Families of curves and variation in moduli

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Abstract

In this paper we study the class of smooth complex projective varieties B such that any *modular* morphism $B \to \mathcal{M}_g$ is constant for any $g \geqslant 2$, giving structural properties and examples. Then we investigate the concept of the *moduli dimension* of a variety B; we bound it by the dimension of the maximal rationally connected quotient of B. In the end, we consider also (generically smooth) families of curves *of compact type* over rational and elliptic curves.

Introduction

In the theory of algebraic curves a powerful tool is the study of families, which is particularly well framed in the theory of moduli spaces. In this paper we deal with the variation of families of complex projective curves over a given base B, *i.e.*, "how many" curves there can be in a family over it. In particular we look for varieties over which there are only isotrivial (smooth) families, that is, families where all curves are isomorphic, or, at the other extreme, there are families with "maximal variation in moduli".

Our starting point is the well known isotriviality of all families of smooth curves over $B = \mathbb{P}^1, \mathbb{C}, \mathbb{C}^*$, and elliptic curves, recalled in the first section. After fixing the notation and recalling the basic facts, we look at two possible directions to generalize it: first allowing higher dimensional bases, then by allowing some (mildly) singular curves in the family.

The second section is dedicated to show the "invariance" of families of smooth curves under birational transformations of the (smooth complex projective) base using an extension criterion of Abramovich and Vistoli ([AV02]) and Hironaka's resolution of indeterminacy. This allow us to claim the "birational invariance" of some of our results.

In the third section we investigate the class $\mathcal{B}_0$ of all (smooth complex projective) varieties over which there are only isotrivial families. After extablishing some properties we give a list of examples of such varieties, and among them some birational classes of surfaces. To complete the proof of proposition 3.3 we provide a simplified proof of a (slightly weaker form of a) recent result by Stix ([Sti05]).

We look in the fourth part at how curves in a family over a given base can vary: using the "moduli variation" of families we give some properties of the "moduli dimension" of a variety and relate it to the dimension of its maximal rationally connected quotient.

In the last section we show that in case B is $\mathbb{P}^1$ or an elliptic curve also families of compact type curves, with smooth generic member, with no rational or elliptic components are isotrivial.

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1 Notation and basic results

By a *variety* we generally mean a smooth irreducible complex projective variety, unless explicitly specified. A *family of curves* over a base B is a flat morphism $f\colon \mathcal{C} \to B$ of varieties with proper 1-dimensional connected fibers; any such family, with *smooth fibers*, induces a morphism, called *modular map*, $\varphi_f\colon B \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the coarse moduli space of genus g smooth curves.

We recall that $\mathcal{M}_g$ is a *coarse* moduli space because morphisms $\psi\colon B \to \mathcal{M}_g$ do not correspond bijectively to families of curves over B, for all B, via a "universal family" (*i.e.*, in a functorial way). Not all morphisms $\psi\colon B \to \mathcal{M}_g$ are modular, and there are as well non isomorphic families of curves over B inducing the same modular map $B \to \mathcal{M}_g$.

One way to overcome this unpleasant feature of $\mathcal{M}_g$ is working with $[\mathcal{M}_g]$, the (Deligne-Mumford) *moduli stack* of genus g smooth curves (see [DM69]) and its fundamental group (as a stack) $\widetilde{\pi_1}(\mathcal{M}_g)$. The moduli stack $[\mathcal{M}_g]$ is a fine moduli space, hence morphisms $\psi\colon B \to [\mathcal{M}_g]$ are, functorially, in bijection with families $\mathcal{C} \to B$ of smooth genus g curves over B.

Another useful device to deal with coarseness of $\mathcal{M}_g$ is the concept of a *m-level structure*: for each integer $m \geq 3$ there is an étale covering $p\colon \mathcal{M}_g^{(m)} \to [\mathcal{M}_g]$, where the algebraic scheme $\mathcal{M}_g^{(m)}$ is the fine moduli space for curves of genus g with *m-level structure*, see [Pop77], Lecture 10, and [HM98], pp. 37−38.

