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Linear stability for line bundles over curves.

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Abstract. Let C be a smooth irreducible projective curve and let $(L, H^0(L))$ be a complete and globally generated linear series on C. Denote by M_L the syzygy bundle, kernel of the evaluation map $H^0(L) \otimes \mathcal{O}_C \to L$. In this work we restrict our attention to the case of globally generated line bundles L over a curve with $h^0(L) = 3$. The purpose of this short note is to connect Mistretta-Stoppino Conjecture on the equivalence between linear (semi)stability of L and slope (semi)stability of M_L with the existence of extensions of line bundles of L by certain quotients Q of M_L. Also, we give numerical conditions to produce examples of line bundles L which are linearly semistables but with syzygy bundle M_L unstable, that is, we find numerical conditions to look for counter-examples to Mistretta-Stoppino Conjecture of rank 2.

1. Introduction

Let C be a smooth irreducible projective curve of genus g. A globally generated g^r_d over C is a pair (L, V), where L is a line bundle of degree d on C and $V \subseteq H^0(L)$ is a linear subspace of dimension $r+1$ such that the evaluation map $V \otimes \mathcal{O}_C \to L$ is surjective. The rank r kernel $M_{V,L}$ of the evaluation map fits into the following exact sequence

$$0 \longrightarrow M_{V,L} \longrightarrow V \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0. \tag{1}$$

The bundle $M_{V,L}$ is called a syzygy bundle (or dual span bundle). When $V = H^0(L)$, we will denote the bundle $M_{H^0(L),L}$ by M_L. The vector bundle $M_{V,L}$ and its dual $M^\vee_{V,L}$ have been studied from various points of view. The study of the stability of $M_{V,L}$ is related with the study of Brill-Noether varieties and the Minimal Resolution Conjecture (see [**5**]). L. Ein and R. Lazarsfeld showed in [**4**] that M_L is stable for $d > 2g$, and it is semistable for $d = 2g$ (see §3 for the notions of stability for vector bundles). In [**11**], the authors proved that M_{K_C} is semistable, where K_C is the canonical line bundle. Recently, the semistability of $M_{V,L}$ was proved for general curves (see [**2**]).

In [**10**], D. Mumford introduced linear semistability for projective varieties $X \subset \mathbb{P}^n$ (cf. Definition 3.3). This implies Chow semistability for curves $C \subset \mathbb{P}^n$ (see [**10**]), and Mumford uses this to construct the moduli space of smooth irreducible

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projective curves of genus g. For this reason, it is interest to know when a curve $C \subset \mathbb{P}^n$ is linearly semistable.

Later, linear semistability was generalized for a pair (L, V) over a curve C, and linear semistability of the pair (L, V) is equivalent to linear semistability on the image curve induced by the linear system (L, V) (cf. [**12**]).

On the other hand, (semi)stability of the vector bundle $M_{V,L}$ is a stronger condition than linear (semi)stability of the generated linear series (L, V), *i.e.* (semi)stability of $M_{V,L}$ implies linear (semi)stability of the pair (L, V) (cf. Remark 3.4).

It is interesting to know when linear semistability of the pair (L, V) implies semistability of $M_{V,L}$. In this direction, in [**8**, Conjectures 8.6 and 8.7] the authors give two conjectures about this equivalence, and give some conditions under which the equivalence between semistability of $M_{V,L}$ and linear semistability of (L, V) holds, then they used this equivalence to prove semistability of $M_{V,L}$ in some cases see [**8**, Theorem 1.3]. In particular they conjecture that linear semistability of a complete linear system L is equivalent to slope semiastability of the syszygy bundle M_L. Afterwards, in [**3**], the first and third named authors proved this conjecture holds when C is a general Brill-Noether curve and when C is hyperelliptic.

Previously, the second named author of the present work proved in [**7**, Lemma 2.2] that semistability of $M_{V,L}$ is equivalent to linear semistability of (L, V) when $d \geq 2g + 2c$ and $V \subseteq H^0(L)$ is a subspace of codimension $c \leq g$. Using this equivalence he showed that for a general subspace $V \subseteq H^0(L)$ of codimension $c \leq g$, $M_{V,L}$ is semistable (see [**7**, Theorems 2.7 and 2.8]).

