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# Holomorphic symmetric differentials and parallelizable compact complex manifolds

**Abstract.** We provide a characterization of complex tori using holomorphic symmetric differentials. With the same method we show that compact complex manifolds of Kodaira dimension 0 having some symmetric power of the cotangent bundle globally generated are quotients of parallelizable manifolds, therefore have an infinite fundamental group.

**Keywords.** Positivity of vector bundles, complex and Kaehler manifolds, complex tori, parallelizable manifolds.

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## 1 - Introduction

The relations between differentials and topology of an algebraic variety were known since the time of Kähler and Severi (cf. [**Kä32**], [**Sev42**], [**Sev50**]).

In recent years a lot of progress has been made towards understanding the relationship between the fundamental group $\pi_1(X)$ of a compact Kähler manifold $X$ and holomporphic symmetric differentials $H^0(X, S^k\Omega^1_X)$, with $\Omega^1_X$ the holomorphic cotangent bundle of $X$. In particular it is asked by Hélène Esnault whether a compact Kähler manifold $X$ with infinite fundamental group always carries a non vanishing $H^0(X, S^k\Omega^1_X)$ for some $k > 0$, and this has an affirmative answer, at least in the case where the fundamental group has a finite dimensional representaion with infinite image (cf. [**BKT13**]).

On the other hand, one could wonder whether the converse is true, *i.e.* whether a variety with some (or many) holomporphic symmetric differentials always have an infinite fundamental group. Because of Hodge decomposition it

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is immediate to observe that if a compact Käler manifold has a non vanishing holomorphic 1-form, then it has an infinite fundamental group. Also, the presence of particular rank-1 holomorphic symmetric differentials on a projective variety implies that the fundamental group of a projective variety is infinite (cf. [**BDO11**]), and on a complact complex manifold the presence of a nowhere degenerate holomorphic section of $S^2\Omega^1_X$ as well implies that the fundamental group is infinite (cf. [**BiDu18**]).

However, this is not the case in general for higher order symmetric differentials: there are varieties $X \subset \mathbb{P}^N$, which are general complete intersections of high degree in $\mathbb{P}^N$ and dimension $n \leqslant N/2$, that have ample cotangent bundle $\Omega^1_X$ (cf. [**BrDa18**]) and are simply connected. These varieties in particular have some symmetric powers of the cotangent bundle with as many holomorphic sections as possible. These varieties have *semiample* (both weakly and strongly, according to the definitions below) cotangent bundle and maximal Kodaira dimension $k(X) = n$, *i.e.* they are of general type. We will show that this cannot hold in case of a smooth projective variety of smaller Kodaira dimension.

If $X$ is a projective variety of Kodaira dimension $k(X) = 0$, in earlier works inspired by the definition of base loci (cf. [**BKK$^+$15**]) and Iitaka fibrations for vector bundles, we showed that having a globally generated symmetric differential bundle $S^k\Omega^1_X$ for some $k$ is equivalent to being isomorphic to an abelian variety, and having a generically generated symmetric differential bundle $S^k\Omega^1_X$ for some $k$ is equivalent to being birational to an abelian variety (cf. [**MU17**] and [**Mis18**]). In particular in those two cases the fundamental group $\pi_1(X)$ is infinite.

The purpose of this work is to show the following generalisation of these results to the cases of a compact complex manifold, a compact Kähler manifold, and a smooth projective variety:

Theorem 1.1. *Let $X$ be a compact complex manifold of dimension $n$ and Kodaira dimension $k(X)$.*

*i. If $k(X) = 0$ and $\Omega^1_X$ is strongly semiample, then the fundamental group of $X$ is infinite.*

*ii. If $k(X) = 0$ and $X$ is Kähler, then $\Omega^1_X$ is strongly semiample if and only if $X$ is biholomorphic to a complex torus.*

*iii. If $k(X) = 0$ and $X$ is projective, then $\Omega^1_X$ is weakly semiample if and only if $X$ is an étale quotient an abelian variety by the action of a finite group.*

*iv. If $k(X) < n$, $X$ is projective, and $\Omega^1_X$ is weakly semiample, then the fundamental group of $X$ is infinite.*

## 2 - Notations and basic lemmas

Let $X$ be a compact complex manifold, let $E$ be a holomorphic vector bundle on $X$. Let $\pi\colon \mathbb{P}(E) \to X$ be the projective bundle of 1-dimensional quotients of $E$. It comes with a *tautological* quotient $\pi^*E \twoheadrightarrow \mathcal{O}_{\mathbb{P}(E)(1)}$, where $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a line bundle on $\mathbb{P}(E)$. Throughout this work we use Grothendieck's (quotient) notation for projective spaces: any map $Y \to \mathbb{P}(E)$ is induced by a quotient $f^*E \to L$ with $L$ a line bundle on $Y$, and given a vector space $V$, the projective space $\mathbb{P}(V)$ will be the space of rank one quotients of $V$, or hyperplanes in $V$. This is the reason for the signs appearing in Chern classes formula (1) below.