Albeit our main concern is with $\mathcal{M}_g$, we will use $[\mathcal{M}_g]$ and $\mathcal{M}_g^{(m)}$ as tools to state and proof some results.

Definition 1.1 Let $\mathcal{C} \to B$ be a family of smooth curves over a scheme B. We say that $\mathcal{C}$ is a *trivial* family if there exist a smooth curve C such that $\mathcal{C} \cong B \times C$ as scheme over B. We say that $\mathcal{C}$ is an *isotrivial* family if for all $b \in B$ the fibers $\mathcal{C}_b$ are isomorphic.

Remark 1.2 To prove that a family of smooth curves is isotrivial it is enough to show that all fibers in an open subset of B are isomorphic, in fact a family $f\colon \mathcal{C} \to B$ is isotrivial if and only if its modular map $\varphi_f\colon B \to \mathcal{M}_g$ is constant.

1.1 Isotriviality over small genus curves

All families of smooth curves, of genus at least two, over $B = \mathbb{P}^1, \mathbb{C}, \mathbb{C}^*$, or an elliptic curve, are isotrivial (see [Par68] and [Ara71], or [Bea81], cf. also [Oor74], Corollary 2.3 and Remark 2.5). Here we recall the argument given by Beauville in [Bea81] (pag. 97), because it is an inspiration for us and we shall explicitely refer to it (see also [MB85], Lemme 5).

Theorem 1.3 *Let $f\colon \mathcal{C} \to B$ be a family of smooth curves of genus $g \geq 2$. If B is an elliptic curve or $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$, then the family f is isotrivial.*

Proof Let $\tilde{f}\colon \widetilde{\mathcal{C}} \to \widetilde{B}$ be the pullback of the family along the universal covering space of B. Since the local system $R^1\tilde{f}_*\mathbb{Z}$ is constant over $\widetilde{B}$, then the modular map composed with Torelli's embedding $\widetilde{B} \to \mathcal{M}_g \hookrightarrow \mathcal{A}_g$ in the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties lifts to the Siegel upper semispace $\mathcal{H}_g$, which is hyperbolic. By the hypothesis on B, its covering space $\widetilde{B}$ is either $\mathbb{P}^1$ or $\mathbb{C}$. Hence any morphism $\widetilde{B} \to \mathcal{H}_g$ is constant, and the map $B \to \mathcal{A}_g$ is constant as well. So the family is isotrivial by Torelli. $\square$

Remark 1.4 The key point is the hyperbolicity of the universal cover, so the same proof applies to show that on a projective space of any dimension, as well as on any Abelian variety, all families are isotrivial.

Remark 1.5 Theorem 1.3 states, in particular, that for $g \geqslant 2$ there are no finite *modular* maps $\mathbb{P}^1 \to \mathcal{M}_g$. But there exist g's and rational curves $\mathbb{P}^1 \to \mathcal{M}_g$ which are not modular, as was shown (in any characteristic) by Oort ([Oor74], Theorem 3.1).

Moreover, for all $g > 1$ there exist complete curves B and non isotrivial families of smooth genus g curves over B (examples by Kodaira, see [HM98], pag. 56).

2 Families of curves and birational maps

We study in this section the behaviour of families of curves above birational varieties. We need a basic result about extending families of curves defined on dense open subsets. Among the literature ([AV02],[dJO97],[Moc99],[MB85],[Sai04]), we have chosen the following lemma (see [AV02], section 2.4, and also [Moc99], Lemma 5.5).

Lemma 2.1 (Purity Lemma) *Let $[\mathcal{M}]$ be a separated Deligne-Mumford stack, $[\mathcal{M}] \to \mathcal{M}$ the coarse moduli space. Let X be a smooth variety, and let $f\colon X \to \mathcal{M}$ be a morphism. Let $Z \subset X$ be a closed subset of codimension at least 2, and*

$U = X \setminus Z$. Suppose there is a lifting $[f_U]\colon U \to [\mathcal{M}]$ of f over U. Then the lifting extends uniquely to a morphism $X \to [\mathcal{M}]$.