The purpose of this work is to investigate further on the relationship between linear semistability of a complete linear series and semistability of M_L in case $\dim H^0(C, L) = 3$. For any such linear series we have the following:

THEOREM 1.1. *Let L be a globally generated line bundle with $h^0(L) = 3$, then L is linearly semistable.*

Therefore any line bundle L with $h^0(L) = 3$ such that M_L is not semistable would provide a counter-example to Mistretta-Stoppino conjecture. This must be produced on a non Brill-Noether general curve, after [**3**].

The important point in the proof of the Theorem and in the following analysis is a characterization of destabilizing quotients of M_L and the extensions they induce:

LEMMA 1.2. *Let Q be a quotient line bundle of M_L with $\deg Q < -\deg L/2$, then there exist an unique globally generated extension*

$$0 \to Q \to F \to L \to 0 \ ,$$

with $h^0(F) = 3$.

Our theorem provides a natural and intrinsic characterization of the kind of singularities of the image curve induced by linear system (see Corollary 3.8):

COROLLARY 1.3. Let L be a globally generated line bundle with $h^0(L) = 3$ inducing a birational map $\phi_L : C \to \mathbb{P}^2$. Denote $\bar{C} \subset \mathbb{P}^2$ its image and d the degree of $\bar{C}$ in $\mathbb{P}^2$. Then, for any point $p \in \bar{C}$, we have multiplicity $m_p \leq \frac{d}{2}$.

The article is organized as follows: In section 2, we construct a globally generated extension (Theorem 1.2). In section 3, we prove that $(L, H^0(L))$ is linearly semistable (Theorem 1.1) and we show Corollary 1.3.

Notation: K_C denote the canonical line bundle on C. Given a vector bundle E over C we denote by d_E (or $\deg(E)$) the degree of E, and by n_E the rank (or $\mathrm{rk}(E)$) of E. The slope of E is defined as the rational number $\mu(E) := \frac{d_E}{n_E}$. Given vector bundles M and N on C, the Ext functor $\mathrm{Ext}^1(M,N)$ is canonically isomorphic to the cohomology space $H^1(M^* \otimes N)$ and classifies extensions of M by N.

2. Extensions of line bundles

Let L be a globally generated line bundle over a curve C, and assume that $h^0(L) = r+1 = 3$. Let M_L be the rank 2 syzygy bundle of L, that is, we have an exact sequence of bundles

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0. \tag{2}$$

Recall that as L is globally generated and non trivial then $\deg L > 0$, so $\mu(M_L) = -\deg L/2 < 0$.

In this section, we are interested in constructing globally generated non-trivial extensions of L by Q, where Q is a quotient line bundle of M_L. This will allow us to analyze possible destabilizations of M_L, or linear destabilizations of $|L|$.

We have the following

Lemma 2.1. *Let Q be a quotient line bundle of M_L such that $\deg Q < -\deg L/2$, then there exist an unique non trivial extension*

$$0 \to Q \to F \to L \to 0,$$

with $h^0(F) = 3$.

Proof. Let $\mathrm{Ext}^1(L,Q)$ be the space which parametrizes extensions of the form

$$u : 0 \to Q \to F_u \to L \to 0.$$

For $u \in \mathrm{Ext}^1(L,Q) \simeq H^1(Q \otimes L^\vee) \simeq H^0(K_C \otimes L \otimes Q^\vee)$ we have the coboundary map in cohomology $\partial_u : H^0(L) \to H^1(Q)$.

Consider the line bundle S defined by $S := \mathrm{K}er(M_L \to Q)$, that is, there exists the following exact sequence

$$0 \to S \to M_L \to Q \to 0. \tag{3}$$

Let

$$\partial : \mathrm{Ext}^1(L,Q) = H^1(L^\vee \otimes Q) \to \mathrm{Hom}(H^0(L), H^1(Q)) \simeq H^0(L)^\vee \otimes H^1(Q)$$

be the map which associates u to its coboundary map

$$\partial_u : H^0(L) \to H^1(Q).$$

As $\deg Q < -\deg L/2 < 0$, then $H^0(Q) = 0$. It follows from the long exact sequence in cohomology

$$0 \to H^0(Q) \to H^0(F_u) \to H^0(L) \xrightarrow{\partial_u} H^1(Q)$$

that $u \in \mathrm{Ker}(\partial)$ if and only if $h^0(F_u) = h^0(L) = 3$.

