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LINEAR SERIES ON CURVES: STABILITY AND CLIFFORD INDEX

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We study concepts of stability associated to a smooth complex curve together with a linear series on it. In particular we investigate the relation between stability of the associated dual span bundle and linear stability. Our results imply that stability of the dual span holds under a hypothesis related to the Clifford index of the curve. Furthermore, in some of the cases, we prove that a stronger stability holds: cohomological stability. Finally, using our results we obtain stable vector bundles of slope 3, and prove that they admit theta-divisors.

Keywords: Slope stability; linear stability; cohomological stability; Clifford index; Dual span bundles; linear series; Butler's conjecture.

Mathematics Subject Classification 2010: 14H51, 14H60

1. Introduction

Let C be an irreducible projective smooth complex curve, and let $\mathcal{L}$ be a globally generated line bundle on C (suppose $\mathcal{L} \neq \mathcal{O}$ throughout the article). Consider a generating subspace $V \subseteq H^0(\mathcal{L})$. The *Dual Span Bundle* (DSB for short) $M_{V,\mathcal{L}}$ associated to this data is the kernel of the evaluation morphism ev:

$$0 \to M_{V,\mathcal{L}} \to V \otimes \mathcal{O}_C \xrightarrow{\text{ev}} \mathcal{L} \to 0.$$

This is a vector bundle of rank $\dim V - 1$ and degree $-\deg \mathcal{L}$. When $V = H^0(\mathcal{L})$ we denote it by $M_{\mathcal{L}}$. Note that we make here an abuse of notation: properly speaking the dual span bundle is the dual bundle of $M_{V,\mathcal{L}}$, which is indeed spanned by V^*. This bundle is also called *transform* [17], or *Lazarsfeld bundle* [22].[a]

In this paper we treat various kinds of stability conditions, associated to these data, namely:

- vector bundle stability, which we simply call stability, or slope stability, of $M_{V,\mathcal{L}}$ (Definition 2.1);
- linear stability of the triple $(C, V, \mathcal{L})$ (Definition 3.1);
- cohomological stability of $M_{V,\mathcal{L}}$ (Definition 7.1).

Stability of DSBs has been studied intensively and with many different purposes, and it has been conjectured by Butler that it should hold under generality conditions. This conjecture has been verified in many cases, and used to prove results on Brill–Noether theory and moduli spaces of coherent systems (cf. [4, 5, 9, 23]). These conditions satisfy the following implications:

$$\text{cohomological stability} \ \Rightarrow\ \text{vector bundle stability} \ \Rightarrow\ \text{linear stability}, \tag{1.1}$$

which hold for semistability as well; moreover, as we are in characteristic 0, we have that cohomological semistability is equivalent to vector bundle semistability [13].

The purpose of this paper is twofold. Firstly we are interested in finding conditions under which the last implication in (1.1) can be reversed, i.e. linear stability is a sufficient condition for the stability of the DSB. The question of DSB's stability is considered by Butler in [9], and that work is the starting point of our investigation. It turns out that the Clifford index of the curve (definition in Sec. 3) plays a central role. In the last part of the paper we give some counterexamples proving that the implication does not hold in general, and we state some conjectures.

Secondly, we want to prove some new stability results. We find conditions for the three stabilities to hold, involving again the Clifford index of the curve C. These results are achieved both using the arguments of the first part of the paper, and by different arguments for linear stability and cohomological stability.

Let us go deeper in the description of our results.

As for the first question, the convenience of reducing the stability of a DSB to the linear stability lies in the fact that linear stability is often less hard to prove. Moreover, it has a clear geometric meaning in terms of relative degrees of projections of the given morphism. So, the question can be reformulated this way: to what extent the knowledge of the geometry of a morphism is sufficient to detect the stability of the associated DSB?

Another motivation for considering this problem comes from the work of the second author on fibered surfaces. To a fibered surface with a family of morphisms on the fibers, one can associate a certain divisorial class on the base curve. There are

[a]It is worth noticing that usually a Lazarsfeld or Mukai–Lazarsfeld bundle, named after [16], is a vector bundle on a K3 surface S, obtained after evaluating global sections of a line bundle on a curve contained in S.

two methods that prove the positivity of this class, one assuming linear stability, the other assuming the stability of the DSB on a general fiber. The comparison between these methods leads naturally to comparing the two assumptions.

Let us now assume that C has genus $g \geq 2$. We obtain the following results.

Theorem 1.1. *Let $\mathcal{L} \in \mathrm{Pic}(C)$ be a globally generated line bundle, and $V \subseteq H^0(C, \mathcal{L})$ be a generating space of global sections such that*

$$\deg \mathcal{L} - 2(\dim V - 1) \leq \mathrm{Cliff}(C). \tag{1.2}$$

Then linear (semi)stability of $(C, \mathcal{L}, V)$ is equivalent to (semi)stability of $M_{V,\mathcal{L}}$ in the following cases:

(1) $V = H^0(\mathcal{L})$ (*complete case*);
(2) $\deg \mathcal{L} \leq 2g - \mathrm{Cliff}(C) + 1$;
(3) $\mathrm{codim}_{H^0(\mathcal{L})} V < h^1(\mathcal{L}) + g/(\dim V - 2)$;
(4) $\deg \mathcal{L} \geq 2g$, *and* $\mathrm{codim}_{H^0(\mathcal{L})} V \leq (\deg \mathcal{L} - 2g)/2$.

This theorem is proved by applying a Castelnuovo-type result, relating evaluation of sections of a line bundle $\mathcal{A}$ tensored with the canonical bundle and the image of the morphism induced by global sections of $\mathcal{A}$, to an exact sequence obtained from a possible destabilization of the bundle $M_{V,\mathcal{L}}$. The geometrical idea for this construction is simple and is carried out in Sec. 4, but the computations in order to make this argument work are quite long (Secs. 5 and 6), and give rise to the bounds imposed in points (1)–(4) of the theorem.

Let us now describe the stability results that we obtain.

By standard linear series techniques, we can prove in Sec. 3 the following dependence of linear stability on the Clifford index of the curve.

Proposition 1.2. *Let C be a curve of genus $g \geq 2$. Let $\mathcal{L} \in \mathrm{Pic}(C)$ be a globally generated line bundle such that $\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \leq \mathrm{Cliff}(C)$. Then $\mathcal{L}$ is linearly semistable. It is linearly stable unless $\mathcal{L} \cong \omega_C(D)$ with D an effective divisor of degree 2, or C is hyperelliptic and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$.*

Using this result and Theorem 1.1, we obtain stability of DSB in the following cases.

Theorem 1.3. *Let $\mathcal{L} \in \mathrm{Pic}(C)$ be a globally generated line bundle such that*

$$\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \leq \mathrm{Cliff}(C). \tag{1.3}$$

Then $M_{\mathcal{L}}$ is semistable, and it is strictly semistable only in one of the following cases:

(i) $\mathcal{L} \cong \omega_C(D)$ *with D an effective divisor of degree 2;*
(ii) *C is hyperelliptic and* $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$.

In particular, this implies the following results.

Corollary 1.4. *Let $\mathcal{L}$ be a globally generated line bundle over C with*

$$\deg \mathcal{L} \geq 2g - \mathrm{Cliff}(C).$$

Then the vector bundle $M_{\mathcal{L}}$ is semistable. It is stable unless (i) or (ii) holds.

Corollary 1.5. *Let $\mathcal{L}$ be any line bundle that computes the Clifford index of C. Then $M_{\mathcal{L}}$ is semistable; it is stable unless C is hyperelliptic.*

Moreover, using a result contained in [2], we can prove, applying Theorem 1.1, the following.

Proposition 1.6. *Let C be a curve such that $\mathrm{Cliff}(C) \geq 4$. Let $V \subset H^0(\omega_C)$ be a general subspace of codimension 1 or 2. Then M_{V,ω_C} is stable.*

Some of the results above were previously known: Theorem 5.3 is a refinement of a result contained in Paranjape's Ph.D. thesis [19], and Corollaries 5.4 and 5.5 follow from [6]. Corollary 5.4 has also been proved in [10].

It is worthwhile remarking that in [17] the first author proves stability of some bundles $M_{V,\mathcal{L}}$ by a similar argument: showing first that if $M_{V,\mathcal{L}}$ is unstable then $(C, \mathcal{L}, V)$ needs to be linearly unstable, and then showing that for general $V \subset H^0(\mathcal{L})$ these are not linearly unstable.

We hope these methods can be of use in order to verify the DSB's stability in more cases, and generalized to investigate the stability of bundles which are dual spans of higher rank vector bundles.

Moreover, we prove in Sec. 7 that, in some of the cases of Theorem 1.1, a stronger condition holds: *cohomological stability* (Definition 7.1).

Theorem 1.7. *Let $(\mathcal{L}, V)$ be a g^r_d on a smooth curve C, inducing a birational morphism. Suppose that*

- $d \leq 2r + \mathrm{Cliff}\, C$;
- $\mathrm{codim}_{H^0(\mathcal{L})} V \leq h^1(\mathcal{L})$.

Then $M_{V,\mathcal{L}}$ is cohomologically semistable. It is cohomologically stable except in the following cases:

(i) *$d = 2r$ and C is hyperelliptic;*
(ii) *$\mathcal{L} \cong \omega_C(D)$ with D an effective divisor of degree 2.*

This theorem is proved by extending the techniques of [13] (see also [22]). As a consequence, we have cohomological stability of M_{ω_C} for C non-hyperelliptic, and of projections from a general point for $\mathrm{Cliff}(C) \geq 2$ (Corollary 7.7).

