On stability of tautological bundles and their total transforms

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Abstract. Through the use of linearized bundles, we prove the stability of tautological bundles over the symmetric product of a curve and of the kernel of the evaluation map on their global sections.

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1. Introduction

Let C be a smooth projective curve, and E a globally generated vector bundle on C. The vector bundle $M_E := \ker(H^0(C, E) \otimes \mathcal{O} \to E)$ is considered in many works in the literature (cf. [EL92], [But94] [Mis06], [Mis08], [MS12], [BBPN15], and many others), mostly when E is a line bundle, and its stability or semistability plays a crucial role. In recent works (cf. [ELM13], [FO12], [Fey16]) the same problem arises on higher dimensional varieties and with E being a higher rank vector bundle, and stability of M_E is proven with various advanced techniques.

The vector bundle M_E is sometimes called Lazarsfeld-Mukai bundle, but other times this term is used for other similar bundles on K3 surfaces, so we will call this *total transform bundle* of the vector bundle E on the variety X.

The purpose of this work is to give some examples of stability of the total transform bundle of a stable vector bundle on a higher dimensional variety, using elementary techniques. The examples treated are vector bundles on symmetric products of curves, and can be summarized as the following:

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Theorem 1.1. *Let C be a smooth projective curve of genus $g \geqslant 1$ over an algebraically closed field of characteristic* 0*. Let L be a degree d line bundle on C. Let S^nC be the symmetric product of C, $\tilde{H}$ the natural polarization on S^nC, and $L^{[n]}$ the tautological bundle on S^nC. Then*

1. *if $d \geqslant n$ the vector bundle $L^{[n]}$ is $\tilde{H}$-stable;*
2. *if $d \geqslant n + 2g$ the total transform $M_{L^{[n]}}$ is $\tilde{H}$-stable.*

The article is organized as follows: in section 2 we recall the basic definitions and properties of group actions and linearizations that we make use of, and we prove that G-linearized vector bundles have G-linearized destabilizations (when unstable). In section 3.1 we prove the stability of the tautological vector bundles $L^{[n]}$ on the symmetric product of a curve. In section 3.2 we prove the stability of the total transform of those tautological bundles.

The technique we use is basically the observation that stability, or rather poly-stability, is invariant when passing to a finite covering. We consider the quotient map from the product to the symmetric product of the curve. Then we obtain linearized vector bundles, and we show that having a linearization poses remarkable restrictions on a possible destabilization.

1.1. Notations

By a variety we mean a smooth projective variety X over an algebraically closed field. A polarization will be an ample divisor on X. We will denote by $Z_1 \equiv Z_2$ numerical equivalence between cycles Z_1 and Z_2.

2. Stability and group actions

Let X be a variety with an action of an algebraic group G. We recall that a G-linearized sheaf on X is a sheaf E on X with an isomorphism $\Phi_g \colon E \xrightarrow{\sim} g^*E$ for all $g \in G$, satisfying the usual cocycle conditions.

A morphism $\psi \colon E \to F$ of G-linearized sheaves is G-equivariant if the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ \Phi_g \wr \downarrow & & \downarrow \wr \Phi'_g \\ g^*E & \xrightarrow{g^*\psi} & g^*F \end{array}$$

commutes for all $g \in G$.

Definition 2.1. Let H be a divisor on X, we say that H is *numerically G-invariant*, if for all $g \in G$ we have $g^*H \equiv H$.

Definition 2.2. Let E be a G-linearized sheaf on X, we say that a subsheaf $F \subset E$ is a *G-equivariant subsehaf* if we have the equality of subsheaves of g^*E

$$\Phi_g(F) = g^*F$$

for all $g \in G$. That is, the G-linearization of E induces a G-linearization on F such that the embedding $F \subset E$ is G-equivariant.

We recall that the slope of a torsion free sheaf E on X with respect to a polarization H is

$$\mu_H(E) = \frac{c_1(E).H^{n-1}}{\mathrm{rk}E} \ .$$

A sheaf is called semistable (respectively stable) if $\mu_H(F) \leqslant \mu_H(E)$ (respectively $\mu_H(F) < \mu_H(E)$) for every subsheaf $F \subset E$ with smaller rank. Furthermore every torsion free sheaf E admits a unique maximal semistable subsheaf $F \subset E$ such that the slope of F is maximal among the slopes of subsheaves of E and any subsheaf with same slope as F is contained in F. A sheaf is semistable if and only if it coincides with its maximal semistable subsheaf.