Definition 2.1. Let $E$ be a holomorphic vector bundle on a compact complex manifold $X$.

*i.* We say that the vector bundle $E$ is *strongly semiample* if the symmetric product $S^kE$ is a globally generated vector bundle on $X$, for some $k > 0$.

*ii.* We say that the vector bundle $E$ is *semiample* or *weakly semiample* if $\mathcal{O}_{\mathbb{P}(E)}(1)^{\otimes k} = \mathcal{O}_{\mathbb{P}(E)}(k)$ is a globally generated line bundle on $\mathbb{P}(E)$, for some $k > 0$.

*iii.* We say that the vector bundle $E$ is *Asymptotically Generically Generated* or *AGG* if there exixsts an open dense subset $U \subseteq X$ such that the map $ev_x\colon H^0(X, S^kE) \to S^kE(x)$ is surjective for some $k > 0$ and for all $x \in U$.

Remark 2.2. If $L$ is a line bundle on a compact complex variety $X$, then the Iitaka-Kodaira dimension $k(X, L)$ of $L$ is the growth rate of the dimension of holomorphic sections $H^0(X, L^{\otimes k})$. In particular $k(X, L) = 0$ if and only if $h^0(X, L^{\otimes k}) \leqslant 1$ for all $k > 0$ and it is equal to 1 for some $k > 0$.

The main lemmas we will use do follow Fujiwaras constructions in [**Fuj92**].

Lemma 2.3. *Let $E$ be a holomorphic vector bundle over a compact complex manifold $X$. Suppose that $E$ admits a morphism $h\colon E \to L$ to a line bundle $L$ such that the induced map $S^mh\colon S^mE \to L^{\otimes m}$ is surjective and splitting for some $m > 0$. Then $h\colon E \to L$ is surjective and splitting as well.*

Proof. First, remark that as $S^mh$ is surjective then $h$ must be surjective. Let us prove that $h$ splits by recursive induction on $m$. Suppose $m \geqslant 2$. Decompose $S^mh$ as $\alpha \circ \beta\colon S^mE \to S^{m-1}E \otimes L \to L^{\otimes m}$, where

$$\alpha = (S^{m-1}h) \otimes 1_L\colon S^{m-1}E \otimes L \to L^{\otimes m} \ ,$$

and $\beta(v_1 \cdot ... \cdot v_m) = \frac{1}{m} \sum (v_1 \cdot ... \cdot \check{v}_i \cdot ... \cdot v_m) \otimes h(v_i) \in S^{m-1}E \otimes L$.

As $S^m h$ splits, then $\alpha$ splits, and then $S^{m-1}h = \alpha \otimes 1_{L^{-1}}$ splits and we can apply recursive induction. □

## 3 - Parallelizable manifolds

A complex parallelizable manifold is a complex manifold with trivial cotangent bundle. It is known that a compact complex manifold is parallelizable if and only if it is a quotient of a complex Lie group by a discrete subgroup, in particular a compact Kähler manifold is parallelizable if and only if it is a torus (cf. [**Wan54**]).

Lemma 3.1. *Let $X$ be a compact complex manifold admitting a finite étale cover which is parallelizable, then $\pi_1(X)$ is infinite.*

Proof. First remark that by Galois closure any finite étale cover $X' \to X$ admits a cover $X'' \to X'$ such that $X'' \to X$ is a finite étale Galois cover, furthermore if $X'$ is parallelizable then $X''$ is parallelizable as well. So we can suppose that the cover $X' \to X$ is finite étale Galois, and $X'$ parallelizable. Therefore $\pi_1(X') \subseteq \pi_1(X)$ and we need to show that a compact parallelizable manifold has infinite fundamental group.