It is natural to wonder whether the implication, linear stability $\Rightarrow$ stability of DSB, holds more generally. No examples, to our knowledge, were known where the first stability condition holds while the second one does not. The answer to this question is negative in general, and it turned out to be fairly easy to produce linearly stable line bundles whose DSB is not semistable: this is the content of Sec. 8.

Finally (Sec. 9), we show that on a general curve C of even genus $g = 2k$, there exist stable, and even cohomologically stable, DSBs of slope -3 (Proposition 9.1). We show that these bundles admit a generalized theta-divisor in Proposition 9.3, and we formulate some questions on the behavior of these bundles.

Notation. We will work over the complex numbers, and C will be a smooth projective curve, unless explicitly specified.

Let D be a divisor on C. As customary, we shall write $H^i(D)$ for $H^i(\mathcal{O}_C(D))$, and if $\mathcal{F}$ is a vector bundle we shall use the notation $\mathcal{F}(D)$ for $\mathcal{F} \otimes \mathcal{O}_C(D)$.

2. Preliminary Results on Vector Bundle Stability

Given a vector bundle $\mathcal{E}$ on C its slope is the rational number $\mu(\mathcal{E}) := \deg \mathcal{E}/\operatorname{rank} \mathcal{E}$.

Definition 2.1. The vector bundle $\mathcal{E}$ is *stable* (respectively, *semistable*) if for any proper subbundle $\mathcal{F} \subset \mathcal{E}$, we have that $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

Throughout the paper we will consider the following setting.

Let $M_{V,\mathcal{L}} = \ker(V \otimes \mathcal{O} \twoheadrightarrow \mathcal{L})$ be the associated dual span bundle. Let $\mathcal{S} \subset M_{V,\mathcal{L}}$ be a saturated proper subbundle. Then there exist a vector bundle $F_{\mathcal{S}}$ and a subspace $W \hookrightarrow V$ fitting into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S} & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F_{\mathcal{S}} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0
\end{array}$$

Indeed, define $W \hookrightarrow V$ by $W^* := \operatorname{Im}(V^* \to H^0(\mathcal{S}^*))$; then W^* generates $\mathcal{S}^*$. Then define $F_{\mathcal{S}}^* := \ker(W^* \otimes \mathcal{O} \twoheadrightarrow \mathcal{S}^*)$.

Remark 2.2. Let us summarize some properties of these objects, which is well known to experts; see for instance [8]. With the notation above, the following properties hold.

(1) The sheaf $F_{\mathcal{S}}$ is globally generated and $h^0(F_{\mathcal{S}}^*) = 0$.
(2) The induced map $\alpha : F_{\mathcal{S}} \to \mathcal{L}$ is not the zero map.
(3) If $\mathcal{S}$ is a maximal destabilizing subbundle of $M_{V,\mathcal{L}}$, then $\deg F_{\mathcal{S}} \leq \deg \mathcal{I}$, where $\mathcal{I}$ is $\operatorname{Im}(\alpha)$, and equality holds if and only if $\operatorname{rank} F_{\mathcal{S}} = 1$.

The only point worth verifying is the last. We can form the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S} & \longrightarrow & W\otimes\mathcal{O}_C & \longrightarrow & F_{\mathcal{S}} & \longrightarrow & 0\\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & M_{W,I} & \longrightarrow & W\otimes\mathcal{O}_C & \longrightarrow & \mathcal{I} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V\otimes\mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0.
\end{array}$$

If we require maximality of the subbundle $\mathcal{S}$, and destabilization, we have

$$\mu(\mathcal{S}) = \frac{-\deg F_{\mathcal{S}}}{\dim W - \operatorname{rank} F_{\mathcal{S}}} \geq \mu(M_{W,\mathcal{I}}) = \frac{-\deg \mathcal{I}}{\dim W - 1}.$$

So, if $\operatorname{rank} F_{\mathcal{S}} > 1$, we have

$$\deg F_{\mathcal{S}} \leq \frac{\dim W - \operatorname{rank} F_{\mathcal{S}}}{\dim W - 1} \deg \mathcal{I} < \deg \mathcal{I}.$$

3. Linear Stability and Clifford Index

Here we give a natural generalization of the notion of linear stability of a curve and a linear series on it, introduced by Mumford in [18] (cf. [24]).

Definition 3.1. Let $\mathcal{L}$ be a degree d line bundle on C, and $V \subseteq H^0(\mathcal{L})$ be a generating subspace of dimension $r+1$. We say that the triple $(C, \mathcal{L}, V)$ is *linearly semistable* (respectively, stable) if any linear series of degree d' and dimension r' contained in $|V|$ satisfies $d'/r' \geq d/r$ (respectively, $d'/r' > d/r$).

In case $V = H^0(\mathcal{L})$, we shall talk of the stability of the couple $(C, \mathcal{L})$. It is easy to see that in this case it is sufficient to verify that the inequality of the definition holds for any *complete* linear series in $|V|$.

Remark 3.2. It is clear that the following conditions are equivalent:

(1) The triple $(C, \mathcal{L}, V)$ is linearly stable;
(2) The bundle $M_{V,\mathcal{L}}$ is not destabilized by any bundle $M_{V',\mathcal{L}'}$, with $V' \subseteq V$, and $V' \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{L}' \subset \mathcal{L}$.

Using the Clifford theorem and Riemann–Roch theorem it is not hard to prove that $(C, \mathcal{L})$ is linearly stable for any line bundle $\mathcal{L}$ of degree $\geq 2g+1$.

Let C be of genus $g \geq 2$. We now present a more general result relating linear stability to the Clifford index of the curve. Let us recall that *the Clifford index* of a curve C of genus $g \geq 4$ is the integer:

$$\operatorname{Cliff}(C) := \min\{\deg(\mathcal{L}) - 2(h^0(\mathcal{L}) - 1) \,|\, \mathcal{L} \in \operatorname{Pic}(C), h^0(\mathcal{L}) \geq 2, h^1(\mathcal{L}) \geq 2\}.$$

When $g = 2$, we set $\operatorname{Cliff}(C) = 0$; when $g = 3$ we set $\operatorname{Cliff}(C) = 0$ or 1 according to whether C is hyperelliptic or not.

Let $\gamma(C)$ be the gonality of the curve C. The following inequalities hold:

$$\gamma(C) - 3 \le \mathrm{Cliff}(C) \le \gamma(C) - 2;$$

the case $\mathrm{Cliff}(C) = \gamma(C) - 2$ holding for general $\gamma(C)$-gonal curves in the moduli space of smooth curves $\mathcal{M}_g$. Furthermore $\mathrm{Cliff}(C) = 0$ if and only if C is hyperelliptic.

Proposition 3.3. *Let C be a curve of genus $g \ge 2$. Let $\mathcal{L} \in \mathrm{Pic}(C)$ be a globally generated line bundle such that $\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \le \mathrm{Cliff}(C)$. Then $\mathcal{L}$ is linearly semistable. It is linearly stable unless $\mathcal{L} \cong \omega_C(D)$ with D an effective divisor of degree 2, or C is hyperelliptic and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$.*

Proof. Recall that it is sufficient to check linear stability for complete linear subsystems of $|\mathcal{L}|$. Let $\mathcal{P} \hookrightarrow \mathcal{L}$ be a line bundle generated by a subspace of $H^0(\mathcal{L})$. Observe that

$$H^1(\mathcal{P})^* = H^0(\omega \otimes \mathcal{P}^*) \supseteq H^0(\omega \otimes \mathcal{L}^*) = H^1(\mathcal{L})^*.$$

Let us distinguish three cases:

(1) $h^1(\mathcal{L}) \ge 2$. In this case $\mathcal{L}$ computes the Clifford index:

$$\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1) - \mathrm{Cliff}(C).$$

Hence $h^1(\mathcal{P}) \ge h^1(\mathcal{L}) \ge 2$ and $\mathcal{P}$ contributes to the Clifford index $\mathrm{Cliff}(C)$, so $\deg \mathcal{P} \ge 2(h^0(\mathcal{P}) - 1) + \mathrm{Cliff}(C)$. Then we have the inequalities

$$\frac{\deg \mathcal{P}}{h^0(\mathcal{P}) - 1} \ge 2 + \frac{\mathrm{Cliff}(C)}{h^0(\mathcal{P}) - 1} \ge 2 + \frac{\mathrm{Cliff}(C)}{h^0(\mathcal{L}) - 1} = \frac{\deg \mathcal{L}}{h^0(\mathcal{L}) - 1},$$

where the last inequality is strict unless $\mathrm{Cliff}(C) = 0$ and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$, in which case the curve is hyperelliptic, and $\mathcal{L}$ is linearly semistable but not linearly stable (it can be shown that the dual of the g^1_2 maps to $M_{\mathcal{L}}$ in this case).

(2) If $h^1(\mathcal{L}) = 1$, then either $h^1(\mathcal{P}) = 1$ or $h^1(\mathcal{P}) \ge 2$. In the last case $\mathcal{P}$ contributes to the Clifford index, so

$$\deg \mathcal{P}/(h^0(\mathcal{P}) - 1) \ge 2 + \mathrm{Cliff}(C)/(h^0(\mathcal{L}) - 1) \ge \deg \mathcal{L}/(h^0(\mathcal{L}) - 1),$$

with strict inequality unless C is hyperelliptic and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$.

If $h^1(\mathcal{P}) = h^1(\mathcal{L}) = 1$, then, as $\deg \mathcal{P} < \deg \mathcal{L}$, we have that $\deg \mathcal{P}/(\deg \mathcal{P} + 1 - g) > \deg \mathcal{L}/(\deg \mathcal{L} + 1 - g)$.

