# Linear stability for line bundles over curves.

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ABSTRACT. Let C be a smooth irreducible projective curve and let  $(L, H^0(L))$ be a complete and globally generated linear series on C. Denote by  $M_L$  the syzygy bundle, kernel of the evaluation map  $H^0(L) \otimes \mathcal{O}_C \to L$ . In this work we restrict our attention to the case of globally generated line bundles L over a curve with  $h^0(L) = 3$ . The purpose of this short note is to connect Mistretta-Stoppino Conjecture on the equivalence between linear (semi)stability of L and slope (semi)stability of  $M_L$  with the existence of extensions of line bundles of L by certain quotients Q of  $M_L$ . Also, we give numerical conditions to produce examples of line bundles L which are linearly semistables but with syzygy bundle  $M_L$  unstable, that is, we find numerical conditions to look for counter-examples to Mistretta-Stoppino Conjecture of rank 2.

### 1. Introduction

Let C be a smooth irreducible projective curve of genus g. A globally generated  $g_d^r$  over C is a pair (L, V), where L is a line bundle of degree d on C and  $V \subseteq H^0(L)$  is a linear subspace of dimension r + 1 such that the evaluation map  $V \otimes \mathcal{O}_C \to L$  is surjective. The rank r kernel  $M_{V,L}$  of the evaluation map fits into the following exact sequence

(1) 
$$0 \longrightarrow M_{V,L} \longrightarrow V \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0.$$

The bundle  $M_{V,L}$  is called a syzygy bundle (or dual span bundle). When  $V = H^0(L)$ , we will denote the bundle  $M_{H^0(L),L}$  by  $M_L$ . The vector bundle  $M_{V,L}$  and its dual  $M_{V,L}^{\vee}$  have been studied from various points of view. The study of the stability of  $M_{V,L}$  is related with the study of Brill-Noether varieties and the Minimal Resolution Conjecture (see [5]). L. Ein and R. Lazarsfeld showed in [4] that  $M_L$  is stable for d > 2g, and it is semistable for d = 2g (see §3 for the notions of stability for vector bundles). In [11], the authors proved that  $M_{K_C}$  is semistable, where  $K_C$  is the canonical line bundle. Recently, the semistability of  $M_{V,L}$  was proved for general curves (see [2]).

In [10], D. Mumford introduced linear semistability for projective varieties  $X \subset \mathbb{P}^n$ (cf. Definition 3.3). This implies Chow semistability for curves  $C \subset \mathbb{P}^n$  (see [10]), and Mumford uses this to construct the moduli space of smooth irreducible

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projective curves of genus g. For this reason, it is interest to know when a curve  $C \subset \mathbb{P}^n$  is linearly semistable.

Later, linear semistability was generalized for a pair (L, V) over a curve C, and linear semistability of the pair (L, V) is equivalent to linear semistability on the image curve induced by the linear system (L, V) (cf. [12]).

On the other hand, (semi)stability of the vector bundle  $M_{V,L}$  is a stronger condition than linear (semi)stability of the generated linear series (L, V), *i.e.* (semi)stability of  $M_{V,L}$  implies linear (semi)stability of the pair (L, V) (cf. Remark 3.4).

It is interesting to know when linear semistability of the pair (L, V) implies semistability of  $M_{V,L}$ . In this direction, in [8, Conjectures 8.6 and 8.7] the authors give two conjectures about this equivalence, and give some conditions under which the equivalence between semistability of  $M_{V,L}$  and linear semistability of (L, V)holds, then they used this equivalence to prove semistability of  $M_{V,L}$  in some cases see [8, Theorem 1.3]. In particular they conjecture that linear semistability of a complete linear system L is equivalent to slope semiastability of the syszygy bundle  $M_L$ . Afterwards, in [3], the first and third named authors proved this conjecture holds when C is a general Brill-Noether curve and when C is hyperelliptic.