We have the following property

Proposition 2.3. *Let $F \hookrightarrow E$ be the maximal semistable subsheaf with respect to the polarization H, where E is a G-linearized torsion free sheaf, and H is a numerically G-invariant ample divisor. Then $F \hookrightarrow E$ is a G-equivariant subsheaf.*

Proof. Consider the following diagram

$$\begin{array}{ccc} F & \hookrightarrow & E \\ \wr\downarrow & \circlearrowleft & \downarrow\wr \\ \varphi(F) & \hookrightarrow & g^*E \\ g^*F & \nearrow & \end{array}$$

where the isomorphism $\varphi = \Phi_g \colon E \tilde{\rightarrow} g^*E$ is given by the G-linearizaion of E. We want to show that $\varphi(F)$ and g^*F are the same subbundle of g^*E, *i.e.* the linearization of E induces a linearization of F.

We will show that they both are semistable subsheaves of maximal slope of g^*E. First notice that g^*F is a subbundle of g^*E, and its degree is given by

$$c_1(g^*F).H^{n-1} = g^*c_1(F).g^*H^{n-1} = g^*(c_1(F).H^{n-1}) = c_1(F).H^{n-1} \ .$$

By the same computation we see that the slope of a sheaf is invariant by the action of G, hence also g^*F is semistable, and it is the semistable maximal subsheaf of g^*E.

As φ is an isomorphism of sheaves, then $c_1(\varphi(A)) = c_1(A)$ and $\mu_H(\varphi(A)) = \mu_H(A)$ for all subsheaves $A \subset E$. Therefore $\varphi(F)$ is the maximal semiastable subsheaf of g^*E. Then $\varphi(F) = g^*F$, as we have proven that g^*F is the maximal subsheaf of g^*E. Hence, the linearization on E induces a linearization on F, and clearly the inclusion morphism is G-equivariant.

□

We see from this proposition that having a group action poses strict conditions on linearized sheaves for being destabilized, and this is what we use to investigate the stability of tautological sheaves. Next lemma will be useful in the following sections:

Lemma 2.4. *Let I be a finite set together with a transitive G action. Let E_i be coherent sheaves on X, and let $E = \oplus_{i\in I} E_i$ carry a G-linearization Φ such that $\Phi_g(E_i) = g^* E_{g(i)}$ for every $i \in I$ and $g \in G$. Let $\pi_i \colon E \to E_i$ be the i-th projection. Then for every G-equivariant subsheaf $F \subset E$ such that $\pi_i(F) = 0$ for some i, we already have $F = 0$.*

Proof. To prove that $F = 0$ we need to prove that $\pi_j(F) = 0$ for all $j \in I$. Let $g \in G$ be an element of the group G, and call $j = g(i)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} F & \hookrightarrow & E & \xrightarrow{\pi_i} & E_i \\ \wr\downarrow \Phi_g & & \wr\downarrow \Phi_g & & \wr\downarrow \Phi_g \\ g^*F & \hookrightarrow & g^*E & \xrightarrow{g^*\pi_j} & g^*E_j \end{array}$$

where the left square commutes because the subsheaf $F \subset E$ is G-equivariant, and the right square commutes because of the assumption of the compatibility between Φ and the G-action on I. Now, as $\pi_i(F) = 0$, the composition of the top row is 0, so the composition of the bottom row is 0 as well. Then $g^*\pi_j(g^*F) = 0$, so $\pi_j(F) = 0$. Applying this to all g, we have that $\pi_j(F) = 0$ for all $j \in I$. □

3. Symmetric product of a curve

In this section we define tautological sheaves and their total transforms.

Let C be a smooth projective curve, $X = C^n$ the cartesian product of C, and $G = \mathfrak{S}_n$ the symmetric group acting on X permutating the factors.

As C is a smooth curve, the symmetric product $X/G := S^nC$ is a smooth variety. In fact it coincides with the Hilbert scheme of length-n 0-dimensional subschemes of C.

We will denote the elements of S^nC, wich are n-tuples of points of C order free, by $x_1 + \cdots + x_n$.