(3) If $h^1(\mathcal{L}) = 0$, then $h^1(\mathcal{P}) = 0$, or $h^1(\mathcal{P}) = 1$, or $h^1(\mathcal{P}) \ge 2$. In the last case $\mathcal{P}$ contributes to the Clifford index, and we can reason as above. If $h^1(\mathcal{P}) = h^1(\mathcal{L}) = 0$, then, as $\deg \mathcal{P} < \deg \mathcal{L}$, we have $\deg \mathcal{P}/(\deg \mathcal{P} - g) > \deg \mathcal{L}/(\deg \mathcal{L} - g)$.

At last, suppose that $h^1(\mathcal{P}) = 1$. Then of course $\deg \mathcal{P} \leq 2g - 2$. Consider the exact sequence

$$0 \to H^0(\mathcal{P}) \to H^0(\mathcal{L}) \to H^0(D, \mathcal{O}_D) \to H^1(\mathcal{P}) \to 0,$$

where D is an effective divisor such that $\mathcal{P}(D) \cong \mathcal{L}$. From this sequence, remarking that the inclusion $H^0(\mathcal{P}) \subset H^0(\mathcal{L})$ must be strict, we deduce that $\deg \mathcal{L} - \deg \mathcal{P} = h^0(D, \mathcal{O}_D) \geq 2$.

We thus have the following chain of inequalities

$$\frac{\deg \mathcal{P}}{h^0(\mathcal{P}) - 1} = \frac{\deg \mathcal{P}}{\deg \mathcal{P} + 1 - g} \geq \frac{\deg \mathcal{L}}{\deg \mathcal{L} - g} = \frac{\deg \mathcal{P} + h^0(D, \mathcal{O}_D)}{\deg \mathcal{P} + h^0(D, \mathcal{O}_D) - g}.$$

In fact $\deg \mathcal{P}/(\deg \mathcal{P} + 1 - g) \leq (\deg \mathcal{P} + h^0(D, \mathcal{O}_D))/(\deg \mathcal{P} + h^0(D, \mathcal{O}_D) - g)$ if and only if $\deg \mathcal{P} \leq (h^0(D, \mathcal{O}_D))(g-1)$ and as $\deg \mathcal{P} \leq 2g - 2$ the inequality is always verified and is strict unless $\deg \mathcal{P} = 2g - 2$ and $h^0(D, \mathcal{O}_D) = 2$. In this last case we have that $h^0(\mathcal{P}^* \otimes \omega_C) = 1$ so $\mathcal{P} \cong \omega_C$ and $\mathcal{L} \cong \omega_C(D)$ as wanted. □

Remark 3.4. A similar result on non-complete canonical systems was obtained in [2]: it states that the triple (C, ω_C, V), where V is a general subspace $V \subset H^0(\omega_C)$ of codimension $c \leq \mathrm{Cliff}(C)/2$, is linearly semistable. Note that the condition on codimension is analogous to the condition of Proposition 3.3: $\deg \omega_C = 2g - 2 \leq 2(\dim V - 1) - \mathrm{Cliff}(C)$.

4. The Slope of Determinant Bundles

Let us state the following well-known fact (see for instance [15, 5.0.1]).

Proposition 4.1. *Let $\mathcal{F}$ be a globally generated vector bundle of rank $r \geq 2$. Let $\mathcal{A} = \det(\mathcal{F})$. For a general choice of a subspace $T \subset H^0(\mathcal{F})$ of dimension $r - 1$, evaluation on global sections of $\mathcal{F}$ gives the following exact sequence*:

$$0 \to T \otimes \mathcal{O}_C \to \mathcal{F} \to \mathcal{A} \to 0. \tag{4.1}$$

The following argument will be a key point in our proof. It is largely inspired by [9].

Proposition 4.2. *Let $\mathcal{F}$ be a globally generated vector bundle of rank $r \geq 2$ and $h^0(\mathcal{F}^*) = 0$. If the sequence* (4.1) *is* exact on global sections, *then* $\deg \mathcal{A} = \deg \mathcal{F} \geq \gamma(h^0(\mathcal{A}) - 1)$, *where γ is the gonality of the curve C.*

Proof. Let us consider the sequence (4.1) tensored with ω_C. By taking the cohomology sequence, as $h^0(\mathcal{F}^*) = 0$, we can conclude that the homomorphism $H^0(\mathcal{F} \otimes \omega_C) \to H^0(\mathcal{A} \otimes \omega_C)$ is not surjective. From this, we derive that the multiplication homomorphism

$$H^0(\mathcal{A}) \otimes H^0(\omega_C) \to H^0(\mathcal{A} \otimes \omega_C) \tag{4.2}$$

fails to be surjective. Indeed, let us consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \bigoplus^{\operatorname{rank}\mathcal{F}-1} \omega_C & \longrightarrow & \bigoplus^{\operatorname{rank}\mathcal{F}-1} \omega_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{\mathcal{F}} \otimes \omega_C & \longrightarrow & H^0(\mathcal{F}) \otimes \omega_C & \longrightarrow & \mathcal{F} \otimes \omega_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{\mathcal{A}} \otimes \omega_C & \longrightarrow & H^0(\mathcal{A}) \otimes \omega_C & \longrightarrow & \mathcal{A} \otimes \omega_C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Remark that the middle column is exact by our assumptions on the exact sequence (4.1). By taking global sections, we have the commutative diagram

$$\begin{array}{ccc}
H^0(\mathcal{F}) \otimes H^0(\omega_C) & \longrightarrow & H^0(\mathcal{F} \otimes \omega_C) \\
\downarrow & & \downarrow \\
H^0(\mathcal{A}) \otimes H^0(\omega_C) & \longrightarrow & H^0(\mathcal{A} \otimes \omega_C)
\end{array}$$

where the first vertical arrow is surjective, while the second, as it is shown above, is not. Hence the bottom horizontal arrow cannot be surjective.

From a result of Castelnuovo type due to Green [14, Theorem 4.b2] (see also [11]) we have that, for any base point free line bundle $\mathcal{A}$, the sequence (4.2) fails to be surjective only if the image of the morphism induced by $\mathcal{A}$ is a rational normal curve in $\mathbb{P}(H^0(\mathcal{A})^*)$. Hence we have that $\deg \mathcal{A} \geq \gamma(h^0(\mathcal{A}) - 1)$, where γ is the gonality of C, as wanted. □

We now state a consequence on dual span bundles that will be a key point in our arguments. As usual, let $\mathcal{L}$ be a line bundle on C and $V \subseteq H^0(\mathcal{L})$ be a generating subspace. Let $\mathcal{S} \subset M_{V,\mathcal{L}}$ be a saturated subbundle, and $F_{\mathcal{S}}$ and $\mathcal{A} = \det F_{\mathcal{S}}$ be as in Remark 2.2.

Lemma 4.3. *With the notation above, suppose that* $\operatorname{rank} F_{\mathcal{S}} \geq 2$. *If* $F_{\mathcal{S}}$ *fits in an exact sequence*

$$0 \to \bigoplus^{\operatorname{rank} F_{\mathcal{S}}-1} \mathcal{O}_C \to F_{\mathcal{S}} \to \mathcal{A} \to 0,$$

which is also exact on global sections, then the following properties hold.

(1) *If* $\deg \mathcal{L} \leq \gamma(\dim V - 1)$, *then* $\mu(\mathcal{S}) \leq \mu(M_{V,\mathcal{L}})$. *Furthermore, we have equality if and only if*
   - $W = H^0(F_{\mathcal{S}})$,
   - $\gamma = \deg \mathcal{A}/(h^0(\mathcal{A}) - 1)$,
   - $\gamma = \deg \mathcal{L}/(\dim V - 1)$.

(2) *If* $\deg \mathcal{L} < \gamma(\dim V - 1)$, *then* $\mu(\mathcal{S}) < \mu(M_{V,\mathcal{L}})$.

Proof. Note that, as $\operatorname{rank} F_{\mathcal{S}} \geq 2$,

$$\operatorname{rank} \mathcal{S} = \dim W - \operatorname{rank} F_{\mathcal{S}} \leq h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} = h^0(\mathcal{A}) - 1.$$

So if we have

$$\mu(\mathcal{S}) = \frac{-\deg \mathcal{A}}{\dim W - \operatorname{rank} F_{\mathcal{S}}} \geq \mu(M_{V,\mathcal{L}}) = \frac{-\deg \mathcal{L}}{\dim V - 1},$$

then

$$\gamma \leq \frac{\deg \mathcal{A}}{h^0(\mathcal{A}) - 1} \leq \frac{\deg \mathcal{A}}{\dim W - \operatorname{rank} F_{\mathcal{S}}} \leq \frac{\deg \mathcal{L}}{\dim V - 1} \leq \gamma.$$

So the inequality $\mu(\mathcal{S}) \geq \mu(M_{V,\mathcal{L}})$ cannot hold strict, and it is an equality if and only if $W = H^0(F_{\mathcal{S}})$, and $\gamma = \deg \mathcal{A}/(h^0(\mathcal{A}) - 1) = \deg \mathcal{L}/(\dim V - 1)$. □

5. Stability of DSBs in the Complete Case

The main result of this section is the first part of Theorem 1.1.

Theorem 5.1. *Let* $\mathcal{L} \in \operatorname{Pic}(C)$ *be a globally generated line bundle such that*

$$\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \leq \operatorname{Cliff}(C).$$

Then $\mathcal{L}$ *is linearly (semi)stable if and only if* $M_{\mathcal{L}}$ *is (semi)stable.*

Proof. Clearly $M_{\mathcal{L}}$ (semi)stable implies $\mathcal{L}$ is linearly (semi)stable. Let us prove the other implication, thus suppose $\mathcal{L}$ linearly (semi)stable.