Previously, the second named author of the present work proved in [7, Lemma 2.2] that semistability of  $M_{V,L}$  is equivalent to linear semistability of (L, V) when  $d \geq 2g + 2c$  and  $V \subseteq H^0(L)$  is a subspace of codimension  $c \leq g$ . Using this equivalence he showed that for a general subspace  $V \subseteq H^0(L)$  of codimension  $c \leq g$ ,  $M_{V,L}$  is semistable (see [7, Theorems 2.7 and 2.8]).

The purpose of this work is to investigate further on the relationship between linear semistability of a complete linear series and semistability of  $M_L$  in case dim  $H^0(C, L) = 3$ . For any such linear series we have the following:

THEOREM 1.1. Let L be a globally generated line bundle with  $h^0(L) = 3$ , then L is linearly semistable.

Therefore any line bundle L with  $h^0(L) = 3$  such that  $M_L$  is not semistable would provide a counter-example to Mistretta-Stoppino conjecture. This must be produced on a non Brill-Noether general curve, after [3].

The important point in the proof of the Theorem and in the following analysis is a characterization of destabilizing quotients of  $M_L$  and the extensions they induce:

LEMMA 1.2. Let Q be a quotient line bundle of  $M_L$  with deg  $Q < - \deg L/2$ , then there exist an unique globally generated extension

$$0 \to Q \to F \to L \to 0 ,$$

with  $h^0(F) = 3$ .

Our theorem provides a natural and intrinsic characterization of the kind of singularities of the image curve induced by linear system (see Corollary 3.8):

COROLLARY 1.3. Let L be a globally generated line bundle with  $h^0(L) = 3$ inducing a birational map  $\phi_L : C \to \mathbb{P}^2$ . Denote  $\overline{C} \subset \mathbb{P}^2$  its image and d the degree of  $\overline{C}$  in  $\mathbb{P}^2$ . Then, for any point  $p \in \overline{C}$ , we have multiplicity  $m_p \leq \frac{d}{2}$ .

The article is organized as follows: In section 2, we construct a globally generated extension (Theorem 1.2). In section 3, we prove that  $(L, H^0(L))$  is linearly semistable (Theorem 1.1) and we show Corollary 1.3.

**Notation:**  $K_C$  denote the canonical line bundle on C. Given a vector bundle E over C we denote by  $d_E$  (or deg(E)) the degree of E, and by  $n_E$  the rank (or rk(E)) of E. The slope of E is defined as the rational number  $\mu(E) := \frac{d_E}{n_E}$ . Given vector bundles M and N on C, the Ext functor  $\text{Ext}^1(M, N)$  is canonically isomorphic to the cohomology space  $H^1(M^* \otimes N)$  and classifies extensions of M by N.

### 2. Extensions of line bundles

Let L be a globally generated line bundle over a curve C, and assume that  $h^0(L) = r + 1 = 3$ . Let  $M_L$  be the rank 2 syzygy bundle of L, that is, we have an exact sequence of bundles

(2) 
$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0.$$

Recall that as L is globally generated and non trivial then deg L > 0, so  $\mu(M_L) = - \deg L/2 < 0$ .

In this section, we are interested in constructing globally generated non-trivial extensions of L by Q, where Q is a quotient line bundle of  $M_L$ . This will allow us to analyze possible destabilizations of  $M_L$ , or linear destabilizations of |L|.

We have the following

LEMMA 2.1. Let Q be a quotient line bundle of  $M_L$  such that deg  $Q < - \deg L/2$ , then there exist an unique non trivial extension

$$0 \to Q \to F \to L \to 0,$$

with  $h^0(F) = 3$ .

**PROOF.** Let  $\text{Ext}^1(L,Q)$  be the space which parametrizes extensions of the form

$$u: 0 \to Q \to F_u \to L \to 0.$$

For  $u \in \operatorname{Ext}^1(L,Q) \simeq H^1(Q \otimes L^{\vee}) \simeq H^0(K_C \otimes L \otimes Q^{\vee})$  we have the coboundary map in cohomology  $\partial_u : H^0(L) \to H^1(Q)$ .