Now suppose that $X' = G/\Gamma$ with $G$ a complex Lie group and $\Gamma$ a discrete subgroup. If $\Gamma$ is finite, then $G$ is compact as well, therefore $G$ is a complex torus and has an infinite fundamental group, and $G \to X'$ is a finite étale covering, so $\pi_1(X') \supseteq \pi_1(G)$ is infinite. On the other hand if $\Gamma$ is infinite, then the covering $G \to X'$ yields a group extension $1 \to \pi_1(G) \to \pi_1(X') \to \Gamma \to 1$ therefore $\pi_1(X')$ is infinite. □

Theorem 3.2. *Let $X$ be a compact complex manifold, let $E$ be a holomorphic vector bundle on $X$. Suppose that $E$ is strongly semiample, and that its determinant has Iitaka-Kodaira dimension $k(X, \det E) = 0$. Then there exists a finite Galois cover $f\colon X' \to X$ such that $f^*E$ is trivial.*

Proof. First remark that $\det E$ is a torsion line bundle. In fact we recall that a line bundle that has Iitaka-Kodaira dimension 0 is trivial if globally generated, as it cannot have more than 1-dimensional space of global sections.

As some symmetric power $S^m E$ is globally generated, for any point $x \in X$ we find sections $\sigma_1, \dots, \sigma_N \in H^0(X, S^m E)$ linearly independent and providing a basis for the fiber $S^m E(x)$, with $N = \mathrm{rk} S^m E$, therefore we obtain a section

$\sigma_1 \wedge ... \wedge \sigma_N \in H^0(X, (\det E)^{\otimes M})$ of the line bundle $\det(S^m E) = (\det E)^{\otimes M}$ which does not vanish on $x \in X$. So $(\det E)^{\otimes M}$ is globally generated and of Kodaira dimension 0, and therefore it is trivial. This shows that $\det E$ is torsion.

Then also the symmetric power $S^m E$ is a trivial vector bundle, in fact if the sections $\sigma_1, \ldots, \sigma_N \in H^0(X, S^m E)$ chosen above give a basis of $S^m E(x)$ on a point $x \in X$, as the determinant is trivial, then they provide a basis at all points $y \in X$, so the induced map $\mathcal{O}_X^{\oplus N} \to S^m E$ is an isomorphism.

Now consider $\pi \colon \mathbb{P}(E) \to X$, the projective bundle of 1-dimensional quotients of $E$, and its tautological quotient $\pi^* E \twoheadrightarrow \mathcal{O}_{\mathbb{P}(E)(1)}$. As $S^m E$ is globally generated, then so is $\pi^* S^m E$ and $\mathcal{O}_{\mathbb{P}(E)}(m)$. Therefore $\mathcal{O}_{\mathbb{P}(E)}(m)$ induces a map $\Phi \colon \mathbb{P}(E) \to \mathbb{P}^{N-1}$. Let us show that this map induces many sections of $\pi$.

Let $x \in X$ be a point, and consider the diagram:

$$\begin{array}{ccccc} \mathbb{P}(E(x)) & \hookrightarrow & \mathbb{P}(E) & \xrightarrow{\Phi} & \mathbb{P}^{N-1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \\ \{x\} & \hookrightarrow & X & & \end{array}$$

Now $\Phi_{|\mathbb{P}(E(x))} \colon \mathbb{P}(E(x)) \to \mathbb{P}^{N-1}$ is induced by the linear system on $\mathbb{P}(E(x))$ given by the image of the restriction map

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \to H^0(\mathbb{P}(E(x)), \mathcal{O}_{\mathbb{P}(E(x))}(m)) \ .$$

But as $S^m E$ is trivial this map is an isomorphism: in fact we have natural isomorphisms

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \cong H^0(X, S^m E) \cong S^m E(x) \ , \text{ and}$$

$$H^0(\mathbb{P}(E(x)), \mathcal{O}_{\mathbb{P}(E(x))}(m)) \cong S^m H^0(\mathbb{P}(E(x)), \mathcal{O}(1)) \cong S^m E(x) \ .$$

Therefore the map $\Phi$, when restricted to the projective space $\mathbb{P}(E(x))$, is a Veronese embedding, in particular it is injective, and letting $x \in X$ vary the map $\Phi_{|\mathbb{P}(E(x))} \colon \mathbb{P}(E(x)) \to \mathbb{P}^N$ has a fixed image. Therefore for every piont $w \in \mathbb{P}^N$, a fiber $\Phi^{-1}(w)$ meets a fiber $\pi^{-1}(x) = \mathbb{P}(E(x))$ exactly in one point, and actually $\pi$ is trivial as projective bundle. Therefore $W = \Phi^{-1}(w)$ provides a section of $\pi \colon \mathbb{P}(E) \to X$, with $\pi$ inducing an isomorphism $W \cong X$.