By dualizing the sequence (2) and twisting by Q, we get the exact sequence of vector bundles

$$0 \to L^\vee \otimes Q \to H^0(L)^\vee \otimes Q \to M_L^\vee \otimes Q \to 0.$$

Since $H^0(Q) = 0$, we obtain the following long exact sequence in cohomology:

$$0 \to H^0(M_L^\vee \otimes Q) \to H^1(L^\vee \otimes Q) \xrightarrow{\partial} H^0(L)^\vee \otimes H^1(Q).$$

Therefore it is enough to show that $h^0(M_L^\vee \otimes Q) = 1$ in order to conclude that $\dim \ker(\partial) = 1$. Now, dualizing the exact sequence (3) and twisting by Q we get

$$0 \to \mathcal{O}_C \to M_L^\vee \otimes Q \to S^\vee \otimes Q \to 0$$

and as $\deg Q < -\deg L/2 = \mu(M_L)$ then $\mu(M_L) < \deg S$ (cf. Remark 3.1 below). Therefore $h^0(S^\vee \otimes Q) = 0$, so $h^0(M_L^\vee \otimes Q) = 1$ and $\dim \ker(\partial) = 1$. We recall the fact that if $\lambda \in \mathbb{C}^*$ then u and λu determine the same extension up to isomorphism [**9**, Lemma 3.3]. Therefore there is only one non-trivial extension $u : 0 \to Q \to F \to L \to 0$ such that $h^0(F) = 3$. ☐

Following the notation of the above theorem we have:

Lemma 2.2. *The vector bundle F defined by the extension u is globally generated.*

Proof. By contradiction, assume that $0 \neq u \in \ker(\partial)$ is such that the corresponding rank two vector bundle F is not globally generated. Then, the three sections of F generate a subsheaf $\tilde{F} = \mathrm{Im}(\mathrm{ev}\colon H^0(F) \otimes \mathcal{O}_C \to F) \subset F$. As $\tilde{F}$ is a torsion free sheaf on a curve, then $\tilde{F}$ is a globally generated vector bundle.

Since L is globally generated and $H^0(L) = H^0(F)$, we have a surjective map $\tilde{F} \twoheadrightarrow L$, then $\mathrm{rk}(\tilde{F}) = 2$, otherwise $\tilde{F} \cong L$ and the extension u would split. Moreover, since F is not globally generated, there exists an effective divisor D contained in the support $\mathrm{Bs}(F) := \{p \in C \mid \mathrm{ev}\colon H^0(F) \otimes O_C \to F \text{ not surjective at } p\}$, and F fits into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(-D) & \longrightarrow & \tilde{F} & \longrightarrow & L & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & Q & \longrightarrow & F & \longrightarrow & L & \longrightarrow & 0. \end{array} \tag{4}$$

Since $h^0(F) = 3$ and $\tilde{F}$ is globally generated of rank 2 and non trivial, it follows that $h^0(\tilde{F}) = 3$.

Then, for any point $p \in D$ there is an exact sequence

$$u_p : 0 \to Q(-p) \to E \to L \to 0,$$

with $h^0(E) = 3$, such that the extension u is induced from u_p in the following sense: the inclusion $Q(-D) \subset Q$ gives rise to a map $\phi\colon \mathrm{Ext}^1(L, Q(-D)) \to \mathrm{Ext}^1(L, Q)$. After diagram (4) the map ϕ associates to the extension $\tilde{u}\colon 0 \to Q(-D) \to \tilde{F} \to L \to 0$ the extension $u\colon 0 \to Q \to F \to L \to 0$. Now, for any $p \in D$, the map ϕ factors through the maps $\mathrm{Ext}^1(L, Q(-D)) \to \mathrm{Ext}^1(L, Q(-p)) \to \mathrm{Ext}^1(L, Q)$, therefore the diagram (4) can be extended to a commutative diagram:

(5)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q(-D) & \longrightarrow & \tilde{F} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Q(-p) & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Q & \longrightarrow & F & \longrightarrow & L & \longrightarrow & 0.
\end{array}$$

As $h^0(\tilde{F}) = h^0(F) = 3$, then $h^0(E) = 3$ as well.