$$\begin{tikzcd} & & {[\mathcal{M}]} \arrow[d] \\ U \arrow[r] \arrow[urr, "{[f_U]}"] & X \arrow[r, "f"'] \arrow[ur, dotted] & \mathcal{M} \end{tikzcd}$$

Remark 2.2 Hence, if $[\mathcal{M}] = [\mathcal{M}_g]$ is the moduli stack of smooth genus g curves, any morphism $\phi\colon X \to \mathcal{M}_g$ which is modular on a "big" open set, *i.e.*, $U \subset X$ with $X \setminus U$ of codimension at least two, is modular itself; moreover the family of curves on X extending that on U is uniquely determined by it. In this restricted setting, this is part of a result of Moret-Bailly (see [MB85], Théorème).

We show that families over birational bases can be identified by first studiyng the special case of a blowing up with smooth center, and then using Hironaka's resolution of indeterminacy to handle arbitrary birational maps.

Lemma 2.3 *Any family $\mathcal{C}$ of smooth genus g curves over the blowing up $\hat{X}$ of a smooth projective variety X along a smooth center Z is a pull-back of a unique family on X.*

Proof The proof goes as follows: we show that the modular map $\varphi\colon \hat{X} \to \mathcal{M}_g$ factors through a morphism $\psi\colon X \to \mathcal{M}_g$, then that this morphism is a modular map, and it is induced by a family on X which pulls back to the family $\mathcal{C}$.

As the exceptional divisor E is a projective bundle over the smooth center Z of the blowing up, the family $\mathcal{C}$ restricted to a fiber of the blowing up, a projective space, is isotrivial by theorem 1.3. Hence, the modular map $\varphi\colon \hat{X} \to \mathcal{M}_g$ to the coarse moduli space is constant on any fiber of the blowing up. As $\pi_*\mathcal{O}_{\hat{X}} \cong \mathcal{O}_X$ and $\varphi\colon \hat{X} \to \mathcal{M}_g$ contracts every fiber of $\hat{X} \to X$, φ factors through a morphism $\psi\colon X \to \mathcal{M}_g$ ([Deb01], Lemma 1.15).

We have to show that this morphism is modular, *i.e.*, it comes from a family on X, or, equivalently, it lifts to a morphism on the moduli stack. The open set $U := X \setminus Z$ is isomorphic to $\hat{X} \setminus E$, where $\varphi\colon \hat{X} \to \mathcal{M}_g$ is induced by the family $\mathcal{C}$, so there is a lifting of the morphism $U \hookrightarrow X \to \mathcal{M}_g$ to a morphism $U \to [\mathcal{M}_g]$. The closed subset Z has codimension at least 2 in X, hence by the purity lemma 2.1 $\psi\colon X \to \mathcal{M}_g$ lifts to the moduli stack. The pull back to $\hat{X}$ of this family is the family $\mathcal{C}$, as they coincide on the open subset $\hat{X} \setminus E$. $\square$

Remark 2.4 This lemma is implied by a recent theorem of Stix, stating that a family of curves on an open subset U of a normal variety X, extends uniquely to X if the injection $U \hookrightarrow X$ induces an isomorphism $\pi_1(U) \tilde{\to} \pi_1(X)$ on the fundamental groups (see [Sti05]).

We can now see that the sets of families over birational projective varieties are in bijection.

Corollary 2.5 *Given a birational map of smooth projective varieties X and Y, the restrictions to the open isomorphic subsets of X and Y induce a bijection between the sets of families of smooth genus g curves over X and Y.*

Proof So far we have shown that the families on a blowing up (along a smooth center) are in bijection with the families on the base; the general case reduces to this one by means of Hironaka's resolution of indeterminacy ([Hir64], §0.5, Question E and Main Theorem II).

