A remark on stability and restrictions of vector bundles to hypersurfaces

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Abstract

We prove that a vector bundle on a smooth projective variety is (semi)stable if the restriction on a fixed ample smooth subvariety is (semi)stable

1 Introduction

The purpose of this work is to show a property of slope-stability of vector bundles with respect to restriction to a given ample subvariety.

Given a slope-stable vector bundle E on a projective variety X, it is rather difficult to prove that the restriction of E to an ample subvariety is stable. This can be done for general subvarieties of sufficiently high degree (cf. [MR82, Fle84]).

On the converse, it is easy to show that, if the restriction of E on a *general* ample subvariety of high degree is slope-stable, then E is slope-stable as well.

The purpose of this work is to show that if the restriction of $E_{|Y}$ to *one* given smooth ample hypersurface $Y \subset X$ (of any degree) is (semi)stable, then the vector bundle E is (semi)stable on X. This is sometimes stated, however it does not appear explicitly in the literature, so we think it is useful to write it here, for future use and for the interest of the result in itself.

1.1 Notation and main definitions

Throughout this work X is a smooth projective variety of dimension n over an algebraically closed field k.

Intersection products, pull-backs, push-forwards, and Chern classes will be considered in the Chow ring $A^*(X)$ with integral coefficients, and we will identify divisors in $A^1(X)$ with line bundles when useful. For a 0-cycle $W = \sum \lambda_i p_i$, with $p_i \in X$, $\lambda_i \in \mathbb{Z}$ we will denote by

$$< W >_X = \sum \lambda_i \in \mathbb{Z}$$

its degree. For any higher dimensional cycle Z we will set $< W >_X = 0$. Therefore the intersection number of two classes $a, b \in A^*(X)$ with complementary dimensions will be denoted $< a.b >_X$.

We will say that two cycles $a, b \in A^*(X)$ are numerically equivalent when $< a.c >_X = < b.c >_X$ for every cycle $c \in A^*(X)$.

Definition 1.1. Let H be an ample divisor on X. Let F be a torsion free sheaf on X. Let n be the dimension of X.

i. We call slope of F (with respect to the polarization H) the rational number

$$\mu_H(F) := \frac{< c_1(F).H^{n-1} >_X}{\mathrm{rk}F}$$

ii. We say that a vector bundle E on X is (semi)stable if for every torsion free subsheaf $F \subset E$ with $\mathrm{rk}F < \mathrm{rk}E$ the slopes satisfy

$$\mu_H(F) < \mu_H(E) \quad (\mu_H(F) \leqslant \mu_H(E) \text{ for semistability})$$

2 Lemmata

The following is the first useful observation:

Lemma 2.1. *Let F be a torsion free sheaf on X. Suppose that F is locally free outside of a subset $Z \subset X$ of codimension at least 3 in X. If $Y \subset X$ is a hypersurface and H is its class in $A^1(X)$, then*

$$< c_1(F).H^{n-1} >_X = < c_1(F_{|Y}).(H_{|Y})^{n-2} >_Y \quad .$$

Proof. Let us first remark that, if $i \colon Y \hookrightarrow X$ is the immersion of Y in X, then

$$c_1(F_{|Y}) = i^* C_1(F) \ .$$

In fact this is obvious when F is a vector bundle. In general notice that $c_1(F_{|Y})$ and $i^* C_1(F)$ are two line bundles on Y, isomorphic on an open subset $U = Y \setminus Z$ whose complementary $Y \cap Z$ has codimension at least 2 in Y, therefore they are isomorphic. Hence $c_1(F_{|Y}).(H_{|Y})^{n-2} = i^*(c_1(F).H^{n-2})$.

To complete the proof observe that $< w >_Y = < i_* w >_X$ for any cycle $w \in A^*(Y)$, and that by projection formula:

$$i_*(c_1(F_{|Y}).(H_{|Y})^{n-2}) = i_*([Y].i^*(c_1(F).H^{n-2})) = i_*[Y].(c_1(F).H^{n-2}) \ ,$$

where $[Y]$ is the identity class in $A^*(Y)$, and clearly $i_*[Y] = H$.

□

Lemma 2.2. *Let E be a vector bundle on X.*

i. Suppose that $\mu_H(F) < \mu_H(E)$ for all subsheaf $F \subset E$ such that the quotient E/F is torsion free, then E is stable.

ii. Suppose that $\mu_H(F) < \mu_H(E)$ for all subsheaf $F \subset E$ such that $F \cong F^{**}$, where the dual sheaf of F is defined as the sheaf of homomorphisms $F^* = \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$. Then E is stable.