By contradiction let $\mathcal{S}$ be a maximal stable destabilizing subbundle of $M_{\mathcal{L}}$, i.e. $\mathcal{S}$ stable, $\mu(\mathcal{S}) \geq \mu(M_{\mathcal{L}})$ maximal ($>$ for semistability), and $\operatorname{rank} \mathcal{S} < \operatorname{rank} M_{\mathcal{L}}$. Note that

$$\deg \mathcal{L} \leq \operatorname{Cliff}(C) + 2(h^0(\mathcal{L}) - 1) \leq \gamma - 2 + 2(h^0(\mathcal{L}) - 1) \leq \gamma(h^0(\mathcal{L}) - 1), \tag{5.1}$$

with equality if and only if either $\gamma = 2$ and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$, or $h^0(\mathcal{L}) = 2$ and $\deg \mathcal{L} = \gamma = \operatorname{Cliff}(C) + 2$.

By the assumption on linear (semi)stability, we have that $\operatorname{rank} F_{\mathcal{S}} \geq 2$. We prove the following.

Claim. *The bundle* $F_{\mathcal{S}}$ *admits a determinant sequence* (4.1) *exact on global sections.*

Then, by (5.1) and Remark 2.2(ii), we can apply Lemma 4.3. So for such $\mathcal{S}$ and $F_{\mathcal{S}}$ we have that $\mathcal{S}$ cannot destabilize $M_{\mathcal{L}}$. It can strictly destabilize (i.e. $\mu(\mathcal{S}) = \mu(M_{\mathcal{L}})$) only in the case where $\deg \mathcal{L} = \gamma(h^0(\mathcal{L}) - 1)$. By the consequences of (5.1), this strict destabilization can happen only if either $\gamma = 2$ and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$, or $h^0(\mathcal{L}) = 2$ and $\deg \mathcal{L} = \gamma = \mathrm{Cliff}(C) + 2$. In the last case we have that $\operatorname{rank} M_{\mathcal{L}} = 1$. In the first one, we have that C is hyperelliptic and $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$; it is well known that $\mathcal{L}$ is strictly linearly semistable in this case (the dual of the g^1_2 providing a strict destabilization as noted above). In any case we cannot have a strict destabilization if $\mathcal{L}$ is supposed linearly *stable*.

To prove the *claim* let us remark that by Proposition 4.1 such a short exact sequence exists. What we need to show is that it is exact on global sections; this is equivalent to showing that $h^0(\mathcal{A}) \leq h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} + 1$.

Observe that $H^0(F_{\mathcal{S}}) \twoheadrightarrow H^0(\mathcal{A})$ if and only if $H^1(\mathcal{O}^{\oplus \operatorname{rank} F_{\mathcal{S}} - 1}) \hookrightarrow H^1(F_{\mathcal{S}})$. Let us show first that this is numerically possible: we prove that $g(\operatorname{rank} F_{\mathcal{S}} - 1) = h^1(\mathcal{O}^{\oplus \operatorname{rank} F_{\mathcal{S}} - 1}) < h^1(F_{\mathcal{S}})$ indeed. In fact

$$h^1(F_{\mathcal{S}}) = h^0(F_{\mathcal{S}}) - \deg F_{\mathcal{S}} + g \cdot \operatorname{rank} F_{\mathcal{S}} - \operatorname{rank} F_{\mathcal{S}}.$$

Hence $h^1(F_{\mathcal{S}}) > g \cdot \operatorname{rank} F_{\mathcal{S}} - g$ if and only if $h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} > \deg F_{\mathcal{S}} - g$.

As $h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} \geq \operatorname{rank} \mathcal{S}$, we can show that $\operatorname{rank} \mathcal{S} > \deg F_{\mathcal{S}} - g$, i.e. that

$$\frac{\deg F_{\mathcal{S}}}{\operatorname{rank} \mathcal{S}} < 1 + \frac{g}{\operatorname{rank} \mathcal{S}}.$$

By hypothesis $\mu(\mathcal{S}) = -\deg F_{\mathcal{S}} / \operatorname{rank} \mathcal{S} \geq -\deg \mathcal{L} / (h^0(\mathcal{L}) - 1)$, hence

$$\frac{\deg F_{\mathcal{S}}}{\operatorname{rank} \mathcal{S}} \leq \frac{\deg \mathcal{L}}{h^0(\mathcal{L}) - 1} = 1 + \frac{g - h^1(\mathcal{L})}{\operatorname{rank} M_{\mathcal{L}}} < 1 + \frac{g}{\operatorname{rank} \mathcal{S}}.$$

As the cokernel of $\varphi : H^1(\mathcal{O}^{\oplus \operatorname{rank} F_{\mathcal{S}} - 1}) \to H^1(F_{\mathcal{S}})$ is exactly $H^1(\mathcal{A})$, and the inequality above is strict, then if $h^1(\mathcal{A}) \leq 1$ the map φ is injective as we need, and $H^0(F_{\mathcal{S}}) \twoheadrightarrow H^0(\mathcal{A})$.

Let us show that if $h^1(\mathcal{A}) \geq 2$, then the map is surjective as well: in this case we have the inequality

$$\deg \mathcal{A} - 2(h^0(\mathcal{A}) - 1) \geq \mathrm{Cliff}(C) \geq \deg(\mathcal{L}) - 2(h^0(\mathcal{L}) - 1).$$

As $\deg F_{\mathcal{S}} = \deg \mathcal{A} < \deg \mathcal{L}$ (see Remark 2.2), then $2(h^0(\mathcal{L}) - h^0(\mathcal{A})) \geq \deg \mathcal{L} - \deg \mathcal{A} > 0$, hence $h^0(\mathcal{A}) < h^0(\mathcal{L})$.

Remark that, by the assumption made on $\mathcal{S}$,

$$\frac{\deg \mathcal{A}}{\operatorname{rank} \mathcal{S}} = \frac{-\deg \mathcal{S}}{\operatorname{rank} \mathcal{S}} \leq \frac{\deg \mathcal{L}}{h^0(\mathcal{L}) - 1},$$

hence $\operatorname{rank} \mathcal{S} \geq \deg \mathcal{A} \cdot (h^0(\mathcal{L}) - 1) / \deg \mathcal{L}$.

Now assume that $H^0(F_{\mathcal{S}}) \to H^0(\mathcal{A})$ is not surjective, i.e. that $h^0(\mathcal{A}) > h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} + 1$.

Then we have that $h^0(\mathcal{A}) - 1 > h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} \geq \operatorname{rank} \mathcal{S} \geq \deg \mathcal{A} \cdot (h^0(\mathcal{L}) - 1)/\deg \mathcal{L}$, hence

$$\deg \mathcal{A} < \frac{\deg \mathcal{L}}{h^0(\mathcal{L}) - 1}(h^0(\mathcal{A}) - 1) \leq \left(2 + \frac{\operatorname{Cliff}(C)}{h^0(\mathcal{L}) - 1}\right)(h^0(\mathcal{A}) - 1)$$
$$= 2(h^0(\mathcal{A}) - 1) + \operatorname{Cliff}(C)\frac{h^0(\mathcal{A}) - 1}{h^0(\mathcal{L}) - 1} \leq 2(h^0(\mathcal{A}) - 1) + \operatorname{Cliff}(C),$$

so $2(h^0(\mathcal{A}) - 1) + \operatorname{Cliff}(C) \leq \deg \mathcal{A} < 2(h^0(\mathcal{A}) - 1) + \operatorname{Cliff}(C)$ and we get a contradiction. □

Remark 5.2. It is worth noticing that the claim in the proof of Theorem 5.1 above is a point where Butler's argument in [9] fails to be complete.

The consequences of this theorem, as stated in Sec. 1, follow easily.

Theorem 5.3. *Let $\mathcal{L} \in \operatorname{Pic}(C)$ be a globally generated line bundle such that*

$$\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) \leq \operatorname{Cliff}(C). \tag{5.2}$$

Then $M_{\mathcal{L}}$ is semistable, and it is strictly semistable only in one of the following cases:

(i) *$\mathcal{L} \cong \omega_C(D)$ with D an effective divisor of degree* 2,
(ii) *C is hyperelliptic and* $\deg \mathcal{L} = 2(h^0(\mathcal{L}) - 1)$.

Proof. It follows immediately from Theorem 5.1 and Proposition 3.3. □

Corollary 5.4. *Let $\mathcal{L}$ be a globally generated line bundle over C with*

$$\deg \mathcal{L} \geq 2g - \operatorname{Cliff}(C).$$

Then the vector bundle $M_{\mathcal{L}}$ is semistable. It is stable unless (i) *or* (ii) *holds.*

Proof. Observe that if $\mathcal{L}$ is a globally generated line bundle over C with $\deg \mathcal{L} \geq 2g - \operatorname{Cliff}(C)$, then $\operatorname{Cliff}(C) \geq 2g - \deg \mathcal{L}$, so

$$\operatorname{Cliff} C + 2(h^0(\mathcal{L}) - 1) \geq \operatorname{Cliff} C + 2(\deg \mathcal{L} - g) \geq \deg \mathcal{L},$$

then use Theorem 5.3. □

Corollary 5.5. *Let $\mathcal{L}$ be any line bundle that computes the Clifford index of C. Then $M_{\mathcal{L}}$ is semistable; it is stable unless C is hyperelliptic.*

Proof. It follows from Theorem 5.3, recalling that any line bundle computing the Clifford index is globally generated. □

6. The Non-Complete Case

The aim of this section is to extend the methods described above, when possible, to the non-complete case.