If $q\colon X \dashrightarrow Y$ is a birational map by Hironaka there exists a sequence of blowing up's of smooth projective varieties with smooth centers $\pi\colon \widetilde{X} = X_n \to \cdots \to X_0 = X$ such that $\pi \circ q\colon \widetilde{X} \to Y$ is a regular morphism (birational and surjective). Resolving also the indeterminacy of the birational inverse of q we get a morphism $\widetilde{Y} \to Y$ fitting in the following commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr] & \widetilde{Y} \arrow[l] \arrow[d] \\ X \arrow[r, dashed, "q"'] & Y \end{tikzcd}$$

Given a family on X, we can pull it back to $\widetilde{X}$ and then to $\widetilde{Y}$, and we know that this last family comes from a unique family on Y. Vice-versa, given a family on Y, we can pull it back to $\widetilde{X}$ and we know it comes from a unique family on X. By construction this is a bijection between families over X and Y, and two such families in bijection coincide above the isomorphic open sets of the birational map. $\square$

3 Varieties admitting no non-isotrivial families

Are there smooth projective varieties B such that all families of smooth genus $g > 1$ curves over B are isotrivial?

Definition 3.1 We say that a variety B has *moduli dimension* 0 if for all $g > 1$ all families of genus g smooth curves over B are isotrivial. The class of smooth projective varieties of moduli dimension 0 is called $\mathcal{B}_0$.

We first state some "set-theoretic" properties of $\mathcal{B}_0$ and then we give a list of varieties belonging to it.

Theorem 3.2 *The class $\mathcal{B}_0$ has the following properties:*

1. *if a variety B is dominated by a variety $B' \in \mathcal{B}_0$ then B itself is in $\mathcal{B}_0$;*
2. *the class $\mathcal{B}_0$ is closed under "fibrations with sections":* i.e., *it contains any variety B admitting a dominant morphism $p\colon B \twoheadrightarrow Y$ having a section $\sigma\colon Y \to B$, and such that the base Y and generic fiber of p are in $\mathcal{B}_0$. In particular, $\mathcal{B}_0$ is closed under products;*

3. *The class $\mathcal{B}_0$ is closed under birational maps (of smooth projective varieties).*

Proof

1. Just pull back any family to B'.

2. Let $f\colon \mathcal{C} \to B$ be a family of curves over B, and let $\mathcal{C}_b := f^{-1}(b)$ be the fiber on a point $b \in B$. For a generic $y \in Y$ the family $\mathcal{C}_{|p^{-1}(y)}$ is isotrivial, then the fibers $\mathcal{C}_b$ and $\mathcal{C}_{\sigma p(b)}$ are isomorphic for a generic $b \in B$. Furthermore the family $\mathcal{C}_{|\sigma(Y)}$ is isotrivial too, since $Y \cong \sigma(Y) \in \mathcal{B}_0$. So given generic $b_1, b_2 \in B$ we have $\mathcal{C}_{b_1} \cong \mathcal{C}_{\sigma p(b_1)} \cong \mathcal{C}_{\sigma p(b_2)} \cong \mathcal{C}_{b_2}$.

3. Consider a birational map $X \dashrightarrow Y$ of smooth projective varieties and assume $X \in \mathcal{B}_0$. By corollary 2.5 any family $\mathcal{C}$ of curves on Y is an extension of one coming from X: hence it is isotrivial, being an extension of an isotrivial family.

$\square$

We collect here some "examples" of varieties belonging to the class $\mathcal{B}_0$. Note that this list is redundant, for instance rationally connected varieties (point 3) and K3 surfaces (point 5) are simply connected (point 6), but we can show that they belong to $\mathcal{B}_0$ without using it.

Proposition 3.3 *The following varieties belong to $\mathcal{B}_0$:*

1. *curves of genus at most one;*

2. *Abelian varieties and projective spaces;*

3. *rationally connected varieties, in particular rational and unirational varieties;*

4. *ruled surfaces over a curve of genus at most one;*

5. *surfaces of Kodaira dimension zero;*

6. *all varieties B with finite $\pi_1(B)$ (first homotopy group).*

Proof

1. This is theorem 1.3.

2. Any Abelian variety or projective space has respectively $\mathbb{C}^n$ or $\mathbb{P}^n$ as universal cover, so the theorem follows by the very proof of 1.3.

3. Let B be a rationally connected variety, and $\mathcal{C} \to B$ a family of curves. Any two points of B lie on a rational curve, hence the fibers upon them are isomorphic by theorem 1.3.