Such a section yields a quotient $E \to L$, where $L = \mathcal{O}_{\mathbb{P}(E)}(1)_{|W}$, and so $L^{\otimes m} = \mathcal{O}_{\mathbb{P}(E)}(m)_{|W} \cong \mathcal{O}_W$ is trivial on $W$, as $W$ is a fiber of $\Phi$.

By Lemma 2.3 the vector bundle $E$ splits as $F \oplus L$, and considering the cyclic étale covering $h \colon W' \to W$ induced by $L^{\otimes m} \cong \mathcal{O}_W$, then we obtain on $W'$ a splitting $h^* E = h^* F \oplus h^* L = h^* F \oplus \mathcal{O}_{W'}$. Repeating recursively the argument for the vector bundle $h^* F$ on $W'$, we obtain a finite covering where

$E$ becomes trivial. Then by Galois closure we obtain a finite Galois covering where $E$ becomes trivial. □

## 4 - Fundamental groups

From Theorem 3.2 we obtain the first part of Theorem 1.1:

Corollary 4.1. *Let $X$ be a compact complex manifold of Kodaira dimension $k(X) = 0$ such that $\Omega^1_X$ is a strongly semiample vector bundle. Then $X$ admits a finite étale Galois covering which is parallelizable. In particular $\pi_1(X)$ is infinite.*

Proof. This follows directly applying Theorem 3.2 to the cotangent bundle $\Omega^1_X$, and Lemma 3.1. □

In the compact Kähler case we can prove the second part of Theorem 1.1. The proof is actually very similar to the projective case, which is treated in [**MU17**] and does generalise easily in this case:

Theorem 4.2. *Let $X$ be a compact Kähler manifold of Kodaira dimension $k(X) = 0$ such that $\Omega^1_X$ is a strongly semiample vector bundle. Then $X$ is biholomorphic to a complex torus.*

Proof. Let us apply Theorem 3.2 to the cotangent bundle $\Omega_X$ and obtain $\gamma\colon X' \to X$ which is an étale Galois cover. Now $X'$ is a compact Kähler parallelizable manifold, so it is a torus $T$, and carries a finite group action such that $\gamma\colon T \to T/G = X$. Now as the covering is étale $\gamma^*\Omega^1_X = \Omega^1_T$ and $\gamma^* S^m\Omega^1_X = S^m\Omega^1_T$. Therefore

$$\gamma^* H^0(X, S^m\Omega^1_X) = H^0(T, S^m\Omega^1_T)^G \subseteq H^0(T, S^m\Omega^1_T) = S^m H^0(T, \Omega^1_T) \ .$$

For some $m > 0$, the vector bundle $S^m\Omega^1_X$ is globally generated, so

$$\dim H^0(X, S^m\Omega^1_X) \geqslant \mathrm{rk} S^m\Omega^1_X = \mathrm{rk} S^m\Omega^1_T = \dim S^m H^0(T, \Omega^1_T) \ .$$

This implies that $G$ acts trivially on $S^m H^0(T, \Omega^1_T)$, and it can be shown that the action of $G$ on $H^0(T, \Omega^1_T)$ must be then through homotheties. Actually the action of $G$ must be trivial on $H^0(T, \Omega^1_T)$ otherwise the action of $G$ on $T$ could not be free (cf. [**Mis18**], proof of Theorem 4.1). Therefore, as $G$ acts trivially on $H^0(T, \Omega^1_T)$, it acts on $T$ by translations, so the quotient is a torus. □

The third and fourth points in Theorem 1.1 are consequence respectively of the work of Fujiwara [**Fuj92**] and a recent theorem by Andreas Höring [**Hö13**]:

Theorem 4.3 (Höring). *Let $X$ be a projective manifold with strongly semi-ample cotangent bundle, i.e. for some positive integer $m \in \mathbb{N}$ the symmetric product $S^m\Omega^1_X$ is globally generated. Then there exists a finite cover $X' \to X$ such that $X' \cong Y \times A$, where $Y$ has ample canonical bundle and $A$ is an abelian variety.*

Now, Theorem 4.3 is stated only for projective varieties $X$ with strongly semiample cotangent bundle $\Omega^1_X$, however the result still holds for varieties with weakly semiample cotangent bundle, provided that one shows that the canonical bundle $\omega_X = \det \Omega^1_X$ is semiample in this case. This holds in general for weakly semiample vector bundle on projective varieties, and is the object of the following theorem, which is proved in [**Fuj92**]:

Theorem 4.4 (Fujiwara). *Let $X$ be a projective variety, and let $E$ be a weakly semiample vector bundle on $X$. Then the determinant $\det E$ is a semiample line bundle.*

The proof is contained in the work of Fujiwara [**Fuj92**], however, we give a detailed proof here, as it is related to Theorem 3.2 :

Proof. Let $\pi\colon \mathbb{P}(E) \to X$ be the projectivisation of the vector bundle $E$, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the tautological bundle. Fix $x \in X$, we will prove that some power of $\det E$ has a section which does not vanish on $x$. We have the fiber $\pi^{-1}(x) = \mathbb{P}(E(x))$, and the restriction $\mathcal{O}_{\mathbb{P}(E)}(1)_{|\pi^{-1}(x)}$ is the usual very ample line bundle $\mathcal{O}_{\mathbb{P}(E(x))}(1)$. Let us recall that $CH^1(X) \cong \mathrm{Pic}(X)$, and the same for $\mathbb{P}(E)$, cf. Remark 4.5 below. Let $\xi \in \mathrm{Pic}(\mathbb{P}(E)) = CH^1(\mathbb{P}(E))$ be the class of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, and $r = \mathrm{rk}E$. The Chern classes in the Chow ring of $X$ are determined by the relation:

$$(1) \qquad \sum_{i=0}^{r}(-1)^i\pi^*c_i(E)\xi^{r-i} = 0 \ , \text{ with } c_0(E) = 1 \in CH^0(\mathbb{P}(E)) \ .$$

For dimensional reasons we have $\pi_*\xi^k = 0$ if $k \leqslant r-2$, and $\pi_*\xi^{r-1} = 1 \in CH^n(X)$ as $\xi$ restricts to $\mathcal{O}(1)$ on the fibers of $\pi$. Therefore the equation (1) above gives (using projection formula):

$$c_0(E).\pi_*\xi^r - c_1(E).\pi_*\xi^{r-1} = 0 \in CH^1(X) \ ,$$

so $\pi_*\xi^r = c_1(E) \in CH^1(X)$.

Now suppose that $\mathcal{O}_{\mathbb{P}(E)}(m)$ is globally generated, and consider $\Phi\colon \mathbb{P}(E) \to \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{O}(m))) = \mathbb{P}^N$. Notice that $\Phi_{|\mathbb{P}(E(x))}\colon \mathbb{P}(E(x)) \to \mathbb{P}^N$ is induced by the linear system

$$\mathrm{Im}(H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \to H^0(\mathbb{P}(E(x)), \mathcal{O}_{\mathbb{P}(E(x))}(m))) \ ,$$

this is a base point free subsystem of the very ample linear system given by $H^0(\mathbb{P}(E(x)), \mathcal{O}_{\mathbb{P}(E(x))}(m))$ on $\mathbb{P}(E(x))$, therefore $\Phi_{|\mathbb{P}(E(x))}\colon \mathbb{P}(E(x)) \to \mathbb{P}^N$ is a finite map obtained as the composition of a projection after the embedding defined by the very ample linear system (cf. Example 1.1.12 page 14 in [**Laz04**]). So $N \geqslant r-1$ and we can choose $r$ generic sections $\sigma_1, ..., \sigma_r \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m))$, such that $V(\sigma_1)\cap\cdots\cap V(\sigma_r)\cap\mathbb{P}(E(x)) = \emptyset$. Now the intersection $V(\sigma_1)\cap\cdots\cap V(\sigma_r)$ is an effective cycle whose class is $m^r\xi^r \in CH^r(\mathbb{P}(E))$. Therefore the image of $V(\sigma_1)\cap\cdots\cap V(\sigma_r)$ in $X$ is the divisor of a holomorphic section of $(\det E)^{\otimes m^r}$. So the line bundle $(\det E)^{\otimes m^r} = \pi_*(m^r\xi^r)$ has a section which is non zero out of $\pi(V(\sigma_1)\cap\cdots\cap V(\sigma_r))$, in particular it does not vanish on $x \in X$. $\square$

Remark 4.5. It is important to notice that we are working with Chow groups, and Chern classes taking values in the Chow groups. In singular cohomology, we could get a divisor, or a current, which is Poincaré dual to the cohomology class $m^r.c_1(detE)$, but from this we would not get a holomorphic section of $(\det E)^{\otimes m^r}$. This is the reason why we cannot extend directly this result to the compact Kähler case, which would be interesting to investigate in a future work.

