are equivalent. Since $k(X, \Omega_X^1) = 0$ then some symmetric power of the cotangent bundle is the pull-back from a point, hence it is trivial, it follows that (ii) and (iii) are equivalent to point (iv).

(ii), (iii) and (iv) are necessarily satisfied if X is an abelian variety, let us show that they are sufficient as well.

Let us call $d := \dim X$ and suppose $\mathrm{Sym}^m \Omega_X^1 \cong \mathcal{O}_X^{\oplus \sigma_m(d)}$ for some $m > 0$. Then by [6, Theorem I] there is a finite étale Galois cover $f: A \rightarrow X$ where A is an abelian variety. Let us denote G the finite group acting freely on A such that $X = A/G$. In particular A has no fixed points for the action of G . We will prove that within our hypothesis G acts by translations on A and therefore X is an abelian variety.

Since the quotient f is étale, we have $f^* \Omega_X^1 \cong \Omega_A^1 \cong \mathcal{O}_A^{\oplus d}$. Let us call $V := H^0(A, \Omega_A^1)$ the tangent space of the abelian variety A at the origin. Then G acts on V via its action on A , let us denote $\rho: G \rightarrow GL(V)$ this action. We will show that the action ρ is trivial, and that therefore G acts on A by translations.

Let us show that G acts trivially on V . Let us first remark that $\mathrm{Sym}^m(\rho)$ is the trivial action on $\mathrm{Sym}^m V$. In fact, since $\mathrm{Sym}^m \Omega_X^1$ and $\mathrm{Sym}^m \Omega_A^1$ are trivial, then by dimension count

$$f^* H^0(X, \mathrm{Sym}^m \Omega_X^1) \cong H^0(A, \mathrm{Sym}^m \Omega_A^1)^G = H^0(A, \mathrm{Sym}^m \Omega_A^1) = \mathrm{Sym}^m V,$$

therefore $\mathrm{Sym}^m(\rho)$ is the trivial action on $\mathrm{Sym}^m V$.

This implies that G acts by homotheties on V , in fact the action of each element $g \in G$ is diagonalizable as G is a finite group. Hence, using the triviality of the action

on the symmetric product, we just need to show that all the eigenvalues must coincide. Consider two non-zero eigenvectors $v, w \in V$, with eigenvalues respectively λ and μ . By our hypothesis $(\lambda)^m = (\mu)^m = 1$. Then the action of g on the element $v^{m-1} \cdot w \in \text{Sym}^m V$ is given by

$$g(v^{m-1} \cdot w) = \lambda^{m-1} v^{m-1} \cdot \mu w = \left(\frac{\mu}{\lambda}\right) v^{m-1} \cdot w$$

but this has to be trivial by hypothesis, and hence $\lambda = \mu$.

Therefore each $g \in G$ acts on V by $g \cdot v = \lambda_g v$, and on A by $g \cdot x = \lambda_g x + \tau$. Then if $\lambda_g \neq 1$ there is a fixed point $(1 - \lambda_g)^{-1} \tau \in A$, and this cannot be as the quotient is étale by hypothesis. So G acts trivially on V and acts by translations on A , therefore X is an abelian variety.

Notice that in this case all holomorphic 1-forms on V descend to X , and in fact *a posteriori* Ω_X^1 is globally generated. ■

Question 4.15. We can ask whether the above theorem extends to a birational criterion:

- (i) Is it true that $\mathbb{B}(\Omega_X^1) \neq X$ and Kodaira dimension $k(X, K_X) = 0$ implies that X is birational to an abelian variety?
- (ii) Is it true that $\mathbb{B}(\Omega_X^1) \neq X$ and $k(X, \Omega_X^1) = 0$ implies that X is birational to an abelian variety? □

As for the first question we can remark that it can be reduced to smooth minimal models of X (when they exist):

Lemma 4.16. Let X and Y be two birational smooth projective varieties. Then $\mathbb{B}(\Omega_X^1) \neq X$ if and only if $\mathbb{B}(\Omega_Y^1) \neq Y$, that is X has AGG cotangent bundle if and only if Y has AGG cotangent bundle. □

Proof. Suppose $f: X \rightarrow Y$ is a regular and birational morphism between smooth varieties X and Y , with center $Z \subset Y$ and exceptional divisor $E \subset X$. We have an exact sequence on X :

$$0 \rightarrow f^* \Omega_Y^1 \rightarrow \Omega_X^1 \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} is a sheaf supported on the exceptional divisor E of f .

Now if $\mathbb{B}(\Omega_Y^1) \neq Y$ then some symmetric product $S^m \Omega_Y^1$ is generated by global sections over an open subset of Y , so by taking the pull-back of the symmetric product

of 1-forms we have that $S^m \Omega_X^1$ is generated by global sections over an open subset as well, so $\mathbb{B}(\Omega_X^1) \neq X$.

Vice-versa, suppose that $\mathbb{B}(\Omega_X^1) \neq X$: as $\text{codim}_Y Z \geq 2$, we can restrict 1-forms on X to 1-forms on $X \setminus E \cong Y \setminus Z$, and then extend them to 1-forms on Y , so if $S^m \Omega_X^1$ is generated by global sections over an open subset the same happens for $S^m \Omega_Y^1$, the two vector bundles being isomorphic over $X \setminus E \cong Y \setminus Z$. ■

The answer to the first question is positive in case $\dim X \leq 2$:

Theorem 4.17. Let X be a smooth projective variety such that $\dim X \leq 2$ and $k(X, K_X) = 0$. Then X is birational to an abelian variety if and only if $\mathbb{B}(\Omega_X^1) \neq X$. □

Proof. It is obvious in dimension 1. Let us observe first that if X is birational to an abelian variety then the Albanese morphism is a birational map, and the pull-back of holomorphic 1-forms from the Albanese variety to X gives $\mathbb{B}(\Omega_X^1) \neq X$.

Let us prove then that Kodaira dimension $k(X, K_X) = 0$ and $\mathbb{B}(\Omega_X^1) \neq X$ imply birationality to an abelian surface.

By Lemma 4.16 we can suppose that X is a minimal surface. Then it is an abelian surface, or a bielliptic surface, or a $K3$ or an Enriques surface. Now a bielliptic surface is a smooth quotient of an abelian surface, so its cotangent bundle cannot be generically generated by global sections for the same argument as in Theorem 4.14, and the same applies for its symmetric powers. If X is a $K3$ surface, then the symmetric powers of the cotangent bundle have no global sections, as it is proven in Theorem 7.8 in [3], so the same happens on an Enriques surfaces as they are quotients of $K3$'s. ■

Remark 4.18. We notice that other generalizations of Fujiwara's results, in the case of strongly semiample cotangent bundles, were obtained by Hoering in [7]. □

Finally, we notice that in the work of the first author [11] some rational maps are constructed from Grassmannians to moduli spaces of vector bundles over a curve, and it would be interesting to see what kind of constructions could lead considering Kodaira maps from these moduli spaces.

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