The following conjecture is the most natural direct generalization of Theorem 5.1 to the non-complete case. Note that it is weaker than Conjecture 8.6 below. We will not prove it in full generality, but it still holds in many cases.

Conjecture 6.1. *Let* $(C, \mathcal{L}, V)$ *be a triple. If* $\deg \mathcal{L} - 2(\dim V - 1) \leq \mathrm{Cliff}(C)$, *then linear* (*semi*)*stability is equivalent to* (*semi*)*stability of* $M_{V,\mathcal{L}}$.

Remark 6.2. The inequality $\deg \mathcal{L} \leq \mathrm{Cliff}(C) + 2(\dim V - 1)$ holds if and only if

$$\mathrm{codim}_{H^0(L)} V \leq \frac{\mathrm{Cliff}(C) - (\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1))}{2}.$$

The results of this and the previous section can be summarized in the following (equivalent to Theorem 1.1).

Theorem 6.3. *Conjecture* 6.1 *holds in the following cases*:

(1) $H^0(\mathcal{L}) = V$ (*complete case*);
(2) $\deg \mathcal{L} \leq 2g - \mathrm{Cliff}(C) + 1$;
(3) $\mathrm{codim}_{H^0(\mathcal{L})} V < h^1(\mathcal{L}) + g/(\dim V - 2)$;
(4) $\deg \mathcal{L} \geq 2g$, *and* $\mathrm{codim}_{H^0(\mathcal{L})} V \leq (\deg \mathcal{L} - 2g)/2$.

In all of the following result we make this assumption. Let $(C, \mathcal{L}, V)$ be a triple verifying $\deg \mathcal{L} - 2(\dim V - 1) \leq \mathrm{Cliff}(C)$, let $\mathcal{S} \subset M_{V,\mathcal{L}}$ be a proper subbundle such that $\mu(\mathcal{S}) \geq \mu(M_{V,\mathcal{L}})$, let $F_{\mathcal{S}}$ and $\mathcal{A}$ be as in Lemma 4.3.

In order to prove Theorem 6.3, we proceed as in Theorem 5.1, and show that within these numerical hypothesis we can apply Lemma 4.3. That is, we show that for a possible destabilization given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F_{\mathcal{S}} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

the bundle $F_{\mathcal{S}}$ fits into a short exact sequence

$$0 \to \bigoplus^{\mathrm{rank}\, F_{\mathcal{S}} - 1} \mathcal{O}_C \to F_{\mathcal{S}} \to \mathcal{A} \to 0$$

which is exact on global sections.

Lemma 6.4. *If* $h^1(\mathcal{A}) \geq 2$ *then the sequence*

$$0 \to \bigoplus^{\mathrm{rank}\, F_{\mathcal{S}} - 1} \mathcal{O}_C \to F_{\mathcal{S}} \to \mathcal{A} \to 0$$

is exact on global sections.

Proof. If $h^1(\mathcal{A}) \geq 2$ then $\deg \mathcal{A} - 2(h^0(\mathcal{A}) - 1) \geq \text{Cliff}(C) \geq \deg \mathcal{L} - 2(\dim V - 1)$, and then we have that $2(\dim V - h^0(\mathcal{A})) \geq \deg \mathcal{L} - \deg \mathcal{A} = \deg \mathcal{L} - \deg F_{\mathcal{S}} > 0$. Furthermore, if

$$\frac{\deg \mathcal{A}}{\text{rank}\, \mathcal{S}} = \frac{-\deg \mathcal{S}}{\text{rank}\, \mathcal{S}} \leq \frac{\deg \mathcal{L}}{\dim V - 1},$$

then $\text{rank}\, \mathcal{S} \geq \deg \mathcal{A}(\dim V - 1)/\deg \mathcal{L}$.

Then if we had the inequality $h^0(\mathcal{A}) - 1 > h^0(F_{\mathcal{S}}) - \text{rank}\, F_{\mathcal{S}}$, we would have $h^0(\mathcal{A}) - 1 > \text{rank}\, \mathcal{S} \geq \deg \mathcal{A}(\dim V - 1)/\deg \mathcal{L}$, and then

$$\begin{aligned} 2(h^0(\mathcal{A}) - 1) + \text{Cliff}(C) \leq \deg \mathcal{A} &< \deg \mathcal{L} \frac{h^0(\mathcal{A}) - 1}{\dim V - 1} \\ &\leq 2(h^0(\mathcal{A}) - 1) + \text{Cliff}(C) \frac{h^0(\mathcal{A}) - 1}{\dim V - 1} < 2(h^0(\mathcal{A}) - 1) \\ &\quad + \text{Cliff}(C), \end{aligned}$$

which is absurd, so we have $h^0(\mathcal{A}) \leq h^0(F_{\mathcal{S}}) - \text{rank}\, F_{\mathcal{S}} + 1$, hence the sequence

$$0 \to \bigoplus^{\text{rank}\, F_{\mathcal{S}} - 1} \mathcal{O}_C \to F_{\mathcal{S}} \to \mathcal{A} \to 0$$

is exact on global sections. □

To complete the proof of Theorem 6.3 we have to treat the case $h^1(\mathcal{A}) \leq 1$ as well.

Lemma 6.5. *Suppose that* $h^1(\mathcal{A}) \leq 1$. *If we assume that* $\deg \mathcal{L} \leq 2g - \text{Cliff}(C) + 1$, *then the sequence*

$$0 \to \bigoplus^{\text{rank}\, F_{\mathcal{S}} - 1} \mathcal{O}_C \to F_{\mathcal{S}} \to \mathcal{A} \to 0$$

is exact on global sections.

Proof. We want to prove that $h^0(F_{\mathcal{S}}) - \text{rank}\, F_{\mathcal{S}} + 1 \geq h^0(\mathcal{A})$. This is the case if we prove that $h^0(F_{\mathcal{S}}) - \text{rank}\, F_{\mathcal{S}} > \deg \mathcal{A} - g$. As $h^0(F_{\mathcal{S}}) - \text{rank}\, F_{\mathcal{S}} \geq \text{rank}\, \mathcal{S}$, recalling that $\deg F_{\mathcal{S}} = \deg \mathcal{A}$, we are done if we can prove the following.

Claim. $\deg F_{\mathcal{S}} < \text{rank}\, \mathcal{S} + g$.
In fact we have that

$$\begin{aligned} \deg F_{\mathcal{S}} &\leq \frac{\text{rank}\, \mathcal{S}}{\dim V - 1} \deg \mathcal{L} \leq \frac{\text{rank}\, \mathcal{S}}{\dim V - 1} (\text{Cliff}(C) + 2(\dim V - 1)) \\ &= \text{Cliff}(C) \frac{\text{rank}\, \mathcal{S}}{\dim V - 1} + 2\, \text{rank}\, \mathcal{S} < \text{Cliff}(C) + 2\, \text{rank}\, \mathcal{S}. \end{aligned}$$

So, if $\text{rank}\, \mathcal{S} \leq g - \text{Cliff}(C)$ then the claim is verified. Let us show that this is the case when $\text{rank}\, \mathcal{S} > g - \text{Cliff}(C)$ as well. In fact if we had $\deg F_{\mathcal{S}} \geq \text{rank}\, \mathcal{S} + g$

holding together with $\operatorname{rank} \mathcal{S} > g - \operatorname{Cliff}(C)$, then we would have $\deg \mathcal{L} > \deg F_{\mathcal{S}} > 2g - \operatorname{Cliff}(C)$, contrary to the assumption. □

Lemma 6.6. *If* $\operatorname{codim}_{H^0(\mathcal{L})} V < h^1(\mathcal{L}) + g/(\dim V - 2)$, *then the sequence*

$$0 \to \bigoplus^{\operatorname{rank} F_{\mathcal{S}}-1} \mathcal{O}_C \to F_{\mathcal{S}} \to \mathcal{A} \to 0$$

is exact on global sections.

Proof. The case $h^1(\mathcal{A}) \geq 2$ is treated in Lemma 6.4. Let us assume that $h^1(\mathcal{A}) \leq 1$. We proceed as in the proof of Lemma 6.5, and show the same condition.

Claim. $\operatorname{rank} \mathcal{S} > \deg \mathcal{A} - g$.
As shown in Lemma 6.5, this implies that $h^0(F_{\mathcal{S}}) - \operatorname{rank} F_{\mathcal{S}} + 1 \geq h^0(\mathcal{A})$.

To prove the claim, set $c := \operatorname{codim}_{H^0(\mathcal{L})} V$, and observe that

$$\operatorname{rank} \mathcal{S} > \deg \mathcal{A} - g \Leftrightarrow \frac{\deg \mathcal{A}}{\operatorname{rank} \mathcal{S}} < 1 + \frac{g}{\operatorname{rank} \mathcal{S}}$$

and that

$$\frac{\deg \mathcal{A}}{\operatorname{rank} \mathcal{S}} \leq \frac{\deg \mathcal{L}}{\dim V - 1} = 1 + \frac{g + c - h^1(\mathcal{L})}{\dim V - 1}.$$

Now observe that if $c < h^1(\mathcal{L}) + g/(\dim V - 2)$, we have, noting that $\operatorname{rank} \mathcal{S} \leq \dim V - 2$,

$$c - h^1(\mathcal{L}) < g \left(\frac{\dim V - 1}{\operatorname{rank} \mathcal{S}} - 1 \right),$$

and hence that

$$\frac{\deg \mathcal{A}}{\operatorname{rank} \mathcal{S}} < 1 + \frac{g}{\operatorname{rank} \mathcal{S}}.$$ □

As for the last point in Theorem 6.3, it follows directly from [17, Lemma 2.2].