4. A ruled surface is a smooth projective surface S birationally equivalent to $C \times \mathbb{P}^1$, with C a smooth projective curve ([Bea83], Definition III.1). Here we assume that C has genus at most one, so S is in $\mathcal{B}_0$ by proposition 2.5 and part 2 of theorem 3.2.

5. By the birational classification of smooth projective surfaces and part 3 of theorem 3.2 we can restrict to minimal surfaces.

   Any minimal surface of Kodaira dimension zero belongs to one of the following classes ([Bea83], Theorem VIII.2): Abelian surfaces, K3 surfaces, Enriques surfaces, bielliptic surfaces. By what shown before Abelian surfaces and bielliptic surfaces, being quotients of products of elliptic curves, are in $\mathcal{B}_0$. Enriques surfaces are dominated by K3 surfaces ([Bea83], Proposition VIII.17), therefore the claim follows showing that these last are in $\mathcal{B}_0$.

   Let B be a K3 surface. In the Appendix of [MM83] there is a proof of a theorem, due indipendently to Bogomolov and Mumford, stating that every such B contains a (in general singular) rational curve C_0 and a 1-dimensional family of (in general singular) elliptic curve E_t (see also [BHPVdV04], $VIII.23$). Given a family on B, by theorem 1.3 its moduli map $\varphi \colon B \to \mathcal{M}_g$ is constant on C_0 and on each E_t. As C_0 meets the elliptic curves, φ is a regular morphism constant on a Zariski-dense subset, hence it is constant on B, *i.e.*, the family is isotrivial.

6. Any projective variety B with finite fundamental group admits a projective universal cover (which dominates B). Hence we can restrict to the case $\pi_1(B) = 0$. The claim follows by a special case of a recent result of Stix [Sti05] that will be proven in theorem 3.6.

□

In order to complete the proof of the last point of proposition 3.3 we will work with the moduli stack of smooth complete curves $[\mathcal{M}_g]$ of genus $g > 1$. Roughly speaking we are allowed to pretend that the moduli space is smooth and admits a universal family: morphisms to $[\mathcal{M}_g]$ are exactly those induced by families of smooth connected curves of genus g.

We need some concepts from algebraic topology and a couple of lemmas.

Lemma 3.4 *Let $f \colon X \to Z$ be a morphism from an irreducible projective curve X to a quasi-projective variety Z. Then f is constant if and only if for each ample line bundle L on Z we have* $\deg(f^*(L)) = 0$.

Proof Being X an irreducible projective curve, f is non constant if and only if f is finite, and ample line bundles on X are exactly those of positive degree. The lemma follows from the facts that the pull-back of an ample line bundle by a finite morphism is ample and that there are ample line bundles on Z, as it is quasi projective. □

We recall that our varieties have the homotopy type of CW-complexes. In particular, by Riemann uniformization theorem, curves C of positive genus have contractible universal covering space. Hence they are *classifying spaces* for their first homotopy group ([Wei94], Example 6.10.8); being CW-complexes (up to homotopy) this is equivalent to say they are *Eilenberg-Mac Lane* spaces of type $K(\pi,1)$, that is $\pi_n(C)=0$ for all $n\neq 1$. The singular cohomology of a $K(\pi,1)$ CW-complex Z is naturally isomorphic to the group cohomology of $\pi_1(Z)$, that is, $H^n(Z,\mathbb{Z})\xrightarrow{\sim}H^n(\pi_1(Z),\mathbb{Z})$ (see [Wei94], 6.10, and [ML63], *IV*.11).

Lemma 3.5 *Let $f\colon X\to Z$ be a morphism from a curve X to a quasi-projective variety Z. If Z is a $K(\pi,1)$-space, and $\pi_1(f)\colon \pi_1(X)\to\pi_1(Z)$ is constant, then f itself is constant.*

Proof Up to passing to a (possibly ramified) covering of X, we can suppose that X is a curve of genus $g\geqslant 1$, hence X itself is a $K(\pi,1)$-space.