Consider the line bundle S defined by  $S := \operatorname{Ker}(M_L \to Q)$ , that is, there exists the following exact sequence

(3) 
$$0 \to S \to M_L \to Q \to 0.$$

Let

$$\partial : \operatorname{Ext}^{1}(L,Q) = H^{1}(L^{\vee} \otimes Q) \to \operatorname{Hom}(H^{0}(L),H^{1}(Q)) \simeq H^{0}(L)^{\vee} \otimes H^{1}(Q)$$

be the map which associates u to its coboundary map

$$\partial_u : H^0(L) \to H^1(Q).$$

As deg  $Q<-\deg L/2<0,$  then  $H^0(Q)=0.$  It follows from the long exact sequence in cohomology

$$0 \to H^0(Q) \to H^0(F_u) \to H^0(L) \xrightarrow{\partial_u} H^1(Q)$$

that  $u \in \text{Ker}(\partial)$  if and only if  $h^0(F_u) = h^0(L) = 3$ .

By dualizing the sequence (2) and twisting by Q, we get the exact sequence of vector bundles

$$0 \to L^{\vee} \otimes Q \to H^0(L)^{\vee} \otimes Q \to M_L^{\vee} \otimes Q \to 0.$$

Since  $H^0(Q) = 0$ , we obtain the following long exact sequence in cohomology:

$$0 \to H^0(M_L^{\vee} \otimes Q) \to H^1(L^{\vee} \otimes Q) \xrightarrow{\partial} H^0(L)^{\vee} \otimes H^1(Q).$$

Therefore it is enough to show that  $h^0(M_L^{\vee} \otimes Q) = 1$  in order to conclude that dim ker $(\partial) = 1$ . Now, dualizing the exact sequence (3) and twisting by Q we get

$$0 \to \mathcal{O}_C \to M_L^{\vee} \otimes Q \to S^{\vee} \otimes Q \to 0$$

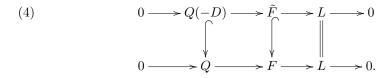
and as deg  $Q < -\deg L/2 = \mu(M_L)$  then  $\mu(M_L) < \deg S$  (cf. Remark 3.1 below). Therefore  $h^0(S^{\vee} \otimes Q) = 0$ , so  $h^0(M_L^{\vee} \otimes Q) = 1$  and dim ker $(\partial) = 1$ . We recall the fact that if  $\lambda \in \mathbb{C}^*$  then u and  $\lambda u$  determine the same extension up to isomorphism [9, Lemma 3.3]. Therefore there is only one non-trivial extension  $u: 0 \to Q \to F \to L \to 0$  such that  $h^0(F) = 3$ .

Following the notation of the above theorem we have:

LEMMA 2.2. The vector bundle F defined by the extension u is globally generated.

PROOF. By contradiction, assume that  $0 \neq u \in \ker(\partial)$  is such that the corresponding rank two vector bundle F is not globally generated. Then, the three sections of F generate a subsheaf  $\tilde{F} = \operatorname{Im}(\operatorname{ev}: H^0(F) \otimes \mathcal{O}_C \to F) \subset F$ . As  $\tilde{F}$  is a torsion free sheaf on a curve, then  $\tilde{F}$  is a globally generated vector bundle.

Since L is globally generated and  $H^0(L) = H^0(F)$ , we have a surjective map  $\tilde{F} \to L$ , then  $\operatorname{rk}(\tilde{F}) = 2$ , otherwise  $\tilde{F} \cong L$  and the extension u would split. Moreover, since F is not globally generated, there exists an effective divisor D contained in the support  $\operatorname{Bs}(F) := \{p \in C \mid \operatorname{ev} \colon H^0(F) \otimes O_C \to F \text{ not surjective at } p\}$ , and F fits into the following commutative diagram

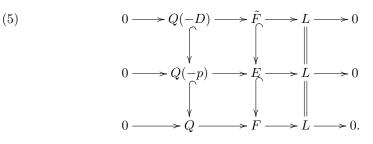


Since  $h^0(F) = 3$  and  $\tilde{F}$  is globally generated of rank 2 and non trivial, it follows that  $h^0(\tilde{F}) = 3$ .