Proposition 6.7. *Let C be a curve such that* $\operatorname{Cliff}(C) \geq 4$. *Let $V \subset H^0(\omega_C)$ be a general subspace of codimension* 1 *or* 2. *Then M_{V,ω_C} is stable.*

Proof. It has been proved in [2] that a general projection from a subspace of dimension smaller than or equal to $\operatorname{Cliff}(C)/2$ is linearly stable. Then, the proof is immediate from Theorem 6.3. □

Remark 6.8. In the complete case, the (semi)stability of M_{ω_C} is well known in the literature, regardless of the Clifford index [20]. In the next section we shall prove a stronger result on the vector bundle M_{ω_C} and on the case of codimension 1; see Corollary 7.7.

7. Cohomological Stability and the Clifford Index

The following definition was introduced by Ein and Lazarsfeld in [13].

Definition 7.1. Let $\mathcal{E}$ be a vector bundle on a curve C. We say that $\mathcal{E}$ is *cohomologically stable* (respectively, *cohomologically semistable*) if for any line bundle $\mathcal{A}$ of degree a and for any integer $t < \operatorname{rank} \mathcal{E}$ we have

$$h^0\left(\bigwedge^t \mathcal{E} \otimes \mathcal{A}^{-1}\right) = 0 \quad \text{whenever } a \geq t\mu(\mathcal{E}) \quad (\text{respectively, } a > t\mu(\mathcal{E})).$$

Remark 7.2. Cohomological (semi)stability implies bundle (semi)stability; indeed, given any proper subbundle $\mathcal{S} \subset \mathcal{E}$ of degree a and rank t, we have an inclusion $\det \mathcal{S} \hookrightarrow \bigwedge^t \mathcal{E}$, hence a non-zero section of $(\det \mathcal{S})^{-1} \otimes \bigwedge^t \mathcal{E}$.

Moreover, observe that cohomological (semi)stability of $\mathcal{E}$ is implied by $\bigwedge^t \mathcal{E}$ being (semi)stable for any integer t; hence cohomological semistability is equivalent to semistability, while cohomological stability can be a stronger condition than stability.

In [13] the two authors prove the cohomological stability of the DSB $M_{\mathcal{L}}$ associated to any line bundle $\mathcal{L}$ on a curve of positive genus g, under the assumption that $\deg \mathcal{L} \geq 2g + 1$.

The main result of this section is Theorem 7.3 stated in Sec. 1, which is a generalization of the result of Ein and Lazarsfeld.

Theorem 7.3. *Let $(\mathcal{L}, V)$ be a g^r_d on a smooth curve C, inducing a birational morphism. Suppose that*

- $d \leq 2r + \operatorname{Cliff} C$;
- $\operatorname{codim}_{H^0(\mathcal{L})} V \leq h^1(\mathcal{L})$.

Then $M_{V,\mathcal{L}}$ is cohomologically semistable. It is cohomologically stable except in the following cases:

(i) *$d = 2r$ and C is hyperelliptic.*
(ii) *$\mathcal{L} \cong \omega_C(D)$ with D an effective divisor of degree* 2.

In order to prove Theorem 7.3, let us first establish this simple generalization of a result used in the proof of [13, Proposition 3.2], and a lemma.

Proposition 7.4. *Let $(\mathcal{L}, V)$ be a g^r_d on a smooth curve C, inducing a birational morphism; let $D_k = p_1 + \cdots + p_k$ be a general effective divisor on C, with $k < r$. The DSB associated to the linear series lies in the following exact sequence of sheaves*:

$$0 \to M_{V(-D_k), \mathcal{L}(-D_k)} \to M_{V,\mathcal{L}} \to \bigoplus_{i=1}^{k} \mathcal{O}_C(-p_i) \to 0.$$

Proof. As D_k is general effective, we have that $\dim V(-D_k) = \dim V - k = r + 1 - k$. Moreover, as the morphism induced by $|V|$ is generically injective, $\mathcal{L}(-D_k)$ is generated by $V(-D_k)$. Let W be the cokernel of the injection $V(-D_k) \subseteq V$.

Using the snake lemma, we can form the top exact row in the diagram below, and the proof is concluded.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{V(-D_k),\mathcal{L}(-D_k)} & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & \bigoplus_{i=1}^{k} \mathcal{O}_C(-p_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(-D_k)\otimes\mathcal{O}_C & \longrightarrow & V\otimes\mathcal{O}_C & \longrightarrow & W\otimes\mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L}(-D_k) & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}_{D_k} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

□

Remark 7.5. With the notation and conditions of the above proposition, if we consider a general effective divisor D of maximal degree $r-1$, we have that $M_{V(-D),\mathcal{L}(-D)}$ is a line bundle which is dual to $\mathcal{O}_C(p_r+\cdots+p_d)$, so

$$M_{V(-D),\mathcal{L}(-D)} \cong \mathcal{O}_C(-p_r-\cdots-p_d).$$

Lemma 7.6. *Let $\mathcal{A}$ be a line bundle on the curve C such that* $\deg\mathcal{A}\le td/r$, *with t, d, and r integers satisfying* $0<t<r<d\le 2r+\operatorname{Cliff}(C)$ *and* $r\ge d-g$. *Suppose that the first inequality* $\deg\mathcal{A}\le td/r$ *is strict if* $\operatorname{Cliff}(C)=0$.

Then $h^0(\mathcal{A})\le t$, except for the case where $\mathcal{A}=\omega_C$, and $t=g-1$, $r=g$, $d=2g$.

Proof. Let us distinguish three cases according to the values of $h^1(\mathcal{A})$:

(A) Suppose that $h^1(\mathcal{A})\ge 2$. Then we can suppose that $\mathcal{A}$ contributes to the Clifford index of C (i.e. that $h^0(\mathcal{A})\ge 2$), and so

$$\begin{aligned}2(h^0(\mathcal{A})-1) &\le \deg\mathcal{A}-\operatorname{Cliff} C \le t\frac{d}{r}-\operatorname{Cliff} C\\ &\le t\left(2+\frac{\operatorname{Cliff} C}{r}\right)-\operatorname{Cliff} C = 2t+\operatorname{Cliff} C\left(\frac{t}{r}-1\right).\end{aligned}$$

The last quantity is strictly smaller than $2t$ if and only if $\operatorname{Cliff} C>0$. So, if C is non-hyperelliptic we are done. If C is hyperelliptic we still have the claim if $d/r<2$, while if $d/r=2$, the claim is true supposing the strict inequality $\deg\mathcal{A}<td/r$.

(B) Suppose that $\mathcal{A}$ is non-special: $h^1(\mathcal{A}) = 0$. Then by Riemann–Roch theorem $h^0(\mathcal{A}) = \deg \mathcal{A} - g + 1$. This quantity is smaller than or equal to t if and only if $\deg \mathcal{A} - g < t$. Hence, it is sufficient to prove that $td/r - g < t$; equivalently, we need to prove that

$$t < r\frac{g}{d-r}.$$

By assumption, we have that $r \geq d - g$, so the above inequality is true, since $t < r \leq rg/(d-r)$.

(C) Finally, let us suppose that $h^1(\mathcal{A}) = 1$. Remember that we are assuming that $d \leq 2r + \operatorname{Cliff} C$. Let us distinguish three cases again.

(C.1) If $d < 2r$ (for instance this is the case if $d > 2g$ as in [13]). Then $\deg \mathcal{A} \leq td/r < 2t$. So, as $\mathcal{A}$ is special, we have that $2(h^0(\mathcal{A}) - 1) \leq \deg \mathcal{A} < 2t$ and we are done.

(C.2) Suppose that $d > 2r$. Observe that

$$r\left(\frac{g-1}{d-r}\right) \geq r - \frac{r}{d-r} > r - 1 \geq t.$$

As in point (B), this is the inequality we need.

(C.3) Let us now suppose that $d = 2r$. In this case, we obtain $r(g-1)/(d-r) \geq t$, so $t \leq g - 1$, and

$$h^0(A) = 1 + \deg A + 1 - g \leq t + 1,$$

with equality holding if and only if $t = g - 1$, $\deg A = 2t = 2g - 2$, which implies that $A = \omega_C$ and $d = 2r = 2g$. □

Proof of Theorem 7.3. By Proposition 7.4 and Remark 7.5, we have that the bundle $M_{V,\mathcal{L}}$ sits in the exact sequence

$$0 \to \mathcal{O}_C(-p_r - \cdots - p_d) \to M_{V,\mathcal{L}} \to \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-p_i) \to 0.$$

Let t be an integer strictly smaller than r. Applying the t-th exterior power, we get the sequence

$$0 \to \bigoplus_{1 \leq i_1 < \cdots < i_{t-1} \leq r-1} \mathcal{O}_C(-p_{i_1} - p_{i_2} - \cdots - p_{i_{t-1}} - p_r - \cdots - p_d)$$

$$\to \bigwedge^t M_{V,\mathcal{L}} \to \bigoplus_{1 \leq j_1 < j_2 < \cdots < j_t \leq r-1} \mathcal{O}_C(-p_{j_1} - p_{j_2} - \cdots - p_{j_t}) \to 0. \tag{7.1}$$

Let us now tensor the above sequence with a line bundle $\mathcal{A}^{-1}$ of degree $-a$.

We shall now suppose that $-a \leq td/r$, in order to prove cohomological stability. We will see in the course of the proof that in case $d/r = 2$ and C hyperelliptic we will need to assume strict inequality, thus proving semistability.