From the fact that $h^0(E) = h^0(F) = h^0(L)$ and $h^0(Q(-p)) = 0$, then $u_p \in \mathrm{Ker}(\mathrm{Ext}^1(L, Q(-p)) \xrightarrow{\tilde{\partial}} \mathrm{Hom}(H^0(L), H^1(Q(-p)))$. Now, we compute the dimension of the space of all extensions $u_p \in \ker(\mathrm{Ext}^1(L, Q(-p)) \xrightarrow{\tilde{\partial}} \mathrm{Hom}(H^0(L), H^1(Q(-p)))$, where

$$\mathrm{Ext}^1(L, Q(-p)) \simeq H^0(K_C \otimes L \otimes Q^\vee(p))^\vee,$$

and

$$\mathrm{Hom}(H^0(L), H^1(Q(-p))) \simeq H^0(L)^\vee \otimes H^0(K_C \otimes Q^\vee(p))^\vee.$$

The map $\tilde{\partial}$ is dual to

$$\mu_1 : H^0(L) \otimes H^0(K_C \otimes Q^\vee(p)) \to H^0(K_C \otimes L \otimes Q^\vee(p)).$$

Then, $\ker(\tilde{\partial}) = \mathrm{cork}(\mu_1)$, where $ker(\tilde{\partial})$ parametrizes non-trivial extensions u inside $\ker(\partial)$, such that the corresponding rank two vector bundle F_u is not globally generated. We know that $\dim(\ker(\partial)) = 1$, then we need to show that $\dim(\ker(\tilde{\partial})) = 0$, that is, we have to show that μ_1 is surjective.

Note that $h^1(K_C \otimes S \otimes Q^\vee(p)) = h^0(S^\vee \otimes Q(-p)) = 0$. Twisting the exact sequence (2) by $K_C \otimes Q^\vee(p)$ and taking cohomology, we obtain that $h^1(M_L \otimes K_C \otimes Q^\vee(p)) = h^1(K_C(p)) = 0$. Hence we have the following exact sequence

$$0 \to H^0(M_L \otimes K_C \otimes Q^\vee(p)) \to H^0(L) \otimes H^0(K_C \otimes Q^\vee(p)) \xrightarrow{\mu_1} H^0(L \otimes K_C \otimes Q^\vee(p)) \to 0.$$

So μ_1 is surjective and $\ker(\tilde{\partial}) = 0$, thus F is a globally generated rank two vector bundle with $h^0(F) = 3$.

□

To finish this section we have the following two result:

Proposition 2.3. [**13**, Proposition 2.6 (b)] The class of non-split vector bundle E fitting in an extension

$$0 \to \mathcal{O}_C \to E \to E_2 \to 0$$

corresponds with injective coboundary maps $H^0(\mathcal{O}_C) \to H^1(E_2^\vee)$ modulo the action of $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Corollary 2.4. Let

$$0 \to \mathcal{O}_C \to E \to E_2 \to 0$$

be an exact sequence. Then E is indecomposable if and only if the coboundary map $H^0(\mathcal{O}_C) \to H^1(E_2^\vee)$ is injective.

We recall two facts:

(1) $H^0(Q) = 0$.

(2) $H^0(L^\vee) = 0$ because L is a globally generated line bundle.

LEMMA 2.5. *The vector bundle F defined by the extension u in lemma 2.2 admits no trivial summands.*

PROOF. Assume that F has trivial summands. From the exact sequence

$$0 \to Q \to F \to L \to 0 \tag{6}$$

we have $F = \mathcal{O}_C \oplus (L \otimes Q)$. Twisting the exact sequence (6) by $Q^\vee$, we get the exact sequence

$$0 \to \mathcal{O}_C \to Q^\vee \oplus L \to L \otimes Q^\vee \to 0,$$

Dualizing the above exact sequence and taking cohomology we obtain

$$0 \to H^0(L^\vee \otimes Q) \to H^0(L^\vee \oplus Q) \to H^0(\mathcal{O}_C) \to H^1(L^\vee \otimes Q).$$

Since $H^0(Q) = H^0(L^\vee) = 0$, it follows that the coboundary map $\delta : H^0(\mathcal{O}_C) \to H^1(L^\vee \otimes Q)$ is injective, but this is imposible by Proposition 2.3. Therefore F has not trivial summands. □

3. Linear stability and stability of Syzygy bundle

In this section we will use the results obtained in the previous section to show that a globally generated line bundle L with 3 sections is linearly semistable. Moreover, we give numerical conditions that can be useful in finding counterexamples to the conjecture proposed by E. C. Mistretta and L. Stoppino [**8**, Conjecture 8.7]. This conjecture affirms the equivalence between the (semi)stability of M_L and the linear (semi)stability of L.