Corresponding statements can be made on semistability.

Proof. To prove stability we have to consider the slope of any subsheaf $F \subset E$.

Suppose that condition (i) holds. Given any subsheaf $F \subset E$ consider the exact sequence $0 \to F \to E \to G \to 0$. If $G = E/F$ is not torsion free consider its torsion $T = T(G)$ and its torsion free quotient $G \twoheadrightarrow G' = G/T$. Then

$$F \subset F' := \ker(E \twoheadrightarrow G') \subset E \ , \text{ and } F'/F = \ker(G \twoheadrightarrow G') = T(G)$$

Now observe that $c_1(T)$ is effective, in fact $c_1(T) = c_1(F') \otimes c_1(F)^*$, and as $F \subset F'$ then the line bundle $c_1(T)$ has a section. Therefore

$$< c_1(F).H^{n-1} >_X \leqslant < c_1(F').H^{n-1} >_X \ \ , \text{ and } \mu_H(F) \leqslant \mu_H(F') < \mu_H(E) \ .$$

Suppose now that condition (ii) holds. Given any subsheaf $F \subset E$ consider the injection of F in its bidual: $F \hookrightarrow F^{**}$. Now F^{**} is a subsheaf of E as well. In fact morphisms from F to E factor through $F \hookrightarrow F^{**}$, as it can be seen from the computation of homomorphism from F to E, recalling that E is a vector bundle:

$$\mathcal{H}om_{\mathcal{O}_X}(F, E) = F^* \otimes E = (F^{**})^* \otimes E = \mathcal{H}om_{\mathcal{O}_X}(F^{**}, E) \ .$$

Therefore $F \subset F^{**} \subset E$ and we can proceed ads above.

□

The following is a well known lemma (cf. [Har94], Lemma 1.5 and Theorem 1.9):

Lemma 2.3. *Let F be a coherent sheaf on X, then*

i. F is torsion-free if and only if it satisfies Serre's condition S1: i.e. *for all schematic points $x \in X$*

$$\operatorname{depth}(F_x) \geqslant \min\{1, \dim \mathcal{O}_{X,x}\} \ .$$

ii. F is reflexive if and only if it satisfies Serre's condition S2: i.e. *for all schematic points $x \in X$*

$$\operatorname{depth}(F_x) \geqslant \min\{2, \dim \mathcal{O}_{X,x}\} \ .$$

Corollary 2.4. *Let F be a reflexive (respectively torsion-free) coherent sheaf on X. The singular locus of F,*

$$\operatorname{sing}(F) = \{x \in X \mid F_x \textit{ is not a free } \mathcal{O}_{X,x}\textit{-module } \} \ ,$$

has codimension at least 3 (respectively at least 2).

Proof. Suppose that F is a reflexive sheaf, let $Z \subset X$ be an integral subscheme of codimension at most 2, and let $z \in X$ be the generic point of Z. Then $\mathcal{O}_{X,z}$ has dimension at most 2. Therefore

$$\operatorname{depth}(F_z) \geqslant \dim \mathcal{O}_{X,z} \ .$$

But as X is a smooth variety, then $\mathcal{O}_{X,x}$ is a regular ring for all $x \in X$, so it satisfies the Auslander-Buchsbaum formula (cf. [AB57]): for any coherent sheaf F and any schematic point $x \in X$

$$\operatorname{depth}(F_x) + \operatorname{dh}(F_x) = \dim \mathcal{O}_{X,x} \ ,$$

where $\operatorname{dh}(F_x)$ is the minimal length of projective resolutions of F_x. In particular for $z \in X$ above, we have $\operatorname{dh}(F_z) = 0$, therefore F_z is a free $\mathcal{O}_{X,z}$-module. So $\operatorname{sing}(F)$ cannot contain any 2-codimensional subscheme.

The same argument applies to the codimension of the singular locus of a torsion-free sheaf.

□

3 Main Theorem

We give in this section the proof of the main theorem stated in the introduction.

Theorem 3.1. *Let X be a smooth projective variety of dimension at least 2. Let H be an ample divisor on X. Let E be a vector bundle on X, and $Y \subset X$ a fixed smooth hypersurface numerically equivalent to a multiple mH, with $m \in \mathbb{Q}^{>0}$. Assume that $E_{|Y}$ is (semi)stable with respect to the polarization $H_{|Y}$. Then E is (semi)stable on X with respect to the polarization H.*

Proof. Let us prove the statement concerning stability, the argument being essentially the same for semistability.