Then, for any point  $p \in D$  there is an exact sequence

$$u_p: 0 \to Q(-p) \to E \to L \to 0,$$

with  $h^0(E) = 3$ , such that the extension u is induced from  $u_p$  in the following sense: the inclusion  $Q(-D) \subset Q$  gives rise to a map  $\phi \colon \operatorname{Ext}^1(L, Q(-D)) \to \operatorname{Ext}^1(L, Q)$ . After diagram (4) the map  $\phi$  associates to the extension  $\tilde{u} \colon 0 \to Q(-D) \to \tilde{F} \to L \to 0$  the extension  $u \colon 0 \to Q \to F \to L \to 0$ . Now, for any  $p \in D$ , the map  $\phi$  factors through the maps  $\operatorname{Ext}^1(L, Q(-D)) \to \operatorname{Ext}^1(L, Q(-p)) \to \operatorname{Ext}^1(L, Q)$ , therefore the diagram (4) can be extended to a commutative diagram:



As  $h^0(\tilde{F}) = h^0(F) = 3$ , then  $h^0(E) = 3$  as well. From the fact that  $h^0(E) = h^0(F) = h^0(L)$  and  $h^0(Q(-p)) = 0$ , then  $u_p \in C$  $\operatorname{Ker}(\operatorname{Ext}^1(L,Q(-p))) \xrightarrow{\tilde{\partial}} \operatorname{Hom}(H^0(L),H^1(Q(-p))).$  Now, we compute the dimension of the space of all extensions  $u_p \in \ker(\operatorname{Ext}^1(L, Q(-p)) \xrightarrow{\tilde{\partial}} \operatorname{Hom}(H^0(L), H^1(Q(-p)))),$ where

$$\operatorname{Ext}^{1}(L, Q(-p)) \simeq H^{0}(K_{C} \otimes L \otimes Q^{\vee}(p))^{\vee},$$

and

$$\operatorname{Hom}(H^0(L), H^1(Q(-p))) \simeq H^0(L)^{\vee} \otimes H^0(K_C \otimes Q^{\vee}(p))^{\vee}$$

The map  $\tilde{\partial}$  is dual to

$$u_1: H^0(L) \otimes H^0(K_C \otimes Q^{\vee}(p)) \to H^0(K_C \otimes L \otimes Q^{\vee}(p)).$$

Then,  $\ker(\tilde{\partial}) = \operatorname{cork}(\mu_1)$ , where  $\ker(\tilde{\partial})$  parametrizes non-trivial extensions *u* inside  $\ker(\partial)$ , such that the corresponding rank two vector bundle  $F_u$  is not globally generated. We know that  $\dim(\ker(\partial)) = 1$ , then we need to show that  $\dim(\ker(\tilde{\partial})) = 0$ , that is, we have to show that  $\mu_1$  is surjective.

Note that  $h^1(K_C \otimes S \otimes Q^{\vee}(p)) = h^0(S^{\vee} \otimes Q(-p)) = 0$ . Twisting the exact sequence (2) by  $K_C \otimes Q^{\vee}(p)$  and taking cohomology, we obtain that  $h^1(M_L \otimes K_C \otimes Q^{\vee}(p)) =$  $h^1(K_C(p)) = 0$ . Hence we have the following exact sequence

$$0 \to H^0(M_L \otimes K_C \otimes Q^{\vee}(p)) \to H^0(L) \otimes H^0(K_C \otimes Q^{\vee}(p)) \xrightarrow{\mu_1} H^0(L \otimes K_C \otimes Q^{\vee}(p)) \to 0$$

So  $\mu_1$  is surjective and ker $(\tilde{\partial}) = 0$ , thus F is a globally generated rank two vector bundle with  $h^0(F) = 3$ .