We can consider the universal family associated to the Hilbert scheme: the universal subscheme $Z \subset S^nC \times C$ consists of all the couples (ξ, x) where ξ is a 0-dimensional subscheme, and x is a point of X lying in ξ. There are two natural projections π_1 and π_2 of $S^nC \times C$ to S^nC and C, respectively.

On the product $S^nC \times C$ we have the following exact sequence:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{S^nC\times C} \to \mathcal{O}_Z \to 0 \ .$$

As C is a curve, then Z is a divisor in $S^nC \times C$, and $\mathcal{I}_Z = \mathcal{O}_{S^nC\times C}(-Z)$. Given a vector bundle E on C, a *tautological bundle* $E^{[n]}$ on S^nC is defined as follows:

$$E^{[n]} := \pi_{1*}(\pi_2^* E \otimes \mathcal{O}_Z) \ ,$$

so, on a point $\xi \in S^nC$ the vaector bundle $E^{[n]}$ has fiber

$$E^{[n]}(\xi) = H^0(\xi, E_{|\xi}) \ .$$

From the exact sequence above we get an exact sequence

$$0 \to \pi_{1*}(\pi_2^* E \otimes \mathcal{O}(-Z)) \to H^0(C, E) \otimes \mathcal{O}_{S^nC} \to E^{[n]} \to R^1\pi_{1*}(\pi_2^* E \otimes \mathcal{O}(-Z)) \ .$$

By projection formula (cf. [Sca05] pages 34 for generalised projection formulas, and 43 for the computation below)

$$H^*(S^nC, E^{[n]}) = H^*(C,E) \otimes S^{n-1}H^*(C,\mathcal{O}_C) \ ,$$

hence, $H^0(S^nC, E^{[n]}) = H^0(C,E)$, and in the sequence above the map $H^0(C,E) \otimes \mathcal{O}_{S^2C} \to E^{[n]}$ is the evaluation map.

When $E^{[n]}$ is globally generated, we call N_E the total transform of $E^{[n]}$, *i.e.*

$$N_E := \pi_{1*}(\pi_2^* E \otimes \mathcal{O}(-Z)) \ .$$

In the following we want to investigate about the stability of the tautological sheaf $L^{[n]}$ of a line bundle L on the curve C, and of its total transform N_L.

Remark 3.1. We recall that, in characteristic 0, if $\varphi\colon X \to Y$ is a finite morphism, and M is a torsion free coherent sheaf on Y, then M is polystable if and only if φ^*M is polystable. More precisely, if M is (poly)stable, then φ^*M is polystable. And, *in any characteristic*, if φ^*M is (semi)stable, then M is (semi)stable (cf. [HL97], chapter 3).

3.1. Tautological sheaf of a line bundle on S^nC

We apply in this section the methods described above to show stability results on S^nC. In order to apply the 2 lemmas stated below, we restrict to characteristic 0 throughout this and the following section. However, we expect that stability of the tautological sheaf hold in any characteristic.

We will consider the ample divisor $H = \Sigma_{i=1}^n p_i^{-1}(p)$ in C^n, for some point $p \in C$, which is the pull-back of the divisor $\tilde{H} = p + S^{n-1}C$ in S^nC. Let us call

$$f_i := p_i^{-1}(p)$$

the hypersurface in C^n. Then

$$H = \Sigma_{i=1}^n f_i \ , \ H^{n-1} = \Sigma_{i=1}^n (n-1)! f_1 \dots \hat{f_j} \dots f_n \ , \ \text{ and } H^n = n! \ ,$$

furthermore the numerical class of $f_1 \dots \hat{f_j} \dots f_n$ is represented by the curve $x_1 \times \cdots \times x_{j-1} \times C \times x_{j+1} \times \cdots \times x_n$, for any $(x_1, \dots, \hat{x_j}, \dots, x_n) \in C^{n-1}$.

We call as usual $\Delta = \{x_1 + \cdots + x_n \in S^nC \mid \exists i \neq j \text{ with } x_i = x_j\}$ the diagonal in S^nC, it is a divisor divisible by 2, as it is the branch locus of the quotient map $\sigma\colon C^n \to S^nC$.