We want to prove that given a subsheaf $F \hookrightarrow E$, we have $\mu_H(F) < \mu_H(E)$, knowing that this property holds on Y.

According to Lemma 2.2 we can suppose that F is reflexive and that $G := E/F$ is torsion-free, so by Corollary 2.4 the singular locus of F,

$$\operatorname{sing}(F) = \{x \in X \mid F_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module } \} \ ,$$

has codimension at least 3, and $\operatorname{sing}(G)$ has codimension at least 2 in X.

Let us restrict the exact sequence $0 \to F \to E \to G \to 0$ to the hypersurface Y:

$$0 \to \mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y) \to F_{|Y} \to E_{|Y} \to G_{|Y} \to 0 \ ,$$

and let us remark that, as $\operatorname{codim}_X \operatorname{sing}(F) \geqslant 3$, then $\operatorname{codim}_Y \operatorname{sing}(F_{|Y}) \geqslant 2$.

As $\operatorname{supp}(\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y)) \subset \operatorname{sing}(G) \cap Y$, and $\operatorname{codim}_X \operatorname{sing}(G) \geqslant 2$ in X, then $\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y)$ is a torsion sheaf on Y, injecting in $F_{|Y}$. So its support must be contained in $\operatorname{sing}(F_{|Y})$.

Hence $c_1(F_{|Y}) = c_1(F_{|Y}/\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y))$, because quotienting by subsheaves concentrated on high codimension subsets does not affect the first Chern class.

On Y we have the following exact sequence

$$0 \to F_{|Y}/\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y) \to E_{|Y} \to G_{|Y} \to 0 \ .$$

Hence by stability of $E_{|Y}$ we have that

$$\frac{< c_1(F_{|Y}).H^{n-2} >_Y}{\mathrm{rk}F} < \frac{< c_1(E_{|Y}).H^{n-2} >_Y}{\mathrm{rk}E} \ ,$$

where $\dim X = n$. As $\mathrm{codim}_X \mathrm{sing}(F) \geqslant 3$, and E is a vector bundle, then by Lemma 2.1 we have:

$$< c_1(F_{|Y}).(mH)^{n-2} >_Y = < c_1(F).(mH)^{n-1} >_X \ \text{, and}$$

$$< c_1(E_{|Y}).(mH)^{n-2} >_Y = < c_1(E).(mH)^{n-1} >_X \ ,$$

so

$$\frac{< c_1(F).H^{n-1} >_X}{\mathrm{rk}F} < \frac{< c_1(E).H^{n-1} >_X}{\mathrm{rk}E} \ .$$

Therefore we get stability on X.

□

4 Applications and questions

By recursive induction it is immediate to prove the following consequence of the main theorem:

Corollary 4.1. *Let X be a smooth projective variety of dimension n, and let E be a vector bundle on X. Let H be an ample divisor, let $Y_i \equiv m_i H$ be hypersurfaces numerically equivalent to a rational multiple of H, for $i = 1, \dots, r$ with $r < n$. Suppose that the complete intersections $Z_s := Y_1 \cap \cdots \cap Y_s$ are irreducible and smooth for all $s \leqslant r$. If $E_{|Z_r}$ is (semi)stable on Z_r, then E is (semi)stable on X (with respect to the polarizations induced by H).*

Restriction to smooth curves obtained in such a way is used in various ways in the literature: in some works of the author (cf. [Mis06, Mis20]) some complete intersection curves are constructed in order to prove the stability of vector bundles on higher dimensional varieties. Also, in [EL92], stability of Picard bundles is proven by restriction to curves which are intersection of theta divisors. Even though it is not needed to fix one subvariety in these cases, Corollary 4.1 could be used in these works.

Question 4.2. Since stability is obtained knowing stability on a fixed smooth subvariety, it is natural to ask about weakening the hypothesis to stability on a fixed singular hypersurface, possibly limiting the kind of singularities. We leave this question to further investigations.

Remark 4.3. In the recent works (cf. [BKK+15, MU19, Mis18, Mis19]) we consider asymptotic base loci of vector bundles, in order to get positivity properties, and construct Iitaka fibrations. It would be interesting to consider restrictions of stable vector bundles to (smooth subvarieties in) their asymptotic base loci as well. Any relationship between asymptotic base loci and stability would be surprising.

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