We will apply lemma 3.4. If L is a line bundle on Z then

$$\deg(f^*(L))=c_1(f^*(L))=f^*(c_1(L))\ ,$$

where $c_1\colon \mathrm{Pic}(X)\to H^2(X,\mathbb{Z})\cong\mathbb{Z}$ and $c_1\colon \mathrm{Pic}(Z)\to H^2(Z,\mathbb{Z})$ are the first Chern class maps on X and Z respectively.

Being X and Z spaces of type $K(\pi,1)$, the singular cohomology homomorphism $f^*\colon H^2(Z,\mathbb{Z})\to H^2(X,\mathbb{Z})$ is naturally isomorphic to the group cohomology homomorphism $(\pi_1(f))^*$:

$$\begin{array}{ccc} H^2(Z,\mathbb{Z}) & \xrightarrow{f^*} & H^2(X,\mathbb{Z}) \\ \wr\downarrow & & \downarrow\wr \\ H^2(\pi_1(Z),\mathbb{Z}) & \xrightarrow{\pi_1(f)^*} & H^2(\pi_1(X),\mathbb{Z}) \end{array}$$

the second line being trivial because $\pi_1(f)$ is trivial by assumption. Hence $f^*(c_1(L))=0$ for any line bundle on Z, and we can apply lemma 3.4.

□

What follows is a special case of a result of Stix (see [Sti05], Corollary 4.11).

Theorem 3.6 *Let X be a smooth projective variety and let $\varphi\colon X\to[\mathcal{M}_g]$ be a morphism such that the induced group homomorphism $\pi_1(\varphi)\colon\pi_1(X)\to\widetilde{\pi_1}(\mathcal{M}_g)$ is trivial. Then the morphism $\varphi\colon X\to[\mathcal{M}_g]$ is constant.*

Proof The proof will follow considering the case where X is a smooth projective curve of genus at least 1. In fact if X is a curve of smaller genus we can consider a (possibly ramified) covering of X of genus at least 1. If X is a higher dimensional variety, and $x,y\in X$ are two points, we can consider any smooth curve passing through x and y, and prove that φ is constant along this curve.

For each integer $m\geq 3$ there is an étale covering $p\colon\mathcal{M}_g^{(m)}\to[\mathcal{M}_g]$ where $\mathcal{M}_g^{(m)}$ is a level structure (see [Pop77], Lecture 10), and is a $K(\pi,1)$ space.

By base change we have a commutative square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\varphi'} & \mathcal{M}_g^{(m)} \\ {\scriptstyle p'}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow{\varphi} & [\mathcal{M}_g] \end{array}$$

with p' and p étale coverings. Passing to fundamental groups,

$$\begin{array}{ccc} \pi_1(\widetilde{X}) & \xrightarrow{\pi_1(\varphi')} & \pi_1(\mathcal{M}_g^{(m)}) \\ {\scriptstyle \pi_1(p')}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_1(p)} \\ \pi_1(X) & \xrightarrow{\pi_1(\varphi)} & \widetilde{\pi_1}(\mathcal{M}_g) \end{array}$$

we see that $\pi_1(\varphi')$ is constant, because $\pi_1(p')$ and $\pi_1(p)$ are injective homomorphisms, and $\pi_1(\varphi)$ is constant by hypothesis.

Since $\widetilde{X}$ is a curve and $\pi_1(\varphi')$ is trivial, we deduce from lemma 3.5 that φ' is constant, hence φ is constant. $\square$

Corollary 3.7 *Let B be a simply connected smooth complex projective variety, then any family of smooth curves on B is trivial.*

Proof By 3.6 any map $B \to [\mathcal{M}_g]$ is constant, hence any family on B is not only isotrivial, but actually trivial. $\square$

4 Moduli dimension of varieties

We have seen some cases of varieties admitting no non isotrivial family of curves of genus $g > 1$. We can go further and investigate which kind of varieties admits no family of maximal variation in moduli.