In order to make the exposition self-contained we recall some facts on vector bundles.

We say that a vector bundle E is stable (semistable) if for all non trivial subbundles $F \subset E$

$$\mu(F) < \mu(E) \qquad \text{(resp. } \leq).$$

If E is not semistable, then we say that E is unstable.

REMARK 3.1. Suppose $0 \to F \to E \to Q \to 0$ is an exact sequence of vector bundles. Then

$$\mu(F) < \mu(E) \iff \mu(E) < \mu(Q)$$

and the same holds for $\geq$ and all other inequalities.

Is well known that for any unstable vector bundle E there exists an unique filtration (*Harder-Narasimhan*)

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E,$$

where the grading $gr_i = E_i/E_{i-1}$ satisfies the following conditions:

(1) the grading gr_i is semistable;

(2) $\mu(gr_i) > \mu(gr_{i+1})$ for $i = 1, \ldots, k-1$.

Remark 3.2. Let E be an unstable vector bundle of rank 2, then there exists a unique sub-line bundle $S \subset E$ with $\mu(S) > \mu(E)$; indeed suppose that there exists subline bundles S_1 and S_2 with $\mu(S_1), \mu(S_2) > \mu(E)$. Then consider the following filtration

$$0 \subset S_i \subset E \quad \text{for } i = 1, 2. \tag{7}$$

We have S_i and E/S_i satisfies the following conditions

(1) S_i and E/S_i are line bundles;

(2) $\mu(S_i) > \mu(E/S_i)$ for $i = 1, 2$.

Hence for $i = 1, 2$ the filtration $0 \subset S_i \subset E$ is Harder-Narasimhan filtrations and by uniqueness we have that $S_1 = S_2$. The bundle S is called the *maximal destabilizing subbundle of E.*

Definition 3.3. Let (L, V) be a globally generated g^r_d over a curve C, that is, $\deg(L) = d$ and $V \subseteq H^0(L)$ with $r + 1 = \dim(V)$. We say that (L, V) is linearly semistable (respectively linearly stable) if for any linear subspace $W \subset V$ of dimension w,

$$\frac{\deg(\tilde{L})}{w-1} \geq \frac{\deg(L)}{r} \qquad (\text{respectively} >),$$

where $\tilde{L}$ is the line bundle generated by W, namely, there exists the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{W,\tilde{L}} & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & \tilde{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,L} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0. \end{array} \tag{8}$$

Remark 3.4. Linear (semi)stability of (L, V) is equivalent to the condition that the bundle $M_{V,L}$ can not be destabilized by subbundles of the form $M_{W,\tilde{L}}$, where $(\tilde{L}, W)$ is a generated subseries of (L, V).

Conjecture 3.5. [**8**, Conjecture 8.7] Let C be a curve, and let L be a globally generated line bundle on C. The linear (semi)stability of $(L, H^0(L))$ is equivalent to (semi)stability for M_L.

Lemma 3.6. *Let L be a globally generated line bundle with $h^0(L) = 3$. Let $W \subset H^0(L)$ be a linear subspace with $\dim(W) = 2$, then $h^0(\tilde{L}) \leq 3$, where $\tilde{L}$ is the line bundle generated by W. Moreover, $h^0(\tilde{L}) = 3$ if and only if W generates L.*

Proof. Let $W \subset H^0(L)$ be a subspace of dimension 2, let $\tilde{L}$ be the line bundle generated by W, which fits into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{L}^{\vee} & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & \tilde{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_L & \longrightarrow & H^0(L) \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

We have $\tilde{L} = L(-D)$ with D an effective divisor, which can be zero. That is, if W generates L then $\tilde{L} = L$ and if W doesn't generates L then there exists an effective divisor $D \neq 0$ such that $\tilde{L} = L(-D)$. Since L is generated and D is effective, we see that $h^0(\tilde{L}) = h^0(L(-D)) = 2$ and this completes the proof. $\square$

THEOREM 3.7. *Let L be a globally generated line bundle with $h^0(L) = 3$, then L is linearly semistable.*

PROOF. By contradiction, assume that L is not linearly semistable, so there exists a linear subspace $W \subset H^0(L)$ of dimension 2 with $\deg(\tilde{L}) < \frac{d}{2}$, where $\tilde{L}$ is the line bundle generated by W, and $d = \deg L$. The line bundle $\tilde{L}$ fits into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{L}^\vee & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & \tilde{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_L & \longrightarrow & H^0(L) \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

From Lemma 3.6, we have $h^0(\tilde{L}) \leq 3$ and $h^0(\tilde{L}) = 3$ if and only if W generates L. In this last case, we have $deg(\tilde{L}) = d > \frac{d}{2}$. Thus $h^0(\tilde{L}) = 2$ because $\deg(\tilde{L}) < \frac{d}{2}$. On the other hand, we can show that $h^0(\tilde{L}) \geq 3$.