We want to prove that $H^0(\bigwedge^t M_{V,\mathcal{L}} \otimes \mathcal{A}^{-1}) = \{0\}$. To this aim, let us consider the global sections of sequence (7.1) tensorized by $\mathcal{A}^{-1}$ and prove that both left and right-hand side are trivial.

The left-hand side is a sum of global sections of line bundles each of degree $-t-d+r-a \leq -t-d+r+(td)/r = (r-t)(1-d/r)$. As $r-t > 0$ by assumption, and $d > r$, this degree is negative and we are done.

Let us now study the right-hand side. Remark that the hypothesis on $h^1(\mathcal{L})$ is equivalent to $r \geq d-g$. Applying Lemma 7.6 to $\mathcal{A}^{-1}$, we have $h^0(\mathcal{A}^{-1}) \leq t$, except for the case where $\mathcal{A}^{-1} = \omega_C$, $t = g-1$, $r = g$, and $d = 2g$.

In this last case a strict cohomological destabilization is given by

$$H^0\left(\bigwedge^{r-1} M_{V,\mathcal{L}} \otimes \omega_C\right) = H^0(M^*_{V,\mathcal{L}} \otimes \mathcal{L}^* \otimes \omega_C) \neq 0,$$

and it can be shown that this can only happen in case (ii) of the hypothesis: $\mathcal{L} \cong \omega_C(D)$ with D an effective divisor of degree 2.

In all the other cases, the pieces of the right-hand side are of the form $H^0(\mathcal{A}^{-1}(-D))$ where D is a general effective divisor of degree t, so they vanish. □

As a consequence, we have immediately the following result.

Corollary 7.7. *If C is non-hyperelliptic, then M_{ω_C} is cohomologically stable. Moreover, if* $\mathrm{Cliff}\, C \geq 2$, *a projection from a general point in $\mathbb{P}^{g-1}$ is cohomologically stable.*

Remark 7.8. What can we say if we drop the assumption of the linear series to induce a birational morphism? Let $(\mathcal{L}, V)$ be a base point free linear series on a curve C. Let $\varphi : C \to \mathbb{P}^r$ be the induced morphism, and $\nu : \overline{C} \to \varphi(C) \subseteq \mathbb{P}^r$ be the normalization of the image curve. Then the morphism φ decomposes as

$$C \xrightarrow{\beta} \overline{C} \xrightarrow{\nu} \varphi(C) \overset{\iota}{\hookrightarrow} \mathbb{P}^r,$$

where β is a finite morphism (of degree b). Let $(\overline{\mathcal{L}}, \overline{V})$ be the linear series induced on $\overline{C}$ by $\iota \circ \nu$. Clearly $V = \beta^*(\overline{V})$, and $\mathcal{L} = \beta^*\overline{\mathcal{L}}$, and $\deg \overline{\mathcal{L}} = (\deg \mathcal{L})/b$.

Proposition 7.4 still holds if we substitute D_k with $\beta^*(\beta(D_k))$, and the points of this divisor fail to impose independent conditions on $H^0(\mathcal{L})$, so that the argument of Theorem 7.3 cannot be pushed through.

Observe that $\beta^* M_{\overline{V},\overline{\mathcal{L}}} = M_{V,\mathcal{L}}$. If $(\overline{\mathcal{L}}, \overline{V})$ satisfies the numerical conditions of Theorem 7.3, then $M_{\overline{V},\overline{\mathcal{L}}}$ is cohomologically stable by Theorem 7.3, so its pullback $M_{V,\mathcal{L}}$ is semistable, but we cannot say anything about its cohomological stability, nor vector bundle stability.

On the other hand, it is worth noticing that linear stability is preserved by finite morphisms: it is easy to verify that $(C, \mathcal{L}, V)$ is linearly (semi)stable if and only if $(\overline{C}, \overline{\mathcal{L}}, \overline{V})$ is linearly (semi)stable.

8. Linear Series of Dimension 2 and Counterexamples

In this section we discuss linear stability for curves with a g^2_d, then exhibit some examples and counterexamples to the implication (linear stability of the triple $(C, \mathcal{L}, V) \Rightarrow$ stability of $M_{V,\mathcal{L}}$).

The first result shows that linear stability is in this case related to the singularity of the image, which is very natural when compared with the results in [18].

Proposition 8.1. *Let $\nu : C \to \mathbb{P}^2$ be a birational morphism. Denote by $\overline{C} \subset \mathbb{P}^2$ its image, and by d the degree of $\overline{C}$ in $\mathbb{P}^2$. The morphism ν is induced by a linearly (semi)stable linear system if and only if all points $p \in \overline{C}$ have multiplicity $m_p < d/2$ (or $m_p \leq d/2$ for semistability).*

Proof. Linear stability (respectively, semistability) is equivalent to the fact that any projection from a point $p \in \mathbb{P}^2$ has degree $> d/2$ (respectively, $\geq d/2$). This degree is precisely $d - m_p$. □

From this result we can easily derive linear stability for any general g^2_d contained in a very ample linear series $\mathcal{L}$: such a linear series induces a birational morphism whose image in $\mathbb{P}^2$ is an integral plane curve with at most nodes as singularities (cf. [1, Exercises B-5 and B-6]). Hence this series is linearly stable (respectively, semistable) as soon as $d > 4$ (respectively, $d \geq 4$). Summing up we have proven the following.

Proposition 8.2. *Let C be a smooth curve with an embedding in $\mathbb{P}^n$ of degree $d > 4$ (respectively, $d \geq 4$). The general projection on $\mathbb{P}^2$ is linearly stable (respectively, semistable).*

Clearly this result goes in the direction of Butler's conjecture. It is not hard to prove the stability of DSB for smooth plane curves, and anytime the degree d is greater than or equal to $4g$. However we do not know in the general case whether or not linear stability implies the stability of the associated DSB.

We now describe an example showing that linear stability is not always equivalent to stability of DSB. Let us start with the following easy lemma.

Lemma 8.3. *Let $|V|$ be a base point free linear series of dimension r contained in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. Then if $r \nmid d$, the dual span bundle $M_{V,\mathcal{O}_{\mathbb{P}^1}(d)}$ is unstable.*

Proof. The bundle $M_{V,\mathcal{O}_{\mathbb{P}^1}(d)} = \ker(V \otimes \mathcal{O}_C \to \mathcal{O}_{\mathbb{P}^1}(d))$ is a rank r vector bundle on $\mathbb{P}^1$ that splits as the direct sum of r line bundles. If r does not divide d, this bundle cannot be (semi)stable. □

Combining the above lemma with Proposition 8.1, we easily get counterexamples, as follows.

Proposition 8.4. *On any curve C there exist non-complete linear systems $V \subset H^0(\mathcal{L})$ such that $(C, \mathcal{L}, V)$ is linearly stable and $M_{V,\mathcal{L}}$ is unstable.*

Proof. Consider any finite morphism $\beta : C \to \mathbb{P}^1$, and choose a map $\eta : \mathbb{P}^1 \to \mathbb{P}^2$ associated to a general base point free $W \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ with odd degree $d > 4$. By Lemma 8.3 the bundle $M_{W,\mathcal{O}_{\mathbb{P}^1}(d)}$ is unstable. Let $\mathcal{L} = \beta^*\mathcal{O}_{\mathbb{P}^1}(d)$ and $V := \beta^*(W) \subset H^0(\mathcal{L})$ be the linear series associated to the composition $\eta \circ \beta$. Clearly also $M_{V,\mathcal{L}} = \beta^* M_{W,\mathcal{O}_{\mathbb{P}^1}(d)}$ is unstable. On the other hand, $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d), W)$ is linearly stable, and, as linear stability respects finite morphisms (Remark 7.8), so is $(C, \mathcal{L}, V)$. □

Remark 8.5. Note that the linear systems produced satisfy the inequality

$$\deg \mathcal{L} \geq \gamma d > \gamma(\dim V - 1),$$

where γ is the gonality of C, so there is no contradiction with our conjectures. Furthermore, the subspace $V \subset H^0(\mathcal{L})$ is not general.

Therefore it seems reasonable to formulate some conjectures respectively on the non-complete and complete case.

Conjecture 8.6. *Let $(C, \mathcal{L}, V)$ be a triple as usual. If $\deg \mathcal{L} \leq \gamma(\dim V - 1)$, where γ is the gonality of C, then linear (semi)stability is equivalent to (semi)stability of $M_{V,\mathcal{L}}$.*

Conjecture 8.7. *For any curve C, and any line bundle $\mathcal{L}$ on C, linear (semi)-stability of $(C, \mathcal{L})$ is equivalent to (semi)stability of $M_{\mathcal{L}}$.*

These conjectures arise implicitly from Butler's article [9] (cf. Remark 5.2).

9. Stable DSB's with Slope 3, and Their Theta-Divisors

In this section we construct explicitly some stable bundles of integral slope on a general curve, and we prove that they admit theta-divisors.

Let us consider a curve C of even genus $g = 2k$, $k \geq 2$, having general gonality $\gamma = k + 1$ and Clifford Index $\mathrm{Cliff}(C) = k - 1$. Let D be a gonal divisor: $h^0(D) = 2$ and $\deg D = k + 1$, and hence $h^1(D) = k$ from the Riemann–Roch formula.

Let $\mathcal{L} = \omega_C(-D)$. We have that $\deg \mathcal{L} = 2g - 2 - k - 1 = 3k - 3$, $h^0(\mathcal{L}) = h^1(D) = k$, $h^1(\mathcal{L}) = h^0(D) = 2$, so $\deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1) = \mathrm{Cliff}(C)$ and $\mathcal{L}$ computes the Clifford Index of C. So the DSB $M_{\mathcal{L}}$ is stable by Corollary 5.5, and has integral slope

$$\mu(M_{\mathcal{L}}) = -\frac{3k - 3}{k - 1} = -3.$$

Moreover, we can state the following result, which was suggested to us by the referee.