We will the use the following computations:

Lemma 3.2. $c_1(L^{[n]}) \equiv (\deg L)\tilde{H} - \frac{\Delta}{2}$

Proof. From Göttsche's appendix in [BS91], we know that

$$c_1(L^{[n]}) = \tilde{L}^{\boxtimes n} - \frac{\Delta}{2} \ ,$$

where $\tilde{L}^{\boxtimes n}$ is the unique line bundle on S^nC such that its pull-back via $\sigma\colon C^n \to S^nC$ is equal to $L^{\boxtimes n} = p_1^*L \otimes \dots p_n^*L$. And we can verify easily that $L^{\boxtimes n} \equiv \deg L(f_1 + \cdots + f_n) = (\deg L)H$ and $\tilde{L}^{\boxtimes n} \equiv (\deg L)\tilde{H}$.

□

Lemma 3.3. $c_1(\sigma^* L^{[n]}).H^{n-1} = n!(\deg L - n + 1)$

Proof. This follows from the equalities $\tilde{H}^n = 1$ and $\frac{\Delta}{2}.\tilde{H}^{n-1} = n - 1$.

In fact a we can choose a representative in the numerical class of $\tilde{H}$ as the divisor $p_j + S^{n-1}C$, with p_j a point of C, then choosing n distinct points $p_1, \dots, p_n$ and n corrensponding divisors $\tilde{H}_j = p_j + S^{n-1}C$ numerically equivalent to $\tilde{H}$, then the divisors $\tilde{H}_j$ intersect properly and $\tilde{H}_1 \cap \cdots \cap \tilde{H}_n = \{p_1 + \cdots + p_n\}$ is a point in S^nC.

In the same way the numerical class of the cycle $\tilde{H}^{n-1}$ is represented by $p_1 + \dots p_{n-1} + C$, and the divisor Δ is the divisor with support $\{x_1 + \cdots + x_n \in S^nC \mid \exists i \neq j \text{ with } x_i = x_j\}$, so it intersects the cycle $p_1 + \cdots + p_{n-1} + C$ exactly in $n - 1$ points (each of which with multiplicity 2).

Therefore $c_1(L^{[n]}).\tilde{H}^{n-1} = ((\deg L)\tilde{H} - \frac{\Delta}{2}).\tilde{H}^{n-1} = (\deg L - n + 1)$, and the statement of the lemma follows, as σ is a degree $n!$ cover.

□

We can now prove the first part of Theorem 1.1 stated in the introduction:

Theorem 3.4. *Let L be a line bundle of degree $d \geqslant n$ on C, then $L^{[n]}$ is an $\tilde{H}$-stable vector bundle on S^nC.*

Proof. Let us suppose that there exists a destabilization of $L^{[n]}$, *i.e.* an injection of sheaves $\tilde{F} \hookrightarrow L^{[n]}$, such that $\mu_{\tilde{H}}(\tilde{F}) \geqslant \mu_{\tilde{H}}(L^{[n]})$, with $\tilde{F}$ torsion free of rank $r < n$. As we remark at the end of the proof, we can assume that $\tilde{F}$ is locally free.

Let us set $F := \sigma^*\tilde{F}$. Pulling the injection $\tilde{F} \hookrightarrow L^{[n]}$ back to C^n by the quotient map $\sigma \colon C^n \to S^nC$ we have an injection $F \hookrightarrow \sigma^* L^{[n]}$.

Let us consider the vector bundle $L^{\boxplus n} = \bigoplus_{i=1}^n p_i^* L$ and remark that it carries a $\mathcal{S}_n$-linearization given by the direct sum of the natural isomorphisms $p_i^* L \tilde{\to} g^* p_{g(i)}^* L$, for $g \in \mathcal{S}_n$.

We have a natural $\mathcal{S}_n$-equivariant injection $\sigma^* L^{[n]} \hookrightarrow \bigoplus_{i=1}^n p_i^* L$. In fact $L^{[n]} \cong (\sigma_* L^{\boxplus n})^{\mathcal{S}_n}$ (cf. [BL11] Proposition 1.1), then by adjunction we get a map

$$\sigma^* L^{[n]} \to L^{\boxplus n}$$

which is injective on the generic point on S^nC (choose n distinct points in C) and therefore it is an injective map of sheaves. Furthermore, it makes $\sigma^* L^{[n]}$ an equivariant subsheaf of $L^{\boxplus n}$ as the linearization induces a linearization on $\sigma^* L^{[n]}$.