To finish this section we have the following two result:

PROPOSITION 2.3. [13, Proposition 2.6 (b)] The class of non-split vector bundle E fitting in an extension

 $0 \to \mathcal{O}_C \to E \to E_2 \to 0$ 

corresponds with injective coboundary maps  $H^0(\mathcal{O}_C) \to H^1(E_2^{\vee})$  modulo the action of  $\mathbb{C}^* = \mathbb{C} - \{0\}.$ 

Corollary 2.4. Let

$$0 \to \mathcal{O}_C \to E \to E_2 \to 0$$

be an exact sequence. Then E is indecomposable if and only if the coboundary map  $H^0(\mathcal{O}_C) \to H^1(E_2^{\vee})$  is injective.

We recall two facts:

- (1)  $H^0(Q) = 0.$
- (2)  $H^0(L^{\vee}) = 0$  because L is a globally generated line bundle.

LEMMA 2.5. The vector bundle F defined by the extension u in lemma 2.2 admits no trivial summands.

**PROOF.** Assume that F has trivial summands. From the exact sequence

$$(6) 0 \to Q \to F \to L \to 0$$

we have  $F = \mathcal{O}_C \oplus (L \otimes Q)$ . Twisting the exact sequence (6) by  $Q^{\vee}$ , we get the exact sequence

$$0 \to \mathcal{O}_C \to Q^{\vee} \oplus L \to L \otimes Q^{\vee} \to 0,$$

Dualizing the above exact sequence and taking cohomology we obtain

$$0 \to H^0(L^{\vee} \otimes Q) \to H^0(L^{\vee} \oplus Q) \to H^0(\mathcal{O}_C) \to H^1(L^{\vee} \otimes Q).$$

Since  $H^0(Q) = H^0(L^{\vee}) = 0$ , it follows that the coboundary map  $\delta : H^0(\mathcal{O}_C) \to H^1(L^{\vee} \otimes Q)$  is injective, but this is imposible by Proposition 2.3. Therefore F has not trivial summands.

## 3. Linear stability and stability of Syzygy bundle

In this section we will use the results obtained in the previous section to show that a globally generated line bundle L with 3 sections is linearly semistable. Moreover, we give numerical conditions that can be useful in finding counterexamples to the conjecture proposed by E. C. Mistretta and L. Stoppino [8, Conjecture 8.7]. This conjecture affirms the equivalence between the (semi)stability of  $M_L$  and the linear (semi)stability of L.

In order to make the exposition self-contained we recall some facts on vector bundles.

We say that a vector bundle E is stable (semistable) if for all non trivial subbundles  $F \subset E$ 

 $\mu(F) < \mu(E) \qquad (\text{resp.} \le).$ 

If E is not semistable, then we say that E is unstable.

Remark 3.1. Suppose  $0 \to F \to E \to Q \to 0$  is an exact sequence of vector bundles. Then

$$\mu(F) < \mu(E) \iff \mu(E) < \mu(Q)$$

and the same holds for  $\geq$  and all other inequalities.

Is well known that for any unstable vector bundle E there exists an unique filtration (*Harder-Narasimhan*)

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E,$$

where the grading  $gr_i = E_i/E_{i-1}$  satisfies the following conditions:

- (1) the grading  $gr_i$  is semistable;
- (2)  $\mu(gr_i) > \mu(gr_{i+1})$  for  $i = 1, \dots, k-1$ .

REMARK 3.2. Let E be an unstable vector bundle of rank 2, then there exists a unique sub-line bundle  $S \subset E$  with  $\mu(S) > \mu(E)$ ; indeed suppose that there exists subline bundles  $S_1$  and  $S_2$  with  $\mu(S_1), \mu(S_2) > \mu(E)$ . Then consider the following filtration

(7) 
$$0 \subset S_i \subset E \quad \text{for } i = 1, 2.$$

We have  $S_i$  and  $E/S_i$  satisfies the following conditions

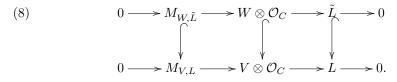
- (1)  $S_i$  and  $E/S_i$  are line bundles;
- (2)  $\mu(S_i) > \mu(E/S_i)$  for i = 1, 2.