Definition 4.1 We call *moduli variation* of a family $f\colon \mathcal{C} \to B$ of genus g curves, the dimension of the image $\varphi_f(B)$ of B in $\mathcal{M}_g$. We say that a family $f\colon \mathcal{C} \to B$ has *maximal variation in moduli* if its moduli variation is equal to $\min(\dim B, \dim \mathcal{M}_g)$. Given a variety B, the maximum moduli variation of families of genus g smooth curves on B for any $g > 1$ is called *moduli dimension* of the variety B, and is noted

$$\mathcal{M}\dim(B) := \sup_{g>1}\{\dim \varphi_f(B) \mid f\colon \mathcal{C} \to B \text{ is a family of smooth genus } g \text{ curves}\}$$

We note $\mathcal{B}_i$ the class of all varieties with moduli dimension i.

In the precedent paragraph we studied the properties of the class $\mathcal{B}_0$. We want to investigate here the classes $\mathcal{B}_i$. At first we can observe:

Lemma 4.2 *The moduli dimension is a birational invariant (for projective varieties).*

Proof Let $X \dashrightarrow Y$ be a birational map. It follows from corollary 2.5 that all families on Y are obtained as extensions of families coming from an open subset of X isomorphic to an open subset of Y, so have the same moduli variation of the families on X. Hence, as this construction is a bijection on the sets of families, $\mathcal{M}\dim(Y) = \mathcal{M}\dim(X)$, $\square$

Remark 4.3 We can define as well the moduli dimension for non complete varieties, and we have in this case that $\mathcal{M}\dim(X) \leqslant \mathcal{M}\dim(U)$ for all open dense subset $U \subset X$ (by restriction to U of families on X). This inequality can be strict. For example, consider a non isotrivial smooth family obtained by removing the singular fibers from a non isotrivial family on $\mathbb{P}^1$ with generic smooth fiber.

As in the case of the class $\mathcal{B}_0$, we look now for some properties of the classes $\mathcal{B}_i$, and for some examples.

Proposition 4.4 *Let X be a variety admitting a dominant morphism $X \to Y$ with smooth fibers in $\mathcal{B}_0$, then the moduli dimension of X is at most* $\dim Y$.

This follows from the more general:

Proposition 4.5 *Let $\pi\colon X \to Y$ be a dominant morphism with smooth generic fibers. If there exists an open set $U \subset Y$ such that the fibers X_y have moduli dimension at most i for all $y \in U$, then the moduli dimension of X is at most $i + \dim Y$.*

Proof Let $\mathcal{C} \to X$ be a family of smooth genus g curves on X. Let S be the image of X in $\mathcal{M}_g$ via the moduli map $\varphi_f\colon X \to \mathcal{M}_g$, we want to bound the dimension of S. Consider a generic point $s \in S$. Then $\dim X = \dim S + \dim \varphi_f^{-1}(s)$. The point $s \in S$ being generic we can chose a point in $x \in \varphi_f^{-1}(s)$ and posing $y := \pi(x)$ and $X_y := \pi^{-1}(y)$, we have $\dim X = \dim Y + \dim X_y$, and

$$\dim X_y = \dim \varphi_f(X_y) + \dim(\varphi_f^{-1}(s) \cap X_y) \leqslant i + \dim(\varphi_f^{-1}(s) \cap X_y).$$

So

$$\dim S = \dim X - \dim \varphi_f^{-1}(s) = \dim Y + \dim X_y - \dim \varphi_f^{-1}(s) \leqslant$$

$$\leqslant \dim Y - \dim \varphi_f^{-1}(s) + i + \dim(\varphi_f^{-1}(s) \cap X_y) \leqslant \dim Y + i\ .$$

$\square$

Remark 4.6 As we have only used a dimension count, this result is still valid when the varieties are not complete, as long as we can bound the moduli dimension of the fibers X_y.

Theorem 4.7 *All smooth projective surfaces in $\mathcal{B}_2$ are of general type.*

Proof By lemma 4.2 we can restrict to consider families of curves on minimal surfaces, so we have to show that minimal surfaces not of general type do not admit families with maximal variation in moduli.

We have shown in proposition 3.3 that surfaces of Kodaira dimension 0 belong to $\mathcal{B}_0$. And we can observe that minimal surfaces of Kodaira dimension $-\infty$ or 1 are fibrations over curves with generic fibers of genus 0 or 1, hence they have moduli dimension at most 1 by proposition 4.5 above. $\square$

Remark 4.8 There exist surfaces in $\mathcal{B}_2$. And more generally for any $i > 0$ there exist i-dimensional varieties in $\mathcal{B}_i$ (see [HM98], Theorem 2.33).