Note that M_L is unstable and that the line bundle $S := \tilde{L}^\vee$ is a destabilizing subbundle of M_L, in fact $-\deg(\tilde{L}) = \mu(S) > \mu(M_L) = -\frac{d}{2}$ by contradiction hypothesis.

Let Q be the quotient line bundle of M_L by S.

Claim: $h^0(\tilde{L}) \geq 3$.
Proof of the **Claim**: first notice that Q satisfies the following:

$$\deg(Q) = \deg(M_L) - \deg(S) = -d - \deg(S) < -d + \frac{d}{2} = -\frac{d}{2} < 0.$$

From Lemmas 2.1 and 2.2, there exists an unique globally generated vector bundle F of rank two which fits into the following exact sequence

$$0 \to Q \to F \xrightarrow{\alpha} L \to 0.$$

Since $H^0(Q) = 0$, we complete the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_F & \longrightarrow & H^0(F) \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & M_L & \longrightarrow & H^0(L) \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0, \end{array}$$

such that α_2 is an isomorphism i.e $H^0(F) \simeq H^0(L)$. By snake lemma, we have an exact sequence

$$\ker(\alpha_1) \to \ker(\alpha_2) \to \ker(\alpha) \to \operatorname{coker}(\alpha_1) \to \operatorname{coker}(\alpha_2) \to ....$$

where $\ker(\alpha_1) = \ker(\alpha_2) = \{0\}$, $\ker(\alpha) = Q$, $\operatorname{coker}(\alpha_2) = \{0\}$ since $\alpha_2 : (H^0(F) \to H^0(L))$ is an isomorphism. Thus, $\ker(\alpha) \simeq \operatorname{coker}(\alpha_1)$, $Q = M_L/M_F$ and $M_F \simeq S$, that is, we have the following exact sequence

$$0 \to S \to H^0(F) \otimes \mathcal{O}_C \to F \to 0.$$

Dualizing the above exact sequence and taking cohomology, we get

$$0 \to H^0(F^\vee) \to H^0(F)^\vee \to H^0(S^\vee) \to \cdots$$

Since F is globally generated without trivial summands (see lemma 2.2 and lemma 2.5), we have $h^0(F^\vee) = 0$. So $h^0(S^\vee) \geq h^0(F) = 3$. This proves the claim.

So from the first part, we have $h^0(\tilde{L}) = h^0(S^\vee) = 2$ and from the Claim we have $h^0(\tilde{L}) \geq 3$, which is a contradiction. Thus, L is linearly semistable and this complete the proof. □

Corollary 3.8. Let L be globally generated line bundle with $h^0(L) = 3$ inducing a birational map $\phi_L : C \to \mathbb{P}^2$. Denote $\bar{C} \subset \mathbb{P}^2$ its image and d the degree of $\bar{C}$ in $\mathbb{P}^2$. Then, for any point $p \in \bar{C}$, we have multiplicity $m_p \leq \frac{d}{2}$.

Proof. The proof follows from Theorem 3.7 and [**8**, Proposition 8.1]. □

Remark 3.9. Let L be a globally generated line bundle over a curve C of degree d such that $h^0(L) = 3$. From the exact sequence

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0$$

we have for any line bundle B the sequence

$$0 \to M_L \otimes B \to H^0(L) \otimes B \to L \otimes B \to 0.$$

So, if there exists a line bundle B on C with $h^0(L \otimes B) < 3h^0(B)$, then $h^0(M_L \otimes B) \neq 0$, which implies that $O_C \hookrightarrow M_L \otimes B$, that is, $B^\vee \hookrightarrow M_L$. We have that M_L is unstable if $2\deg(B) < d$. Thus, should be interesting to find some numerical conditions to find a linearly semistable line bundle L with M_L unstable.

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