In particular the third point of Theorem 1.1 follows directly from Fujiwara's characterization of quotients of abelian varieties appearing in [**Fuj92**], and the fourth point follows applying Höring's theorem above: suppose $X$ is a smooth projective variety with $k(X) < n$ and $\Omega^1_X$ weakly semiample. Then it admits an étale finite cover $X' \to X$ with $X' \cong A \times Y$, the variety $Y$ having ample canonical bundle and $A$ an abelian variety of dimension $n - k(X)$, therefore $\pi(X) \supseteq \pi(A) \times \pi(Y)$ and it is infinite.

## 5 - Examples, questions, and remarks

Example 5.1. Given a surjective morphism $f\colon Y' \to Y$ and a vector bundle $E$ on $Y$, then $E$ is weakly semiample if and only if $f^*E$ is weakly semiample (cf. [**Fuj83**]). However the same cannot be said for strongly semiample bundles, even in the case that $f$ is finite étale and Galois: for example consider a non-trivial 2-torsion line bundle $L$ on a curve $C$, therefore $L$ determines an étale $\mathbb{Z}/2\mathbb{Z}$-cover $f\colon C' \to C$ such that $f^*L = \mathcal{O}_{C'}$. Therefore the vector bundle $E = \mathcal{O}_C \oplus L$ is weakly semiample on $C$, as its pull back $f^*E$ is trivial and so it is weakly (and strongly as well) semiample.

However $E$ is not strongly semiample, as any symmetric power $S^mE$ contains a copy of $L$ as direct factor, and cannot be globally generated. As stated above, the vector bundle $f^*E$ being trivial, it is strongly semiample.

Example 5.2. According to Theorem 4.2 the only compact Kähler manifolds $X$ with strongly semiample cotangent bundle and Kodaira dimension $k(X) = 0$ are compact tori.

Therefore any étale finite quotient of a torus has Kodaira dimension 0 and weakly semiample cotangent bundle, if this quotient is not again a torus, it gives an example of a variety with weakly semiample cotangent bundle but not strongly semiample.

Such an example is any bielliptic surface, which is covered by an abelian surface.

Remark 5.3. In the proof of Theorem 3.2 we see that a vector bundle $E$ on a compact complex variety $X$ such that $S^mE$ is trivial splits as a direct sum $F \oplus L$ with $L$ an $m$-torsion line bundle. As $E = (F \otimes L^{-1} \oplus \mathcal{O}) \otimes L$ then $S^mE = S^m(F \otimes L^{-1} \oplus \mathcal{O})$ has a direct factor $F \otimes L^{-1}$ which is trivial as well. Therefore $E = L^{\oplus \mathrm{rk} E}$ is a direct sum of the same torsion line bundle.

Question 5.4. Is there a compact complex manifold $X$ of Kodaira dimension $k(X) = 0$ with cotangent bundle $\Omega^1_X$ which is not trivial but strongly semiample? According to Corollary 4.1 and Remark 5.3 it must be a cyclic quotient of a parallelizable compact manifold, and cannot be Kähler. Futhermore its tangent bundle shoud decompose as a direct sum of isomorphic torsion line bundles.

Question 5.5. Let $X$ be a compact complex manifold, and let $E$ be a *weakly* semiample vector bundle on $X$. Is $\det(E)$ a semiample line bundle? And if $X$ is compact Kähler?

The techniques used to prove Theorem 4.4 cannot be (directly) used for the compact Kähler case, nevertheless, in order to apply Höring's result to a compact Kähler manifold in Kodaira dimension 0 we would just need that the cotangent bundle be numerically trivial. We leave these questions to futher investigations.

Question 5.6. Let $X$ be a compact complex manifold with a fixed hermitian metric, and let $E$ be a *weakly* semiample vector bundle on $X$. If the Iitaka-Kodaira dimension of the determinant is $k(X, \det(E)) = 0$, is $E$ a *numerically trivial* vector bundle? Numerically trivial means that both $E$ and its dual $E^*$ are nef. And what if $X$ is compact Kähler?

Remark 5.7. A positive answer to Question 5.6 above, would allow to apply Höring's theorem to the case of complex tori, in order to generalize Fujiwara results and Theorem 1.1 as follows: let $X$ be a compact Kähler manifold with Kodaira dimension $k(X) = 0$, then the cotangent bundle is weakly semiample if and only if $X$ is an étale quotient of a complex torus by the action of a finite group.

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