Question 4.9 *Are all n-dimensional varieties in $\mathcal{B}_n$ of general type,* i.e., *of Kodaira dimension n?*

We show now that the moduli dimension of a variety is bounded by the dimension of its maximal rationally connected quotient (MRCQ). First we recall the definition and the basic properties of the MRCQ.

Definition 4.10 Let X be a variety. A *maximal rationally connected quotient (MRCQ)* is a dominant rational map $f\colon X \dashrightarrow Y$ with rationnally connected fibers, such that any other rational map with rationally connected fibers $g\colon X \dashrightarrow Z$ factors through f.

Remark 4.11 Any variety admits a maximal rationally connected quotient $f\colon X \dashrightarrow Y$, and the map f has complete generic fibers (see [Deb01], theorem 5.13).

Proposition 4.12 *For any variety X, the moduli dimension of X is at most the dimension of its maximal rationally connected quotient (MRCQ).*

Proof Let $f\colon X \dashrightarrow Y$ be a MRCQ. Then f is regular on an open set $U \subset X$ where all fibers are complete (rationally connected) varieties, hence belong to $\mathcal{B}_0$. So we can apply proposition 4.5 to $f\colon U \to Y$, and

$$\mathcal{M}\dim(X) \leqslant \mathcal{M}\dim(U) \leqslant \dim(Y)\ .$$

$\square$

Remark 4.13 This inequality can be strict. The moduli dimension of a K3 surface is 0, but its MRCQ, being the surface itself, has dimension 2.

5 Compact type curves

We try now to generalize Beauville's theorem in the other direction, *i.e.*, allowing some kind of singular fibers.

Proposition 5.1 *Let B be a curve of genus at most 1, then all families of compact type curves over B of genus $g > 1$ with smooth generic fiber are isotrivial.*

Proof Consider the coarse moduli space of compact type curves $\mathcal{M}_g^{ct} \subset \overline{\mathcal{M}_g}$, since all compact type curves have a generalized Jacobian, as defined, *e.g.*, at pag. 249 of [HM98], which is a principally polarized Abelian variety, and we still have a Torelli-like map $\mathcal{M}_g^{ct} \to \mathcal{A}_g$. This map is not necessarily injective, (as follows, for instance, by the exact sequence at pag. 250 of [HM98]) but is injective when restricted to the locus of smooth curves.

So given a family of compact type curves over B we have the moduli map $B \to \mathcal{M}_g^{ct} \to \mathcal{A}_g$. Arguing as in theorem 1.3 we see that the map $B \to \mathcal{A}_g$ is constant. But as the generic fiber is smooth, and the jacobian map $\mathcal{M}_g^{ct} \to \mathcal{A}_g$ is injective on the smooth locus, the map $B \to \mathcal{M}_g^{ct}$ is constant on an open set of B, hence it is constant. $\square$

Remark 5.2 The hypothesis of compact type is necessary and cannot be replaced by stability, in fact there are non isotrivial families of curves over $\mathbb{P}^1$ with smooth generic fiber and at least 3 stable singular fibers (see [Bea81], pag. 97).

Question 5.3 *Let B be a complete smooth curve of genus at most 1, are there non isotrivial families of compact type curves over B, such that all irreducible components of the generic fiber have genus at least 2?*

Remark 5.4 The hypothesis on the irreducible components of the generic fiber is necessary. In fact, we can construct non isotrivial families of stable compact type singular curves over any curve B of genus at most 1: take an elliptic curve E, and attach a fixed stable curve of genus $g > 1$, through a fixed point $x \in C$ to a point $p \in E$. Letting the point p vary, we obtain a non isotrivial family of compact type stable curves parametrised by E.

In the same way, attaching 4 genus $g > 1$ curves to $\mathbb{P}^1$, in 4 points, and letting the 4 points vary, we obtain a non isotrivial family of singular compact type curves over $\mathbb{P}^1$.

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