Hence for i = 1, 2 the filtration  $0 \subset S_i \subset E$  is Harder-Narasimhan filtrations and by uniqueness we have that  $S_1 = S_2$ . The bundle S is called the maximal destabilizing subbundle of E.

DEFINITION 3.3. Let (L, V) be a globally generated  $g_d^r$  over a curve C, that is,  $\deg(L) = d$  and  $V \subseteq H^0(L)$  with  $r + 1 = \dim(V)$ . We say that (L, V) is linearly semistable (respectively linearly stable) if for any linear subspace  $W \subset V$ of dimension w,

$$\frac{\deg(\tilde{L})}{w-1} \ge \frac{\deg(L)}{r} \quad \text{(respectively >)},$$

where L is the line bundle generated by W, namely, there exists the following commutative diagram

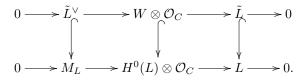


REMARK 3.4. Linear (semi)stability of (L, V) is equivalent to the condition that the bundle  $M_{V,L}$  can not be destabilized by subbundles of the form  $M_{W,\tilde{L}}$ , where  $(\tilde{L}, W)$  is a generated subseries of (L, V).

CONJECTURE 3.5. [8, Conjecture 8.7] Let C be a curve, and let L be a globally generated line bundle on C. The linear (semi)stability of  $(L, H^0(L))$  is equivalent to (semi)stability for  $M_L$ .

LEMMA 3.6. Let L be a globally generated line bundle with  $h^0(L) = 3$ . Let  $W \subset H^0(L)$  be a linear subspace with  $\dim(W) = 2$ , then  $h^0(\tilde{L}) \leq 3$ , where  $\tilde{L}$  is the line bundle generated by W. Moreover,  $h^0(\tilde{L}) = 3$  if and only if W generates L.

PROOF. Let  $W \subset H^0(L)$  be a subspace of dimension 2, let  $\tilde{L}$  be the line bundle generated by W, which fits into the following diagram

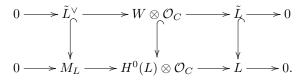


We have  $\tilde{L} = L(-D)$  with D an effective divisor, which can be zero. That is, if W generates L then  $\tilde{L} = L$  and if W doesn't generates L then there exists an effective

divisor  $D \neq 0$  such that  $\tilde{L} = L(-D)$ . Since L is generated and D is effective, we see that  $h^0(\tilde{L}) = h^0(L(-D)) = 2$  and this completes the proof.

THEOREM 3.7. Let L be a globally generated line bundle with  $h^0(L) = 3$ , then L is linearly semistable.

PROOF. By contradiction, assume that L is not linearly semistable, so there exists a linear subspace  $W \subset H^0(L)$  of dimension 2 with  $\deg(\tilde{L}) < \frac{d}{2}$ , where  $\tilde{L}$  is the line bundle generated by W, and  $d = \deg L$ . The line bundle  $\tilde{L}$  fits into the following diagram



From Lemma 3.6, we have  $h^0(\tilde{L}) \leq 3$  and  $h^0(\tilde{L}) = 3$  if and only if W generates L. In this last case, we have  $deg(\tilde{L}) = d > \frac{d}{2}$ . Thus  $h^0(\tilde{L}) = 2$  because  $deg(\tilde{L}) < \frac{d}{2}$ . On the other hand, we can show that  $h^0(\tilde{L}) \geq 3$ .

Note that  $M_L$  is unstable and that the line bundle  $S := \tilde{L}^{\vee}$  is a destabilizing subbundle of  $M_L$ , in fact  $-\deg(\tilde{L}) = \mu(S) > \mu(M_L) = -\frac{d}{2}$  by contradiction hypothesis.