Proposition 9.1. *With the above notation, if C is general, then $M_{\mathcal{L}}$ is cohomologically stable.*

Proof. The statement follows from Theorem 7.3, as soon as it is proved that $\mathcal{L}$ induces a birational morphism.

In fact, since $\mathcal{L}$ is generated, the morphism is an immersion if $h^0(\omega_C(-p-q)) = h^0(\omega_C) - 2$ for all $p, q \in C$. This is equivalent to $h^0(D+p+q) = 2$.

As C is general of genus $g = 2k$, there exists a g^r_d on C if and only if the Brill–Noether number $\rho(d, r, 2k) = r(d-r+1) - 2k(r-1)$ is greater or equal to 0. This implies that the minimal degree of a line bundle with space of global sections of dimension greater or equal to 3 is $2k + 2 - [\frac{2k}{3}]$. We therefore have the required result for $k \geq 4$, i.e. $g \geq 8$.

In case $k = 3$, the line bundle $\mathcal{L}$ induces a morphism in $\mathbb{P}^2$. If this morphism would not be birational, then the curve would be either trigonal (having a degree 3 morphism on a conic in $\mathbb{P}^2$) or bielliptic (having a degree 2 morphism on a cubic in $\mathbb{P}^2$); in any case C would not be general. In fact, for a general C, the image of the morphism induced by $\mathcal{L}$ is a plane sextic curve with four nodes. If $k = 2$, Then $M_{\mathcal{L}}$ is a line bundle and the proof is thus concluded. □

Question 9.2. Does $M_{\mathcal{L}}$ admit a theta-divisor?

We recall that the vector bundle $\mathcal{E}$ with integral slope is said to admit a theta-divisor if

$$\Theta_{\mathcal{E}} = \{\mathcal{P} \in \mathrm{Pic}^{g-1-\mu(\mathcal{E})}(C) \,|\, h^0(\mathcal{P} \otimes \mathcal{E}) \neq 0\} \subsetneq \mathrm{Pic}^{g-1-\mu(\mathcal{E})}(C).$$

If this is the case, then $\Theta_{\mathcal{E}}$ has a natural structure of (possibly non reduced) divisor in $\mathrm{Pic}^{g-1-\mu(\mathcal{E})}(C)$, whose cohomology class is $\mathrm{rank}\,\mathcal{E} \cdot \vartheta$ where ϑ is the class of the canonical theta-divisor in $\mathrm{Pic}^{g-1-\mu(\mathcal{E})}(C)$.

A vector bundle admitting a theta-divisor is semistable, and if the vector bundle admits a theta-divisor and is strictly semistable then the theta-divisor is not integral (cf. [3, 12]).

Proposition 9.3. *If C is general, the vector bundle $M_{\mathcal{L}}$ constructed above admits a theta-divisor.*

Proof. Recall that the genus of the curve is $g = 2k$, $\deg \mathcal{L} = 3k-3$, and $\mu(M_{\mathcal{L}}) = -3$. So $M_{\mathcal{L}}$ admits a theta-divisor if there exists a line bundle $\mathcal{P}$ of degree $\deg \mathcal{P} = g+2$ such that $h^0(\mathcal{P} \otimes M_{\mathcal{L}}) = 0$. Looking at the exact sequence

$$0 \to M_{\mathcal{L}} \otimes \mathcal{P} \to H^0(\mathcal{L}) \otimes \mathcal{P} \to \mathcal{L} \otimes \mathcal{P} \to 0$$

and passing to global sections, we have that $h^0(M_{\mathcal{L}} \otimes \mathcal{P}) = 0$ if and only if the multiplication map $H^0(\mathcal{P}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{P} \otimes \mathcal{L})$ is injective.

Let us write $(\omega_C) - C_{g-3} + C$ for the subset of $\mathrm{Pic}^{g+2}(C)$ consisting of line bundles of the form $\omega_C(-x_1 - x_2 - \cdots - x_{g-3} + y)$ for some points $x_1, \ldots, x_{g-3}, y \in C$. This is a two-codimensional subset of $\mathrm{Pic}^{g+2}(C)$, and its cohomology class is $W_{g-2} = \vartheta^2/2$.

The elements $\mathcal{P} \in (\omega_C) - C_{g-3} + C$ are exactly those satisfying one of the following properties:

(1) $h^0(\mathcal{P}) > 3$;
(2) $h^0(\mathcal{P}) = 3$ and $\mathcal{P}$ has base points.

It can be shown that all elements satisfying one of these properties lie in $\Theta_{M_{\mathcal{L}}}$. The remaining elements of $\mathrm{Pic}^{g+2}(C)$ are line bundles $\mathcal{P} \in \mathrm{Pic}^{g+2}(C)$ which are base point free and satisfy $h^0(\mathcal{P}) = 3$.

Let us show that there exists a line bundle $\mathcal{P} \notin (\omega_C) - C_{g-3} + C$ such that the map $H^0(\mathcal{P}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{P} \otimes \mathcal{L})$ is injective.

Let us start by considering the multiplication map

$$\mu : H^0(D) \otimes H^0(\omega_C(-D)) \to H^0(\omega_C),$$

and suppose that it is injective (then in fact it is an isomorphism): this assumption is true for a general curve, for instance it is true if we suppose that C is a Petri curve.

Let G be a general effective divisor of degree $k+1$. Then observe that, as G imposes general conditions on $H^0(\omega_C(-D))$,

$$h^1(D+G) = h^0(\omega_C(-D-G)) = 0,$$

and hence $h^0(D+G) = 3$. Moreover, the divisor $D+G$ is free from base points. Indeed, as G is general of degree $k+1$ we also have that $h^0(\omega_C(-D-G+p)) = 0$ for any $p \in C$, and so $h^0(D+G-p) = 2$ as wanted.

Hence $\mathcal{O}_C(D+G) \in \mathrm{Pic}^{g+2}(C)$ belongs to the complement of $(\omega_C) - C_{g-3} + C$.

Let us now prove that the map $\nu : H^0(D+G) \otimes H^0(\omega_C(-D)) \to H^0(\omega_C(G))$ is injective. Let $\sigma_1, \sigma_2, \sigma_3$ be a basis for $H^0(D+G)$ such that σ_1 and σ_2 generate $H^0(D) \subset H^0(D+G)$. Then of course the restriction of ν to $\langle \sigma_1, \sigma_2 \rangle \otimes H^0(\omega_C(-D))$ is the map μ, and so it is injective by our assumption.

Let $t = \ell_1 \otimes \sigma_1 + \ell_2 \otimes \sigma_2 + \ell_3 \otimes \sigma_3$ be an element of $\ker \nu$, where the ℓ_is belong to $H^0(\omega_C(-D))$. Note that G is the base locus of σ_1 and σ_2, and clearly $\ell_1\sigma_1 + \ell_2\sigma_2 \in H^0(\omega_C) \subset H^0(\omega_C(G))$. As $-\ell_3\sigma_3 = \ell_1\sigma_1 + \ell_2\sigma_2$ in $H^0(\omega_C(G))$, and σ_3 does not vanish on any of the points of G, we have that ℓ_3 has to vanish on G, but, as observed above, $H^0(\omega_C(-D-G)) = \{0\}$. So $t = \ell_1 \otimes \sigma_1 + \ell_2 \otimes \sigma_2$ but then t is in $\ker \mu = \{0\}$, as wanted. □

Remark 9.4. Using the same notations as in the proof of the theorem above, let us make some remarks.

All line bundles $\mathcal{P} \in \mathrm{Pic}^{g+2}(C) \backslash ((\omega_C) - C_{g-3} + C)$ induce a semistable dual span $M_{\mathcal{P}}$ of rank 2: in fact $M_{\mathcal{P}}$ is clearly a rank 2 bundle, and any possible destabilization $Q \subset M_{\mathcal{P}}$ would be a line bundle of negative degree $-q > \mu(M_{\mathcal{P}}) = -(g+2)/2 = -(k+1) = -\gamma(C)$, and dualizing we would have a globally generated line bundle Q^* of degree $q < \gamma(C)$, which is impossible. By a similar argument it can be shown that $M_{\mathcal{P}}$ is actually stable for a general $\mathcal{P} \in \mathrm{Pic}^{g+2}(C)$.

Furthermore for all such $\mathcal{P}$, the bundle $M_{\mathcal{P}}$ admits a theta-divisor, because all rank 2 stable bundles do (very ample, cf. [7]).

Then we have a map

$$\begin{aligned} \mathrm{Pic}^{g+2}(C) \backslash ((\omega_C) - C_{g-3} + C) &\to \mathrm{Hilb}(\mathrm{Pic}^{3k}(C), 2\vartheta) \\ \mathcal{P} &\mapsto \Theta_{M_{\mathcal{P}}} \subset \mathrm{Pic}^{3k}(C). \end{aligned}$$

Remark 9.5. As it was shown in some cases that there exist DSBs M_Q such that some exterior power $\bigwedge^t M_Q$ has integral slope and does not admit a theta-divisor (cf. [21]), it seems natural to ask the following question.

Question 9.6. Do exterior powers $\bigwedge^t M_{\mathcal{L}}$ admit a theta-divisor? If this is the case, are all theta-divisors integral?

The fact that $M_{\mathcal{L}}$ is cohomologically stable (Proposition 9.1) provides an evidence towards a positive answer to this question.

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