Hence we have a $\mathcal{S}_n$-equivariant injective morphism

$$F \hookrightarrow \bigoplus_{i=1}^n p_i^* L\ .$$

As $A := \bigwedge^r F = \det F$ and $\bigwedge^r \bigoplus_{i=1}^n p_i^* L$ carry linearizations induced by those of F and $\bigoplus_{i=1}^n p_i^* L$, we have a $\mathcal{S}_n$-equivariant morphism

$$\psi \colon A \to \bigwedge^r \bigoplus_{i=1}^n p_i^* L \ .$$

We claim that this morphism must be zero, hence the maps $F \to \sigma^* L^{[n]}$ and $\tilde{F} \to L^{[n]}$ cannot be injective.

Decomposing $\bigwedge^r \bigoplus_{i=1}^n p_i^* L$ as $\bigoplus_{|J|=r} L_J$, where $L_J = p_{j_1}^* L \otimes \cdots \otimes p_{j_r}^* L$, we decompose the map $\psi = \bigoplus \psi_J \colon A \to \bigoplus L_J$ as well.

Using the notations at the beginning of the section: $f_i := p_i^{-1}(p) \subset C^n$, $H = \Sigma_{i=1}^n f_i$, $H^{n-1} = \Sigma_{i=1}^n (n-1)! f_1 \dots \hat{f_j} \dots f_n$, and $H^n = n!$, the numerical class $f_1 \dots \hat{f_j} \dots f_n$ being represented by the curve $x_1 \times \cdots \times x_{j-1} \times C \times x_{j+1} \times \cdots \times x_n$, for any $(x_1, \dots, \hat{x_j}, \dots, x_n) \in C^{n-1}$, we have that

$$c_1(F).H^{n-1} = c_1(\det F).H^{n-1} \geqslant \frac{r}{n} c_1(\sigma^* L^{[n]}).H^{n-1} = \frac{r}{n}(n!)(d-n+1) > 0 \ ,$$

where the last equality is given by Lemma 3.3.

We can suppose that $c_1(F).(f_2.f_3 \dots f_n) > 0$. Hence

$$\deg(A_{|C \times x_2 \times \cdots \times x_n}) > 0 \ , \quad \text{for all } (x_2 \dots, x_n) \in C^{n-1} \ .$$

For all J such that $1 \notin J$, $L_{J|C \times x_2 \times \cdots \times x_n} = \mathcal{O}$. Hence, for all such J, $\psi_J \colon A \to L_J$ is the zero map, as we can see by restriction to the curve $C \times x_2 \times \cdots \times x_n$.

Then we can apply Lemma 2.4: if the map ψ_J vanishes for a J, then the whole map ψ vanishes, and the morphism $F \to \bigoplus_{i=1}^n p_i^* L$ cannot be injective.

The assumption that F is locally free is not limiting, as for any F torsion free of rank r, we have a line bundle A, such that $c_1(A) = c_1(F)$, with $A = \bigwedge^r F$ on the locus where F is locally free, and such that $F \to \bigoplus_{i=1}^n p_i^* L$ induces $A \to \bigwedge^r \bigoplus_{i=1}^n p_i^* L$. This last map being zero, the map $F \to \bigoplus_{i=1}^n p_i^* L$ cannot be injective.

□

We recall that in characteristic 0, if we show that the vector bundle $L^{[n]}$ is stable with respect to the polarization $\tilde{H}$, then the vector bundle $\sigma^* L^{[n]}$ is polystable with respect to H (cf. Remark 3.1). Thus, we have the following

Corollary 3.5. *The vector bundle $\sigma^* L^{[n]}$ on C^n is poly-stable with respect to the polarization H.*

3.2. Transform of the tautological sheaf of a line bundle

In this paragraph we show that when $\deg L > 2g + n$ the total transform

$$N_L = M_{L^{[n]}} = \ker(H^0(C, L) \otimes \mathcal{O}_{S^n C} \twoheadrightarrow L^{[n]})$$

is stable. The proof is inspired by Ein and Lazarsfeld's proof of the stability of M_L for line bundles (cf. [EL92]), in this work the stability of the Picard bundle is shown proving the stability of a total transform:

Theorem 3.6 (Ein-Lazarsfeld). *Let L be a line bundle of degree $d > 2g$ (respectively $d \geqslant 2g$) on a smooth projective curve of genus g, then the total transform M_L is stable (respectively semistable).*

Using a similar argument we show the following:

Lemma 3.7. *Let $x_1, \dots, x_{n-1} \in C$ be fixed distinct points, and let $i \colon C \to S^nC$ mapping $i(x) = x_1 + \dots + x_{n-1} + x$ be the embedding with image $x_1 + \dots + x_{n-1} + C \subset S^nC$. Then $i^*(N_L) = M_{L(-x_1 \cdots - x_n)}$.*

Proof. Let us remark that, for every $P \in \mathrm{Pic}C$, we have an isomorphism

$$M_P \cong pr_{1*}(pr_2^* P \otimes \mathcal{O}_{C\times C}(-\delta))$$

where $\delta \subset C \times C$ is the diagonal. As we remarked at the beginning of this section, the total transform of the tautological bundle N_L, is given by the formula:

$$N_L := \pi_{1*}(\pi_2^* L \otimes \mathcal{O}(-Z)) \ ,$$

where $Z \subset S^nC \times C$ is the universal family and is a divisor, and π_1, π_2 are the projections. Furthermore, the pull-back of the universal family Z through i is the closed subset:

$$Y := i^*Z = (i \times id_C)^{-1}(Z) = \{(x,y) \mid y \in \{x_1, \dots, x_{n-1}, x\}\} \subset C \times C \ .$$

This is a divisor in $C \times C$ with n irreducible components:

$$\delta \ , \ pr_2^{-1}(x_1), \dots, pr_2^{-1}(x_{n-1}) \ ,$$

therefore $\mathcal{O}_{C\times C}(-Y) = \mathcal{O}_{C\times C}(-\delta) \otimes pr_2^* \mathcal{O}_C(-x_1 - \dots - x_{n-1})$.

Now by flat base change,

$$i^* N_L = i^*(\pi_{1*}(\pi_2^* L \otimes \mathcal{O}(-Z))) = pr_{1*}(pr_2^* L \otimes (i \times id_C)^* \mathcal{O}(-Z)) =$$
$$= pr_{1*}(pr_2^* L \otimes (i \times id_C)^* \mathcal{O}(-Z)) = pr_{1*}(pr_2^* L(-x_1 - \dots - x_{n-1}) \otimes \mathcal{O}(-\delta)) =$$
$$= M_{L(-x_1 - \dots - x_{n-1})} \ .$$

□

Finally, we can prove a stability result for the total transforms of tautological sheaves on the symmetric product of a curve, *i.e.* the second part of Theorem 1.1:

Theorem 3.8. *Let C be a smooth curve of genus $g \geqslant 1$, and let L be a line bundle on C of degree $d \geqslant 2g + n$. Then N_L is a stable bundle on S^nC with respect to the polarization $\tilde{H}$.*

Proof. We just need to remark that the class of $x_1 + \dots x_{n-1} + C$ is numerically equivalent to $\tilde{H}^{n-1}$, and that we know stability when restricting to that curve. In greater detail: suppose by contradiction that there is a destabilizing subsheaf $F \subset N_L$ on S^nC (where F can be supposed to be a reflexive sheaf without loss of generality), with

$$\mu_{\tilde{H}} = \frac{F.\tilde{H}^{n-1}}{\mathrm{rk}F} \geqslant \mu_{\tilde{H}}(N_L) \tag{3.1}$$

then by choosing sufficiently general points $x_1, \dots, x_{n-1} \in C$ and considering the injection i defined above, we can suppose that we have an injection $i^*F \subset i^*N_L$ of vector bundles. However as $i^*(N_L) = M_{L(-x_1 \cdots -x_n)}$ by Lemma 3.7, this vector bundle on C being stable by Ein and Lazarsfeld result, and $\mu(i^*N_L) = \frac{N_L.\tilde{H}^{n-1}}{\mathrm{rk} N_L} = \mu_{\tilde{H}}(N_L)$, then equality (3.1) cannot hold.

□

Remark 3.9. As according to Proposition 2.3 G-linearized vector bundles are destabilized by G-linearized subbundles, it would be interesting to apply this in a more general setting of a (finite) group action, or for tautological bundles of higher rank: in fact an earlier (unpublished) version of this work has been recently extended by Andreas Krug in [Kru18].

Question 3.10. In the recent works (cf [BKK$^+$15], [MU19], [Mis18], [Mis19]) we consider globally generated vector bundles and their base loci, in order to get positivity properties, and construct Iitaka fibrations. It would be interesting to understand whether these loci can be used in order to understand positivity properties of tautological bundles and their Iitaka fibrations.

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