Let Q be the quotient line bundle of  $M_L$  by S.

Claim:  $h^0(\tilde{L}) \ge 3$ .

Proof of the **Claim**: first notice that Q satisfies the following:

$$\deg(Q) = \deg(M_L) - \deg(S) = -d - \deg(S) < -d + \frac{d}{2} = -\frac{d}{2} < 0.$$

From Lemmas 2.1 and 2.2, there exists an unique globally generated vector bundle F of rank two which fits into the following exact sequence

$$0 \to Q \to F \stackrel{\alpha}{\to} L \to 0.$$

Since  $H^0(Q) = 0$ , we complete the diagram

$$0 \longrightarrow M_F \longrightarrow H^0(F) \otimes \mathcal{O}_C \longrightarrow F \longrightarrow 0$$

$$\downarrow^{\alpha_1} \qquad \qquad \downarrow^{\alpha_2} \qquad \qquad \downarrow^{\alpha}$$

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0,$$

such that  $\alpha_2$  is an isomorphism i.e  $H^0(F) \simeq H^0(L)$ . By snake lemma, we have an exact sequence

$$\ker(\alpha_1) \to \ker(\alpha_2) \to \ker(\alpha) \to \operatorname{coker}(\alpha_1) \to \operatorname{coker}(\alpha_2) \to \dots$$

where ker( $\alpha_1$ ) = ker( $\alpha_2$ ) = {0}, ker( $\alpha$ ) = Q, coker( $\alpha_2$ ) = {0} since  $\alpha_2$ : ( $H^0(F) \rightarrow H^0(L)$ ) is an isomorphism. Thus, ker( $\alpha$ )  $\simeq$  coker( $\alpha_1$ ),  $Q = M_L/M_F$  and  $M_F \simeq S$ , that is, we have the following exact sequence

$$0 \to S \to H^0(F) \otimes \mathcal{O}_C \to F \to 0$$

Dualizing the above exact sequence and taking cohomology, we get

$$0 \to H^0(F^{\vee}) \to H^0(F)^{\vee} \to H^0(S^{\vee}) \to \cdots$$

Since F is globally generated without trivial summands (see lemma 2.2 and lemma 2.5), we have  $h^0(F^{\vee}) = 0$ . So  $h^0(S^{\vee}) \ge h^0(F) = 3$ . This proves the claim.

So from the first part, we have  $h^0(\tilde{L}) = h^0(S^{\vee}) = 2$  and from the Claim we have  $h^0(\tilde{L}) \geq 3$ , which is a contradiction. Thus, L is linearly semistable and this complete the proof.

COROLLARY 3.8. Let L be globally generated line bundle with  $h^0(L) = 3$ inducing a birational map  $\phi_L : C \to \mathbb{P}^2$ . Denote  $\overline{C} \subset \mathbb{P}^2$  its image and d the degree of  $\overline{C}$  in  $\mathbb{P}^2$ . Then, for any point  $p \in \overline{C}$ , we have multiplicity  $m_p \leq \frac{d}{2}$ .

PROOF. The proof follows from Theorem 3.7 and [8, Proposition 8.1].

REMARK 3.9. Let L be a globally generated line bundle over a curve C of degree d such that  $h^0(L) = 3$ . From the exact sequence

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0$$

we have for any line bundle B the sequence

$$0 \to M_L \otimes B \to H^0(L) \otimes B \to L \otimes B \to 0.$$

So, if there exists a line bundle B on C with  $h^0(L \otimes B) < 3h^0(B)$ , then  $h^0(M_L \otimes B) \neq 0$ , which implies that  $O_C \hookrightarrow M_L \otimes B$ , that is,  $B^{\vee} \hookrightarrow M_L$ . We have that  $M_L$  is unstable if  $2 \operatorname{deg}(B) < d$ . Thus, should be interesting to find some numerical conditions to find a linearly semistable line bundle L with  $M_L$  unstable.

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